One fixed point theorem on space of continuous functions with applications to nonlinear integral equations *

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Abstract

The work is concerned with coupled fixed point theorems for mixed monotone operators on Banach spaces. Especially, some examples and applications to nonlinear integral equations are given here to illustrate the usability of the obtained results.

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1 Introduction

Fixed point theorem for mixed monotone operators is introduced by Guo and Lakshmikantham at first in [3], where Guo and Lakshmikantham give some existence theorems of the coupled fixed points for both continuous and discontinuous operators and offered some applications to the initial value problems of ordinary differential equations with discontinuous right-hand sides. In 1996, Zhang studied fixed point of mixed monotone operators with convexity and concavity, and offered some applications to nonlinear integral equations on unbounded regions and differential equations in Banach spaces [13]. Thereafter many authors have investigated these kinds of operators in Banach spaces and obtained a lot of interesting and important results, which are used extensively in nonlinear differential and integral equations. In recent years, fixed point theory for mixed monotone operators is considered as one of the most important tools of nonlinear analysis that widely applied to optimization, computational algorithms, physics, variational inequalities, ordinary differential equations, integral equations, matrix equations and so on (see, for example,[13, 4, 7, 8, 1, 6]).

The purpose of this paper is to present a fixed point theorem for a mixed monotone operator on a real Banach space. The main result is a generalizations of the results of Zhang in [13]. Moreover, different examples and applications to non-linear integral equations are considered to illustrate the usability of our obtained results.

Now we briefly recall various basic definitions and facts.

Let $(E, \|\cdot\|)$ be a real Banach space and $P \subset E$ be a nonempty closed convex subset. P is a cone in E if the following properties hold:

(1) for any $x \in P$ and $\lambda \geqslant 0$, then $\lambda x \in P$,

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(2) for any $x \in P$, if $-x \in P$ then $x = \vartheta$,
where ϑ denotes the zero element of the Banach space E. If there exists a constant $C > 0$ such that

$$\|x\| \leqslant C\|y\| \text{ for any } x, y \in P \text{ with } \vartheta \leqslant x \leqslant y,$$

then P is called a normal cone.

The Banach space E is partially ordered by P, means $x \leqslant y$ if and only if $y - x \in P$ for any $x, y \in P$. For arbitrary $x_1, x_2 \in E$, the ordered interval is defined by

$$[x_1, x_2] = \{x \in E : x_1 \leqslant x \leqslant x_2\}.$$

Let E be a Banach space, which is partially ordered by a cone P and $F : E \times E \to E$. F is said to a mixed monotone operator if $F(x, y)$ is non-decreasing in x and is non-increasing in y, that is,

$$F(x_1, y) \leqslant F(x_2, y) \text{ holds for any } x_1, x_2, y \in E \text{ with } x_1 \leqslant x_2,$$

and

$$F(x, y_1) \leqslant F(x, y_2) \text{ holds for any } x, y_1, y_2 \in E \text{ with } y_1 \geqslant y_2.$$

An element $(x^*, y^*) \in E \times E$ is said to be a coupled fixed point of the operator F if

$$F(x^*, y^*) = x^* \text{ and } F(y^*, x^*) = y^*.$$

The element $x^* \in E$ is called a fixed point of F if $F(x^*, x^*) = x^*$. Clearly, if x^* is a fixed point of F, then (x^*, x^*) is a coupled fixed point of F.

2 Fixed point theorem

First, we give the following lemma, which is a key result.

Lemma 2.1. *Let $(E, \|\cdot\|)$ be a real Banach space, P be a normal cone in E, $F : P \times P \to P$ be a mixed monotone operator, the map $\varphi : (0, 1) \to (0, 1)$ be well-defined. Suppose that there exist $t_0 \in (0, 1)$, $\varphi(t_0) \in (t_0, 1)$ and $x_0 \in P$ such that*

$$t_0 x_0 \leqslant F(x_0, x_0) \leqslant \frac{1}{t_0} x_0 \tag{2.1}$$

and

$$\frac{1}{\varphi(t_0)} F(t_0 x, \frac{1}{t_0} y) - F(x, y) \geqslant 0 \tag{2.2}$$

holds for any $x, y \in P$.

Then there exist $k \in \mathbb{Z}^+$ and $u_0, v_0 \in P$ such that

$$t_0^{2k} v_0 \leqslant u_0 < v_0 \quad \text{and} \quad u_0 \leqslant F(u_0, v_0) \leqslant F(v_0, u_0) \leqslant v_0.$$

Proof. It is easy to get from (2.1) that

$$F(\frac{1}{t_0} x, t_0 y) \leqslant \frac{1}{\varphi(t_0)} F(x, y) \tag{2.3}$$

holds for any $x, y \in P$.

