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A NOTE ON THE EXISTENCE OF SOLUTIONS FOR CAPUTO FRACTIONAL DIFFERENTIAL EQUATIONS

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ABSTRACT. The existence of a unique bounded continuous solution of a Caputo fractional differential equation is studied in this paper. The results are obtained from an equivalent Volterra integral equation which is derived by inverting the fractional differential equation. The kernel function of this integral equation is weakly singular and hence the standard techniques that are normally used for Volterra integral equations do not apply here. This hurdle is overcome by using a resolvent equation, applying some known properties of the resolvent, and then employing the contraction mapping principle.

1. Introduction

Recently, there has been a lot of focus on fractional derivatives and the differential equations that are created using fractional derivatives, sometimes known as fractional differential equations. In contrast, historical records reveal that the origins of fractional calculus and fractional equations date back 300 years, to the studies of Liouville, Riemann, and Leibniz—three of the most famous persons in mathematics history (see [28]). When the order of the derivative was not exactly an integer, the concept of fractional calculus was based solely on the theoretical question, "How to obtain derivatives and integrals when the order is not strictly an integer?"

The subject of derivatives without integer order was addressed by researchers; as a result, fractional calculus developed quickly. We refer to [14] and [25] for a thorough reading on fractional calculus and fractional differential equations. Since fractional calculus on continuous domains are extended to discrete and hybrid domains, the theories of fractional differential and fractional difference equations have been developed simultaneously. The analysis of fractional equations on continuous, discrete, and hybrid time domains is undoubtedly well-documented. Very significant advancements in the application of fractional equations can be found in tandem with the significant advances in the theory of fractional equations. A brief survey of the literature reveals the vital roles that fractional equations play in signal processing, computer vision theory, biology, physics, economics, and data fitting (see [3, 7, 13, 19, 22, 29, 30, 32]). We will highlight that fractional models yield significantly better results than integer order models in a variety of domains, which is undoubtedly why fractional equations are so popular in the life sciences (see [1, 2]). Recent advances in fractional equation theory and applications show that this area is still expanding and welcoming to new ideas.

Researchers can invert fractional differential equations as integral equations by using the definitions of fractional derivatives. It is also a viable and productive research direction to handle the issue as a

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Volterra integral equation and use classical analysis tools as an alternative to the innovative techniques that have been developed to analyze fractional differential equations. We cite [9, 10, 11, 20], and their references. In the current work, we investigate the existence and uniqueness of solutions to Caputo fractional differential equations under specific conditions, continuing our investigation in the context of the link between fractional and integral equations. In this paper we make use of the contraction mapping principle in proving existence and uniqueness of the solution.

In the next section, we provide some introductory materials that are essential to the development of this paper. Additionally, we provide some guiding remarks which clearly outlines our motivation and some of the challenges. Section 3 contains the main results and some examples as applications to our proven theorems.

2. Setup: The linkage between fractional and Volterra equations

We consider the Caputo fractional differential equation of order q

$$ {}^cD^q x(t) = -f(t,x(t)),\ x(0) = x_0 \in \mathbb{R},\ 0 < q < 1, \tag{2.1} $$

where $f : [0,\infty) \times \mathbb{R} \to \mathbb{R}$ is continuous, and ${}^cD^q$ denotes the Caputo differential operator of order q defined as

$$ {}^cD^q x(t) = \frac{1}{\Gamma(1-q)} \frac{d}{dt} \int_0^t (t-s)^{-q}[x(s)-x_0]ds $$

(see [14, p. 50]). Equation (2.1) can be inverted into the equivalent Volterra integral equation

$$ x(t) = x_0 - \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s,x(s))ds \tag{2.2} $$

where Γ is the regular gamma function. The proof of this inversion can be found in [23, p.54] or [14, p. 78, 86, 103].