Since $\varphi(t_0) \in (t_0, 1)$, there exists $k \in \mathbb{Z}^+$ such that

$$[\varphi(t_0)]^k \geqslant t_0^{k-1}. \tag{2.4}$$

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Taking $u_0 = t_0^k x_0$ and $v_0 = t_0^{-k} x_0$, one finds that $u_0, v_0 \in P$ and $u_0 = t_0^{2k} v_0 < v_0$. Moreover, one obtains from (2.3) and (2.4) that

$$\begin{aligned} F(u_0, v_0) &= F(t_0^k x_0, t_0^{-k} x_0) = F(t_0 t_0^{k-1} x_0, \frac{1}{t_0}\frac{1}{t_0^{k-1}} x_0) \\ &\geqslant \varphi(t_0) F(t_0^{k-1} x_0, \frac{1}{t_0^{k-1}} x_0) \geqslant \cdots \geqslant [\varphi(t_0)]^k F(x_0, \frac{1}{t_0} x_0) \\ &\geqslant [\varphi(t_0)]^k t_0 x_0 \end{aligned}$$

and

$$\begin{aligned} F(v_0, u_0) &= F(t_0^{-k} x_0, t_0^k x_0) = F(\frac{1}{t_0}\frac{1}{t_0^{k-1}} x_0, t_0 t_0^{k-1} x_0) \\ &\leqslant \frac{1}{\varphi(t_0)} F(\frac{1}{t_0^{k-1}} x_0, t_0^{k-1} x_0) \leqslant \cdots \leqslant \frac{1}{[\varphi(t_0)]^k} F(x_0, x_0) \\ &\leqslant \frac{1}{[\varphi(t_0)]^k t_0} x_0. \end{aligned}$$

According to (2.4), one gets that

$$u_0 \leqslant [\varphi(t_0)]^k t_0 x_0 \leqslant F(u_0, v_0) \leqslant F(v_0, u_0) \leqslant \frac{1}{[\varphi(t_0)]^k t_0} x_0 < v_0 \tag{2.5}$$

and the desired results. □

In the sequel, we state and prove the main result.

Theorem 2.1. *Let $(E, \|\cdot\|)$ be a real Banach space, P be a normal cone in E, $F : P \times P \to P$ be a mixed monotone operator, the map $\varphi : (0,1) \to (0,1)$ be an increasing function. Suppose that there exist $t_0 \in (0,1)$, $\varphi(t_0) \in (t_0, 1]$ and $x_0 \in P$ such that the following properties hold*

- $t_0 x_0 \leqslant F(x_0, x_0) \leqslant \frac{1}{t_0} x_0$,
- $\frac{1}{\underbrace{(\varphi \circ \cdots \circ \varphi)}_{n}(t_0)} F(\underbrace{(\varphi \circ \cdots \circ \varphi)}_{n-1}(t_0) x, \frac{1}{\underbrace{(\varphi \circ \cdots \circ \varphi)}_{n-1}(t_0)} y) - F(x, y) \geqslant 0$ *holds for any $x, y \in P$ and any $n \in \mathbb{Z}^+$ with $\underbrace{(\varphi \circ \cdots \circ \varphi)}_{0}(t_0) = t_0$,*
- $\lim_{n\to\infty} (1 - \underbrace{(\varphi \circ \cdots \circ \varphi)}_{n}(t_0)) = 0.$

Then the mixed monotone operator F admits a unique fixed point $x^* \in P$.

Proof. According to Lemma 2.1, there exist $k \in \mathbb{Z}^+$ and $u_0, v_0 \in P$ such that

$$t_0^{2k} v_0 \leqslant u_0 < v_0 \quad \text{and} \quad u_0 \leqslant F(u_0, v_0) \leqslant F(v_0, u_0) \leqslant v_0.$$

Taking $u_0 = t_0^k x_0$ and $v_0 = t_0^{-k} x_0$, we get that $u_0, v_0 \in P$ satisfy that

$$t_0^{2k} v_0 = u_0 < v_0 \quad \text{and} \quad u_0 \leqslant F(u_0, v_0) \leqslant F(v_0, u_0) \leqslant v_0.$$

Construct the sequences

$$u_n = F(u_{n-1}, v_{n-1}) \quad \text{and} \quad v_n = F(v_{n-1}, u_{n-1}) \ \ (n = 1, 2, \cdots).$$

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It follows from Lemma 2.1 that

$$u_0 \leqslant u_1 = F(u_0, v_0) \leqslant v_1 = F(v_0, u_0) < v_0.$$

It is easy to deduce from (2.4) and (2.5) that

$$\begin{aligned} u_2 &= F(u_1, v_1) \geqslant F([\varphi(t_0)]^k t_0 x_0, \frac{1}{[\varphi(t_0)]^k t_0} x_0) \geqslant F(t_0^k x_0, \frac{1}{t_0^k} x_0) = u_1 \\ v_2 &= F(v_1, u_1) \leqslant F(\frac{1}{[\varphi(t_0)]^k t_0} x_0, [\varphi(t_0)]^k t_0 x_0) \leqslant F(\frac{1}{t_0^k} x_0, t_0^k x_0) = v_1. \end{aligned}$$