Let

$$ C(t-s) = \frac{1}{\Gamma(q)}(t-s)^{q-1} \tag{2.3} $$

Then equation (2.2) becomes

$$ x(t) = x_0 - \int_0^t C(t-s) f(s,x(s))ds,\ t \geq 0, \tag{2.4} $$

which is a Volterra integral equation of convolution type. Equations (2.1) and (2.4) are equivalent in the sense that x is a solution of (2.1) if and only if x is a solution of (2.4). The objective of this paper is to show the existence of a unique solution of (2.1). It should be pointed out that mere continuity assumption on f may not guarantee the uniqueness of the solution on the entire interval $[0,\infty)$ or on a finite subinterval (see [14, Remark 6.9]). We will show it by proving the existence of a unique solution of equation (2.4).

If the function C defined in (2.3) were $L^1[0,\infty)$ then under some suitable conditions on C and f, one could easily prove the existence of a unique solution of (2.4). In fact, following [21, Theorem 2.1], one can show that if $f(t,0) = 0$, $|f(t,x) - f(t,y)| \leq k|x-y|$, and

$$k \int_0^t |C(t-s)ds| \leq \alpha < 1,$$

for all $t \geq 0$, then there exists a unique continuous solution of (2.4) and hence of (2.1).

Unfortunately, the function C defined in (2.3) is not $L^1[0,\infty)$. Indeed it is a weakly singular function by the definition given in [8] and in [12, p.25]. Therefore, a technique such as the one used in [21, Theorem 2.1] does not apply here. To overcome this hurdle, we consider a special case when $f(t,x) = x + h(t,x)$, and use the variation of parameters formula (2.6) associated with the Volterra equation (2.5) along with some known properties of the resolvent function which are provided in [24, p. 189-193]. For $f(t,x) = x + h(t,x)$, equation (2.4) becomes

$$x(t) = x_0 - \int_0^t C(t-s)[x(s) + h(s,x(s))]ds, \; t \geq 0, \tag{2.5}$$

The variation of parameters formula for the Volterra equation (2.5) is

$$x(t) = y(t) - \int_0^t R(t-s)h(s,x(s))ds, \; t \geq 0, \tag{2.6}$$

where the function $y(t)$ is given by

$$y(t) = x_0 - \int_0^t R(t-s)x_0 ds, \; t \geq 0, \tag{2.7}$$

and the function $R(t)$, known as the resolvent kernel of $C(t)$, is the solution of the resolvent equation

$$R(t) = C(t) - \int_0^t C(t-s)R(s)ds, \; t \geq 0, \tag{2.8}$$

It is known [24] that a function $x(t)$ is a solution of (2.5) if and only if $x(t)$ is a solution of (2.6), provided that $R(t)$ satisfies (2.8).

The function $C(t)$ defined in (2.3) is completely monotone on $(0,\infty)$ in the sense that $(-1)^m C^{(m)}(t) \geq 0$ for $m = 0,1,2,\ldots$ and $t \in (0,\infty)$. This $C(t)$ satisfies the conditions of Theorem 6.2 of [24], which states that the associated resolvent kernel $R(t)$ satisfies, for all $t \geq 0$,

$$0 \leq R(t) \leq C(t), \;\; R(t) \to 0 \;\; as \;\; t \to \infty, \tag{2.9}$$

and that

$$C(t) \notin L^1[0,\infty) \Rightarrow \int_0^\infty R(t)dt = 1. \tag{2.10}$$

Also, it is stated in [24, Theorem 7.2] that the resolvent $R(t)$ is completely monotone on $0 \leq t < \infty$. The information presented above can be found in [12], which contains a considerable amount of work on the use of resolvent in the study of Caputo fractional differential equation (2.1).

3. Main Results

Let $\mathbb{R}_+ := [0,\infty)$, and $\mathbb{R} := (-\infty,\infty)$. We set

$$Y := \{x : \mathbb{R}_+ \to \mathbb{R},\ x \text{ bounded and continuous}\}.$$

Then Y is a Banach space with the norm $||x|| = \sup_{t\geq 0} |x(t)|$. Suppose the function h in (2.5) satisfies a global Lipschitz condition, i.e.,

(A1) there exists a constant $k > 0$ such that $|h(t,x) - h(t,y)| \leq k|x-y|$ for all $t \geq 0$, and $x, y \in \mathbb{R}$.