Moreover,

$$u_2 = F(u_1, v_1) \geqslant F([\varphi(t_0)]^k t_0 x_0, \frac{1}{[\varphi(t_0)]^k t_0} x_0) \geqslant \varphi(t_0) F([\varphi(t_0)]^k x_0, \frac{1}{[\varphi(t_0)]^k} x_0) \geqslant \varphi(t_0) u_1$$

and

$$v_2 = F(v_1, u_1) \leqslant F(\frac{1}{[\varphi(t_0)]^k t_0} x_0, [\varphi(t_0)]^k t_0 x_0) \leqslant \frac{1}{\varphi(t_0)} F(\frac{1}{[\varphi(t_0)]^k} x_0, [\varphi(t_0)]^k x_0) \leqslant \frac{1}{\varphi(t_0)} v_1.$$

Assume that

$$u_n \geqslant \underbrace{(\varphi \circ \cdots \circ \varphi)}_{n-1}(t_0) u_{n-1}, \qquad v_n \leqslant \frac{1}{\underbrace{(\varphi \circ \cdots \circ \varphi)}_{n-1}(t_0)} v_{n-1}.$$

So, $u_{n+1} = F(u_n, v_n) \geqslant F(u_{n-1}, v_{n-1}) = u_n, v_{n+1} = F(v_n, u_n) \leqslant F(v_{n-1}, u_{n-1}) = v_n$ and

$$\begin{aligned} u_{n+1} &= F(u_n, v_n) \\ &\geqslant F(\underbrace{(\varphi \circ \cdots \circ \varphi)}_{n-1}(t_0) u_{n-1}, \frac{1}{\underbrace{(\varphi \circ \cdots \circ \varphi)}_{n-1}(t_0)} v_{n-1}) \\ &\geqslant \underbrace{(\varphi \circ \cdots \circ \varphi)}_{n}(t_0) F(u_{n-1}, v_{n-1}) \\ &\geqslant \underbrace{(\varphi \circ \cdots \circ \varphi)}_{n}(t_0) u_n, \\ v_{n+1} &= F(v_n, u_n) \\ &\leqslant F(\frac{1}{\underbrace{(\varphi \circ \cdots \circ \varphi)}_{n-1}(t_0)} v_{n-1}, \underbrace{(\varphi \circ \cdots \circ \varphi)}_{n-1}(t_0) u_{n-1}) \\ &\leqslant \frac{1}{\underbrace{(\varphi \circ \cdots \circ \varphi)}_{n}(t_0)} F(v_{n-1}, u_{n-1}) \\ &\leqslant \frac{1}{\underbrace{(\varphi \circ \cdots \circ \varphi)}_{n}(t_0)} v_n. \end{aligned}$$

Thus, the sequences $\{u_n\}$ and $\{v_n\}$ satisfy that

$$u_0 \leqslant u_1 \leqslant \cdots \leqslant u_n \leqslant \cdots \leqslant v_n \leqslant \cdots \leqslant v_1 \leqslant v_0. \tag{2.6}$$

Noting that $u_0 \leqslant t_0^{2k} v_0$, we can get $u_n \geqslant u_0 \geqslant t_0^{2k} v_0 \geqslant t_0^{2k} v_n$ $(n = 1, 2, \cdots)$. Moreover,

$$v_1 = F(v_0, u_0) \geqslant F(t_0^{-2k} u_0, t_0^{2k} v_0) \geqslant F(t_0 u_0, t_0^{-1} v_0) \geqslant \varphi(t_0) u_1.$$

Assume that $v_n \geqslant \underbrace{(\varphi \circ \cdots \circ \varphi)}_{n}(t_0) u_n$. Then

$$\begin{aligned}
& v_{n+1} = F(v_n, u_n) \\
& \geqslant F(\underbrace{(\varphi \circ \cdots \circ \varphi)}_{n}(t_0) u_n, \frac{1}{\underbrace{(\varphi \circ \cdots \circ \varphi)}_{n}(t_0)} v_n) \\
& \geqslant \underbrace{(\varphi \circ \cdots \circ \varphi)}_{n+1}(t_0) F(u_n, v_n) \\
& \geqslant \underbrace{(\varphi \circ \cdots \circ \varphi)}_{n+1}(t_0) u_{n+1}.
\end{aligned}$$

Thus, it holds for any natural number p that

$$\begin{aligned}
\theta & \leqslant u_{n+p} - u_n \leqslant v_n - u_n \leqslant (1 - \underbrace{(\varphi \circ \cdots \circ \varphi)}_{n}(t_0)) v_n \leqslant (1 - \underbrace{(\varphi \circ \cdots \circ \varphi)}_{n}(t_0)) v_0, \\
\theta & \leqslant v_n - v_{n+p} \leqslant v_n - u_n \leqslant (1 - \underbrace{(\varphi \circ \cdots \circ \varphi)}_{n}(t_0)) v_n \leqslant (1 - \underbrace{(\varphi \circ \cdots \circ \varphi)}_{n}(t_0)) v_0.
\end{aligned}$$

Since the come P is normal and $\lim_{n \to \infty} \underbrace{(\varphi \circ \cdots \circ \varphi)}_{n}(t_0) = 1$, one gets that

$$\|u_{n+p} - u_n\| \leqslant (1 - \underbrace{(\varphi \circ \cdots \circ \varphi)}_{n}(t_0)) \|v_0\| \to 0 \ (n \to \infty)$$

and

$$\|v_n - v_{n+p}\| \leqslant (1 - \underbrace{(\varphi \circ \cdots \circ \varphi)}_{n}(t_0)) \|v_0\| \to 0 \ (n \to \infty).$$

So, the sequences $\{u_n\}$ and $\{v_n\}$ are Cauchy sequences. Because E is complete, there exist u^* and v^* such that

$$u_n \to u^* \ (n \to \infty) \quad \text{and} \quad v_n \to v^* \ (n \to \infty).$$

According to (2.6), one obtains that $u_n \leqslant u^* \leqslant v^* \leqslant v_n$ with $u^*,\ v^* \in P$ and

$$\theta \leqslant v^* - u^* \leqslant v_n - u_n \leqslant (1 - (1 - \underbrace{(\varphi \circ \cdots \circ \varphi)}_{n}(t_0))) v_0.$$

The fact that $\lim_{n \to \infty} (1 - \underbrace{(\varphi \circ \cdots \circ \varphi)}_{n}(t_0)) = 1$, implies that $u^* = v^*$. Setting $x^* := u^* = v^*$, one gets that

$$u_{n+1} = F(u_n, v_n) \leqslant F(x^*, x^*) \leqslant F(v_n, u_n) = v_{n+1}.$$

Taking $n \to \infty$, we obtain that $x^* = F(x^*, x^*)$. That is, x^* is a fixed point of F in P.

The uniqueness of x^* is deduced from the definiteness of $x_0 \in P$ and the value of k obtained in Lemma 2.1. □

Remark 2.1. *Compared with the corresponding result in [[11], Theorem 2.1], we focus on one special point $t_0 \in (0,1)$ with $\varphi(t_0) \in (t_0, 1)$ and one element $x_0 \in P$ such that*

$$t_0 x_0 \leqslant F(x_0, x_0) \leqslant \frac{1}{t_0} x_0.$$

The assumption in Theorem 2.1, is weaker than the assumption in [11] to a certain extent.

Remark 2.2. *For Theorem 2.1, it is not easy to find the special point $t_0 \in (0,1)$ and the mapping φ such that $\lim_{n\to\infty}(1 - \underbrace{(\varphi \circ \cdots \circ \varphi)}_{n}(t_0)) = 0$. Compared with the assumption of special point and the mapping in Theorem 2.1, Zhai [12] introduced a fixed point theorem for a class of a mixed monotone operators which is stated as Theorem 2.2.*

Theorem 2.2. *([12]) Let $(E, \|\cdot\|)$ be a real Banach space, P be a normal cone in E. Suppose that $F : P \times P \to P$ is a mixed monotone operator satisfying*

(1) for any $c \in (0,1)$, $x, y \in P$, there exists $\alpha(c,x,y) \in (1, +\infty)$ such that

$$F(cx, y) \leqslant c^{\alpha(c,x,y)} F(x,y);$$

(2) there exists $u_0, v_0 \in P$, $r \in (0,1)$ such that

$$u_0 \leqslant rv_0,\ F(u_0, v_0) \geqslant u_0,\ F(v_0, u_0) \leqslant v_0.$$

Then, F has a unique fixed point $u^* \in [u_0, rv_0]$. Moreover, the successive sequences

$$x_n = F(x_{n-1}, y_{n-1}),\ y_n = F(y_{n-1}, x_{n-1})\ (n = 1, 2, \cdots)$$

for any initial values $x_0, y_0 \in [u_0, rv_0]$, has the following property

$$\lim_{n\to\infty} \|x_n - u^*\| = 0 \text{ and } \lim_{n\to\infty} \|y_n - u^*\| = 0.$$