In preparation for the construction of the main outcomes, we present the following auxiliary result which can be found in any textbook on Lebesgue integrals. Therefore, the proof of the next result is omitted.

Lemma 1. *Suppose $A \in L^1[0,\infty)$ and u is continuous on $[0,\infty)$. Then the convolution of A and u,*

$$b(t) = \int_0^t A(t-s)u(s)ds = \int_0^t A(s)u(t-s)ds,$$

is continuous on $[0,\infty)$.

In the next lemma, we show that the solution $x(t)$ given by (2.6) of (2.5) is bounded on $[0,\infty)$.

Lemma 2. *Suppose that there is a constant $\eta \in (0,1)$ and a continuous function $|\Psi(t)| \leq \eta$ so that*

$$|h(t,x)| \leq |\Psi(t)||x(t)|, \tag{3.1}$$

for all $x \in \mathbb{R}$. Then the solution function $x(t)$ given by (2.5) of (2.6) is bounded on $[0,\infty)$.

Proof. Let $x(t)$ be given by (2.6). Then

$$\begin{aligned} |x(t)| &\leq |x_0|(1 + |\int_0^t R(t-s)ds|) + \left|\int_0^t R(t-s)h(s,x(s))ds\right| \\ &\leq |x_0|(1 + \int_0^\infty R(t)dt) + ||\Psi|| \int_0^t R(t-s)|x(s)|ds \\ &= 2|x_0| + ||\Psi|| \int_0^t R(t-s)|x(s)|ds. \end{aligned}$$

We will finish the proof by contradiction. Assume $x(t)$ is unbounded. Then there is a sequence $\{t_n\} \uparrow \infty$ such that $|x(t)| \leq |x(t_n)|$ if $0 \leq t \leq t_n$. Then from the above inequality we have that

$$\begin{aligned} |x(t)| &\leq 2|x_0| + \eta \int_0^t R(t-s)|x(s)|ds \\ &\leq 2|x_0| + \eta \int_0^\infty R(t)dt|x(t_n)| \\ &= 2|x_0| + \eta|x(t_n)|, \end{aligned}$$

a contradiction for large t_n. Hence $x(t)$ is bounded. ∎

Lemma 3. *Suppose $A \in L^1[0,\infty)$, $f(t,x)$ is continuous on $[0,\infty)\times\mathbb{R}$, the function $h(t,x) = f(t,x) - x$ satisfies condition (A1), and $h(t,0)$ is bounded for $t \geq 0$. Then for each bounded continuous function x on $[0,\infty)$,*

$$z(t) = \int_0^t A(t-s)h(s,x(s))ds$$

is continuous and bounded on $[0,\infty)$.

Proof. The continuity assumption for f on $[0,\infty)\times\mathbb{R}$ implies that the function $h(t,x) = f(t,x) - x$ is also continuous on the same set. That is, if a function $x(t)$ is continuous on $[0,\infty)$, then $\phi(t) := h(t,x(t))$ is continuous on $[0,\infty)$ as well. In the sequel,

$$z(t) = \int_0^t A(t-s)\phi(s)\,ds$$

is continuous on $[0,\infty)$ by Lemma 1. As for boundedness, it follows from (A1) that

$$|h(t,x(t))| \leq k|x(t)| + |h(t,0)|.$$

Let $x(t)$ be any bounded function on $[0,\infty)$. Due to the above condition on $h(t,x(t))$ and the fact that $h(t,0)$ is bounded on $[0,\infty)$, we have that

$$\begin{aligned} |z(t)| &= \left|\int_0^t A(t-s)h(s,x(s))ds\right| \\ &\leq \int_0^t |A(t-s)|\big(k|x(s)| + |h(s,0)|\big)ds \\ &< \infty. \end{aligned}$$