3 Application to nonlinear nonlinear integral equations

As mentioned in Remark 2.2, there are fewer examples to explain how use Theorem 2.1. In this section, we present some examples, where Theorem 2.2 can be applied. Let $D \subset \mathbb{R}^n$ be a simply connected region. We consider the following nonlinear integral equation

$$x(t) = Ax(t) = \int_D K(t,s) g(t, x(s), f(x(s))) ds. \tag{3.1}$$

Theorem 3.1. *Suppose that $D \subset \mathbb{R}^n$ is a simply connected region, $C_B(D)$ is the Banach space with $\|x\| = \sup_{t\in D} |x(t)|$, $g(t,u,v) : D \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}^+$ is continuous, $K : D \times D \to \mathbb{R}^+$ is a continuous function. Assume that*

(i) the mapping $f : C_B(D) \to C_B(D)$ is positive on the domain of f;

(ii) $g(t,u,v)$ is non-decreasing in u and non-increasing in v;

(iii) for any $c \in (0,1)$, any nonnegative continuous functions $u(s), v(s) \in C_B(D)$, there exists $\alpha(t,u,v) \in (1, +\infty)$ such that

$$g(t, cu, v) \leqslant c^{\alpha(t,u,v)} g(t,u,v)$$

and $g(t,u,v) = 0$ whenever $K(t,s) = 0$;

(iv) there exist nonnegative continuous functions $u_0(s), v_0(s) \in C_B(D)$, $r \in (0,1)$ such that

$$u_0 \leqslant rv_0,\ \int_D K(t,s) g(t, u_0(s), f(v_0(s))) ds \geqslant u_0(t),\ \int_D K(t,s) g(t, v_0(s), f(u_0(s))) ds \leqslant v_0(t).$$

Then, the equation (3.1) has a unique solution $u^* \in [u_0, rv_0]$. Moreover, the successive sequences

$$x_n = \int_D K(t,s)g(t, x_{n-1}(s), y_{n-1}(s))ds,\ y_n = \int_D K(t,s)g(t, y_{n-1}(s), x_{n-1}(s))ds\ (n = 1, 2, \cdots)$$

for any initial values $x_0, y_0 \in [u_0, rv_0]$, has the following property

$$\lim_{n\to\infty} \|x_n - u^*\| = 0 \text{ and } \lim_{n\to\infty} \|y_n - u^*\| = 0.$$

Proof. Let $P = C_B^+(D)$ denote the set of nonnegative functions of $C_B(D)$. Then P is a normal cone of $C_B(D)$. The equation (3.1) can be written in the form

$$x = F(x, x),$$

where

$$F(x, y) = \int_D K(t,s)g(t, x(s), f(y(s)))ds.$$

According to the hypothesis of Theorem 3.1, one finds that $F : P \times P \to P$ is a mixed monotone operator. Moreover, for any $c \in (0, 1)$, any nonnegative continuous functions $u(s), v(s) \in C_B(D)$, there exists $\alpha(t, u, v) \in (1, +\infty)$ such that

$$\begin{aligned} F(cx, y) &= \int_D K(t,s)g(t, cx(s), f(y(s)))ds \\ &\leqslant c^{\alpha(c,x,y)} \int_D K(t,s)g(t, x(s), f(y(s)))ds \\ &= c^{\alpha(c,x,y)} F(x, y). \end{aligned}$$

Also, one can choose $u_0, v_0 \in P$ and $r \in (0, 1)$ such that

$$\begin{aligned} F(u_0, v_0) &= \int_D K(t,s)g(t, u_0(s), f(v_0(s)))ds \geqslant u_0(t), \\ F(v_0, u_0) &= \int_D K(t,s)g(t, v_0(s), f(u_0(s)))ds \leqslant v_0(t). \end{aligned}$$

Based on Theorem 2.2, F has a unique nonnegative function $u^* \in P$ such that $F(u^*, u^*) = u^*$. So, u^* is the solution to the equation (3.1). □

Remark 3.1. *Theorem 3.1 conditions onto mapping f is wide range. Consequently, this theorem can be considered as the generalization of such type theorems in [1, 2, 5, 6].*

Remark 3.2. *Theorem 3.1 is a useful tool to deal with the existence and uniqueness of positive solutions for nonlinear integral equation and non-linear fractional partial differential equations*

Now, we present the following nonlinear integral equation.

$$x(t) = Ax(t) = \int_{\mathbb{R}^n} K(t,s)\left[4t^2 + 1 + x^2(s) + \sqrt{1 - x^2(s)}\right]ds. \tag{3.2}$$

Proposition 3.1. *Suppose that $K : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^+$ is a continuous function and $K(t,s) \not\equiv 0$. Then for any fixed $a \in [0, 1)$ the equation (3.2) has a unique solution $x^*(t)$ satisfying $a \leqslant x^* < 1$ and $x^*(t) \not\equiv 0$, provided that one of the following holds*

(i) $0 \leqslant \int_{\mathbb{R}^n} K(t,s)ds \leqslant \frac{1}{4t^2+2}$ *for* $a = 0$;

(ii) $\frac{a}{4t^2+1+a^2} \leqslant \int_{\mathbb{R}^n} K(t,s)ds \leqslant \frac{1}{4t^2+2+\sqrt{1-a^2}}$ *for* $a \in (0, 1)$.