This shows $z(t)$ is bounded on $[0,\infty)$ since $A \in L^1[0,\infty)$. □

Theorem 1. *In addition to the hypothesis of Lemma 2, we assume*

$$\lim_{t\to\infty} |\Psi(t)| = 0. \tag{3.2}$$

Then the solution $x(t)$ of (2.5) on $[0,\infty)$, which is given by (2.6) satisfies

$$\lim_{t\to\infty} x(t) = 0.$$

Proof. By Lemma 2, we know that the solutions are bounded. Thus, there is a positive constant M so that $|x(t)| \leq M$ for all $t \in [0,\infty)$. Additionally, from (2.10),

$$\lim_{t\to\infty} \int_0^t R(s)ds = 1.$$

Then from (2.7),

$$\lim_{t\to\infty} y(t) = x_0(1-\int_0^\infty R(t)dt) = x_0(1-1)=0.$$

Since $R \in L^1[0,\infty)$, it follows from a known result [12, p. 74, Convolution Lemma] that

$$\lim_{t\to\infty}\int_0^t R(t-s)\Psi(s)ds = 0. \tag{3.3}$$

Therefore,

$$|\int_0^t R(t-s)h(s,x(s))ds| \le \int_0^t R(t-s)|h(s,x(s))|ds$$
$$\le M\int_0^t R(t-s)|\Psi(s)|ds \to 0 \text{ as } t\to\infty$$

by (3.3). This implies

$$\lim_{t\to\infty}\int_0^t R(t-s)h(s,x(s))ds = 0.$$

Now from (2.6), we obtain

$$\lim_{t\to\infty} x(t) = \lim_{t\to\infty}\left[y(t)-\int_0^t R(t-s)h(s,x(s))ds\right] = 0.$$

This completes the proof of Theorem 1. □

The next result is devoted to existence of a unique bounded continuous solution for (2.5). In the recent paper [17], authors concentrate on linear and nonlinear Caputo fractional differential equations, and obtain upper and lower estimates for the separation of solutions. It should be highlighted that the following result can be linked with the content given in [17, Subsection 4.2].

Theorem 2. *Suppose that the conditions of Lemma 3 hold by setting $A = R$ and $0 < k < 1$, where R is given by (2.8), and k is the Lipschitz constant of (A1). Then equation (2.5) has a unique bounded continuous solution on $[0,\infty)$.*

Proof. Let L be an operator defined on the Banach space Y as follows. For each $\varphi \in Y$,

$$(L\varphi)(t) = y(t) - \int_0^t R(t-s)h(s,\varphi(s))ds \; t\ge 0,$$

where $y(t)$ is given in (2.7).

It follows from Lemma 1 and Lemma 2 that $(L\varphi)(t)$ is bounded and continuous on $[0,\infty)$. This shows that for each $\varphi \in Y$, $L\varphi \in Y$, showing that $L : Y \to Y$.

Let $\varphi, \psi \in Y$. Then for $t \geq 0$,

$$\begin{aligned}|(L\varphi)(t) - (L\psi)(t)| &\leq \int_0^t R(t-s)|h(s,\varphi(s)) - h(s,\psi(s))|ds \\ &\leq k||\varphi - \psi|| \int_0^\infty R(t)dt \\ &\leq l||\varphi - \psi||\,,\end{aligned}$$

where $l = k\int_0^\infty R(t)dt$. By (2.10), $\int_0^\infty R(t)dt = 1$. Therefore the constant $l < 1$ since $k < 1$. This shows that $L : Y \to Y$ is a contraction mapping. Hence there exists a unique function φ in Y that satisfies $\varphi = L\varphi$. This proves that there exists a unique bounded continuous solution function of (2.5) on $[0,\infty)$. This in turn proves that there exists a unique bounded continuous solution of the Caputo fractional differential equation (2.1) on $[0,\infty)$ for the special case: $f(t,x) = x + h(t,x)$.

□

Remark 1. *Suppose $x_0 = 0$ in (2.1), and $h(t,0) = 0$. Then from (2.7), $y(t) \equiv 0$ for all $t \geq 0$. This implies from (2.6) that $x(t) \equiv 0$ for all $t \geq 0$ is a solution of (2.6) and hence of (2.1). In this case, by Theorem 2, the zero solution $x(t) \equiv 0$ for all $t \geq 0$ is the only continuous solution of the Caputo differential equation (2.1).*

Remark 2. *In this context, we only focus on the scalar fractional differential equations and their associated Volterra integral equations for establishing a qualitative analysis regarding existence, uniqueness, boundedness and asymptotic behaviour of solutions. We shall emphasize that the outcomes of this manuscript can be adapted to vector-valued cases, and one may easily link the main results with stability and asymptotical stability of fractional differential equations and systems. We refer to the pioneering works* [15, 16]*, and references therein as inspiring papers for the qualitative analysis of fractional systems.*

Example 1. *For $q = \frac{1}{2}$, the kernel C defined in (2.3) is*

$$C(t-s) = \frac{1}{\Gamma(\frac{1}{2})}(t-s)^{-\frac{1}{2}}.$$

Clearly, $C \notin L^1[0,\infty)$. Employing the Laplace transform method, one can solve the associated resolvent equation

$$R(t) = C(t) - \int_0^t C(t-s)R(s)ds,$$

and obtain

$$R(t) = \frac{1}{\sqrt{\pi t}} - e^t\, erfc(\sqrt{t}).$$

This R satisfies $\int_0^\infty R(t)dt = 1$. Let $f(t,x) = x + h(t,x)$ in (2.1), where $h(t,x) = \frac{1}{2}e^{-t}x$. Use of the mean value theorem yields

$$|h(t,x) - h(t,y)| \leq |\frac{1}{2}e^{-t}||x-y| \leq k|x-y|.$$

Clearly, $0 < k < 1$, hence the function h satisfies the Lipschitz condition (A1). Therefore, Caputo differential equation (2.1) has a unique bounded continuous solution on $[0,\infty)$ for this function f when $q = \frac{1}{2}$. Also, note that the function h satisfies the condition in Theorem 1, and hence the unique solution $x(t)$ of (2.1) satisfies

$$\lim_{t\to\infty} x(t) = 0.$$

In the next example, we focus on a particular form of a differential equation which can be used for modelling physiological control systems (see [26]). Note that the following model is obtained by replacing the ordinary derivative with Caputo fractional derivative with $q \in (0,1)$.

Example 2. *Consider the fractional differential equation*

$$^cD^q x(t) = -a(t)x(t) + \frac{b(t)}{1+x^2(t)},\ 0 < q < 1 \tag{3.4}$$

with the initial data $x(0) = x_0$. We assume that a and b are continuous, positive-valued functions and $\sup_{t\in\mathbb{R}} b(t) = B < \infty$. If we compare (2.1) with (3.4), then it is obvious that

$$f(t,x(t)) = a(t)x(t) + \frac{p(t)}{1+x^2(t)},$$

where $p(t) = -b(t)$. When we focus on autonomous counterpart of (3.4) by letting $a(t) = 1$ for all $t \geq 0$, we get

$$f(t,x(t)) = x(t) + h(t,x(t)),$$

where

$$h(t,x(t)) = \frac{p(t)}{1+x^2(t)}.$$

In this situation, the fractional equation (3.4) is equivalent to integral equation

$$x(t) = x_0 - \int_0^t C(t-s)\left[x(s) + \frac{p(s)}{1+x^2(s)}\right]ds,\ t \geq 0, \tag{3.5}$$

where $C(t-s)$ is as in (2.3). Then, we have the variation of parameters formula for the Volterra equation (3.6) as follows

$$x(t) = y(t) - \int_0^t R(t-s)\frac{p(s)}{1+x^2(s)}ds,\ t \geq 0,$$

where

$$y(t) = x_0 - \int_0^t R(t-s)x_0 ds$$

and $R(t)$ is defined as in (2.8). Note that $h(t,0) = p(t)$ and

$$\begin{aligned} |h(t,x) - h(t,y)| &\leq |p(t)| \left| \frac{1}{1+x^2} - \frac{1}{1+y^2} \right| \\ &\leq B|x-y|. \end{aligned}$$

In the light of above-given arguments the fractional equation given in (3.4) has a unique bounded solution due to Theorem 2 whenever $B < 1$.