Proof. Let $C_B(\mathbb{R}^n)$ be the Banach space with $\|x\| = \sup_{t\in\mathbb{R}^n}|x(t)|$ and $P = C_B^+(\mathbb{R}^n)$ denote the set of nonnegative functions of $C_B(\mathbb{R}^n)$. Then P is a normal cone of $C_B(\mathbb{R}^n)$. The equation (3.2) can be written in the form

$$x = F(x, x),$$

where

$$F(x, y) = \int_{\mathbb{R}^n} K(t, s)g(t, x(s), f(y(s)))ds$$

and

$$g(t, x(s), f(y(s))) = 4t^2 + 1 + x^2(s) + f(y(s)).$$

where $f(y(s)) = \sqrt{1 - y^2(s)}$. Obviously, $g(t, x, f(y))$ is increasing in $x \in P$ and decreasing in $y \in P$. So, $F : P \times P \to P$ is a mixed monotone operator. Moreover,

$$g(t, cx, y) \leqslant c^{\alpha(c,x,y)}g(t, x, y)$$

for $c \in (0, 1)$, nonnegative continuous functions $x(s), y(s) \in P$ making $g(t, x, y)$ sense, where

$$1 < \alpha(c, x, y) \leqslant \left(\ln\left[\frac{1}{c}\right]\right)^{-1} \ln\left[\frac{1 + x^2(s) + \sqrt{1 - y^2(s)}}{1 + c^2x^2(s) + \sqrt{1 - y^2(s)}}\right].$$

Taking $u_0 = a$ and $v_0 = 1$, one finds that

$$F(u_0, v_0) = F(a, 1) = \int_{\mathbb{R}^n} K(t, s)g(t, a, f(1))ds = (4t^2 + 1 + a^2)\int_{\mathbb{R}^n} K(t, s)ds \geqslant \begin{cases} 0 & \text{for } a = 0, \\ a & \text{for } a \in (0, 1), \end{cases}$$
$$F(v_0, u_0) = F(1, a) = \int_{\mathbb{R}^n} K(t, s)g(t, 1, f(a))ds = (4t^2 + 2 + \sqrt{1 - a^2})\int_{\mathbb{R}^n} K(t, s)ds \leqslant 1.$$

Thus, all hypothesis of Theorem 3.1 are satisfied. So, F has exactly one fixed point x^* with $a \leqslant x^*(t) < 1$. Therefore, the equation (3.2) has a unique solution $x^*(t)$ satisfying $a \leqslant x^* < 1$ and $x^*(t) \not\equiv 0$ for any fixed $a \in [0, 1)$. □

Example 3.1. *The non-linear integral equation*

$$x(t) = \int_0^1 (1 + \sqrt{1 - x^2(s)} + x^{\frac{3}{2}}(s))e^{-2t-2s-1}ds \ (t \in [0, 1]), \tag{3.3}$$

has a unique solution $x^*(s)$ *satisfying* $0 \leqslant x^*(s) < 1$ *and* $x^*(s) \not\equiv 0$.

Proof. Let $C[a, b]$ be the Banach space with $\|x\| = \sup_{t\in[a,b]} |x(t)|$ and $P = C^+[a, b]$ denote the set of nonnegative functions of $C[a, b]$. Then P is a normal cone of $C[a, b]$. The equation (3.3) can be written in the form

$$x = F(x, x),$$

where

$$F(x, y) = \int_0^1 e^{-2t-2s-1}g(s, x(s), y(s))ds$$

and

$$g(t, x(s), y(s)) = 1 + x^{\frac{3}{2}}(s) + \sqrt{1 - y^2(s)}.$$

Obviously, $g(t, x, y)$ is increasing in x and non-increasing in y. So, $F : P \times P \to P$ is a mixed monotone operator. Moreover,

$$g(t, cx, y) \leqslant c^{\alpha(c,x,y)} g(t, x, y)$$

for $c \in (0, 1)$, nonnegative continuous functions $x(s), y(s) \in P$ with making $g(t, x, y)$ sense, where

$$1 < \alpha(c, x, y) \leqslant \left(\ln\left[\frac{1}{c}\right]\right)^{-1} \ln\left[\frac{1 + x^{\frac{3}{2}}(s) + \sqrt{1 - y^2(s)}}{1 + c^{\frac{3}{2}} x^{\frac{3}{2}}(s) + \sqrt{1 - y^2(s)}}\right].$$

Taking $u_0 = 0$ and $v_0 = 1$, one finds that

$$F(u_0, v_0) = F(0, 1) = \int_0^1 e^{-2t-2s-1} g(s, 0, 1) ds = \int_0^1 e^{-2t-2s-1} ds = \frac{e^2 - 1}{2e^2} e^{-2t-1} \geqslant 0,$$
$$F(v_0, u_0) = F(1, 0) = \int_0^1 e^{-2t-2s-1} g(s, 1, 0) ds = 3 \int_0^1 e^{-2t-2s-1} ds = \frac{3(e^2 - 1)}{2e^2} e^{-2t-1} \leqslant 1.$$

Thus, all hypothesis of Theorem 3.1 are satisfied. So, F has exactly one fixed point x^* with $0 \leqslant x^*(t) < 1$. Therefore, the equation (3.3) has a unique solution $x^*(t)$ satisfying $a \leqslant x^* < 1$ and $x^*(t) \not\equiv 0$ for any fixed $a \in [0, 1)$. □

Fractional differential equations can describe many phenomena in various fields of science and engineering such as control, porous media, electrochemistry, etc. Recall that the Riemann-Liouville fractional derivative of order α for a continuous function f is defined by

$$D^\alpha y(t) = \frac{1}{\Gamma(n - \alpha)} \left(\frac{d}{dt}\right)^n \int_0^t \frac{y(s)}{(t - s)^{\alpha - n + 1}} ds,$$

where Γ is the gamma function and $n = [\alpha] + 1$. Next, we give an application of Theorem 3.1 to the initial value problem for the fractional differential equation

$$\begin{cases} D^\nu u(t) + h(t) f(t, u(t)) = 0, \ 0 < t < 1, \ n - 1 < \nu \leqslant n, \\ u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \\ [D^\alpha u(t)]_{t=1} = 0, \ 1 \leqslant \alpha \leqslant n - 2, \end{cases} \tag{3.4}$$

where $n \in \mathbb{N}$ and $n > 3$, D^ν is the standard Riemann-Liouville fractional derivative, $f \in C([0, 1] \times [0, \infty))$ and $h \in C(0, 1) \cap L(0, 1)$ is nonnegative and may be singular at $t = 0$ and/or $t = 1$.

Lemma 3.1. *([2]) Let $f \in C([0, 1] \times [0, \infty), (0, \infty))$ and $h \in C(0, 1) \cap L(0, 1)$ be nonnegative and may be singular at $t = 0$ and/or $t = 1$. Then the problem (3.4) is equivalent to*

$$u(t) = \int_0^1 G(t, s) h(s) f(s, u(s)) ds, \tag{3.5}$$

where

$$G(t, s) = \frac{1}{\Gamma(\nu)} \begin{cases} t^{\nu-1}(1 - s)^{\nu-\alpha-1} - (t - s)^{\nu-1}, & 0 \leqslant s \leqslant t \leqslant 1, \\ t^{\nu-1}(1 - s)^{\nu-\alpha-1}, & 0 \leqslant t \leqslant s \leqslant 1. \end{cases} \tag{3.6}$$

Inspired by Goodrich[2] and Xu, Wei and Dong[10], we establish the existence and uniqueness of solution to the problem (3.4) as an application of Theorem 3.1, which may be regarded as an extension of [10].

Theorem 3.2. *Let $f \in C([0,1]\times[0,\infty),(0,\infty))$ and $h \in C(0,1)\cap L(0,1)$ be nonnegative (may be singular at $t=0$ and/or $t=1$). Suppose that*

(i) $f(s,x)$ is non-decreasing in x;

(ii) for any $c \in (0,1)$ and nonnegative continuous function $x(s) \in C[a,b]$, there exists $\alpha(t,x) \in (1,+\infty)$ such that

$$f(s,cx) \leqslant c^{\alpha(t,x)} f(s,x);$$

(iii) the function $G(t,s) : [0,1]\times[0,1]$ is stated as (3.6) satisfying

$$\int_0^1 G(t,s)h(s)f(s,1)ds \leqslant 1.$$

Then, the problem (3.5) has a unique solution $x^* \in [0,1)$. Moreover, the successive sequences

$$x_n = \int_0^1 G(t,s)h(s)f(s,x_{n-1})ds \ (n=1,2,\cdots)$$

for any initial values $x_0 \in [0,1)$, has the following property

$$\lim_{n\to\infty}\max_{t\in[0,1]}|x_n(t)-x^*(t)| = 0.$$

Proof. Let $C[a,b]$ be the Banach space with $\|x\| = \sup_{t\in[a,b]}|x(t)|$ and $P = C^+[a,b]$ denote the set of nonnegative functions of $C[a,b]$. Then P is a normal cone of $C[a,b]$. The equation (3.5) can be written in the form

$$x = F(x,x),$$

where

$$F(x,y) = \int_0^1 G(t,s)h(s)g(s,x(s),y(s))ds$$

and

$$g(t,x(s),y(s)) = f(s,x(s)).$$

Obviously, $G(t,s)$ is continuous and nonnegative on $[0,1]\times[0,1]$, $g(t,x,y)$ is increasing in x and non-increasing in y. So, $F : P\times P \to P$ is a mixed monotone operator. Moreover,