Remark 3. *If we alter the nonlinear term h in Example 2 as*

$$h(t,x(t)) = \frac{p(t)x(t)}{1+x^2(t)},$$

then (3.4) becomes

$$(3.6) \qquad {}^cD^q x(t) = -x(t) - h(t,x(t)) = -x(t) + \frac{b(t)x(t)}{1+x^2(t)}$$

where $a(t) = 1$ for $t \geq 0$, $p(t) = -b(t)$, and $\sup_{t\in\mathbb{R}} b(t) = B$. Suppose that $\lim_{t\to\infty} b(t) = 0$. Then the solution x of (3.6) satisfies $\lim_{t\to\infty} x(t) = 0$ as a consequence of Theorem 1 if $B < 1$.

In the next application, we are inspired by the Lasota–Wazewska equation without delay (see [33, Example 4.1], also [31])

$$x'(t) = -rx(t) + \eta e^{-\gamma x(t)}, t \in \mathbb{R},$$

which is proposed for modelling survival of red blood cells. Here, $x(t)$ stands for the number of red blood cells at time t, r is the probability of the death of blood cells, and η, γ are positive constants. Motivatedly, we construct the following fractional differential equation by again replacing the ordinary derivative with Caputo fractional derivative with $q \in (0,1)$.

Example 3. *Consider the fractional differential equation*

$$(3.7) \qquad {}^cD^q x(t) = -x(t) - \eta^* e^{-\gamma x(t)},\ 0 < q < 1 \text{ and } x(0) = x_0,$$

where $\eta^* < 0$ and $\gamma > 0$. A direct comparison between (2.1) with (3.7) yields to

$$f(t,x(t)) = x(t) + h(t,x(t)),$$

where $h(t,x(t)) = \eta^* e^{-\gamma x(t)}$. One may easily invert the following Volterra integral equation

$$x(t) = x_0 - \int_0^t C(t-s)\left[x(s) + \eta^* e^{-\gamma x(s)}\right] ds,\ t \geq 0,$$

for $C(t-s)$ is as in (2.3). We illustrate the corresponding variation of parameters formula

$$x(t) = y(t) - \eta^* \int_0^t R(t-s) e^{-\gamma x(s)} ds,\ t \geq 0,$$

where

$$y(t) = x_0 + \int_0^t R(t-s)x_0 ds$$

and $R(t)$ is as in (2.8). Obviously, $h(t,0)$ is bounded and

$$\begin{aligned}|h(t,x)-h(t,y)| &\leq |\eta^*|\left|e^{-\gamma x}-e^{-\gamma y}\right| \\ &\leq |\eta^*|\gamma|x-y|\end{aligned}$$

for any pair $x,y \in \mathbb{R}_+$. Then Theorem 2 implies that (3.7) has a unique continuous bounded solution on $\mathbb{R}_+$ if $|\eta^*|\gamma < 1$.

4. Future Directions

Discrete fractional calculus and fractional difference equations are also one of the popular subjects in theoretical and applied mathematics. Researchers have already established a wide literature on discrete variant of fractional equations. We can refer to the monograph [18] and the papers [4, 5, 6] as cornerstone works on this subject. We shall highlight that the outcomes of our paper can be reconstructed and enhanced with discrete fractional calculus as another research objective. Similar to the fractional differential equations, fractional difference equations are linked to Volterra equations. Thus, in the adaption of the outcomes of this manuscript to discrete fractional calculus one may utilize the monograph [27] since it provides an elaborative reading on Volterra difference equations and their associated resolvents.

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