$$g(t,cx,y) = f(s,cx(s)) \leqslant c^{\alpha(c,x)}g(t,x,y) = c^{\alpha(c,x)}f(s,x(s))$$

for $c \in (0,1)$ and $x \in P$. Taking $u_0 = 0$ and $v_0 = 1$, one finds that

$$F(u_0,v_0) = F(0,1) = \int_0^1 G(t,s)h(s)g(s,0,1)ds = \int_0^1 G(t,s)h(s)f(s,0)ds \geqslant 0,$$
$$F(v_0,u_0) = F(1,0) = \int_0^1 e^{-2t-2s-1}g(s,1,0)ds = \int_0^1 G(t,s)h(s)f(s,1)ds \leqslant 1.$$

Thus, all hypothesis of Theorem 3.1 are satisfied and so F has exactly one fixed point $x^* \in [0,1)$. Therefore, the equation (3.5) has a unique solution $x^*(t) \in [0,1)$ satisfying $x^*(t) \not\equiv 0$. □

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Next, we use Theorem 3.1 to give existence and uniqueness results for a classical fractional-boundary-value problem in [5], which reads as

$$\begin{cases} \frac{D^\nu}{Dt}u(s,t) + f\left(s,t,u(s,t),\frac{\partial}{\partial s}u(s,t)\right) = 0, \\ 0 < \epsilon < T,\ T \geqslant 1,\ t \in [\epsilon, T],\ 0 < \nu < 1,\ s \in [a,b], \\ u(s,\eta) = u(s,T),\ (s,\eta) \in [a,b] \times (\epsilon, T). \end{cases} \tag{3.7}$$

Lemma 3.2. *([5]) Let $(s,t) \in [a,b] \times [\epsilon, T],\ (s,\eta) \in [a,b] \times (\epsilon, t)$ and $0 < \nu < 1$. Then the equation*

$$\frac{D^\nu}{Dt}u(s,t) + f\left(s,t,u(s,t),\frac{\partial}{\partial s}u(s,t)\right) = 0$$

with boundary condition $u(s,\eta) = u(s,T)$, has a solution u_0 if and only if u_0 is a solution of the fractional integral equation

$$u(s,t) = \int_\epsilon^T G(t,\xi) f\left(s,\xi,u(s,\xi),\frac{\partial}{\partial s}u(s,\xi)\right) d\xi$$

where

$$G(t,\xi) = \frac{1}{\Gamma(\nu)} \begin{cases} [t^{\nu-1}(\eta-\xi)^{\nu-1} - t^{\nu-1}(T-\xi)^{\nu-1}]/(\eta^{\nu-1} - T^{\nu-1}) - (t-\xi)^{\nu-1}, & \epsilon \leqslant \xi \leqslant \eta \leqslant t \leqslant T, \\ \left[-t^{\nu-1} - (T-\xi)^{\nu-1}\right]/(\eta^{\nu-1} - T^{\nu-1}) - (t-\xi)^{\nu-1}, & \epsilon \leqslant \eta \leqslant \xi \leqslant t \leqslant T, \\ \left[-t^{\nu-1}(T-\xi)^{\nu-1}\right]/(\eta^{\nu-1} - T^{\nu-1}), & \epsilon \leqslant \eta \leqslant t \leqslant \xi \leqslant T. \end{cases} \tag{3.8}$$

Let $E = C([a,b] \times [\epsilon, T])$ be the Banach space of continuous functions on $[a,b] \times [\epsilon, T]$ with the sup norm, and set

$$P = \{y \in C([a,b] \times [\epsilon, T]) : \min_{(s,t)\in[a,b]\times[\epsilon,T]} y(s,t) \geqslant 0\}.$$

It is pointed out in [5] that P is a normal cone in E. Thus, the Theorem 2.2 in [5] is regarded as a corollary of Theorem 3.1.

Corollary 3.1. *(Theorem 2.2 in [5]) Let $0 < \epsilon < T$ be given. Suppose that the following properties hold:*

(H1) $\frac{\partial}{\partial s}v(s,t) \geqslant 0$ for any $v(s,t) \geqslant 0$;

(H2) $f(s,t,u(s,t),v(s,t)) \in C([a,b] \times [\epsilon, T], [0,\infty), [0,\infty))$ is increasing in u and decreasing in v;

(H3) for $c \in (0,1)$, $u, v \in P$, there exists $\alpha(c,u,v) \in (1,\infty)$ such that

$$f(s,t,cu(s,t),v(s,t)) \leqslant c^{\alpha(c,u,v)} f(s,t,u(s,t),v(s,t))$$

and $f(s,t,u(s,t),v(s,t)) = 0$ whenever $G(s,t) < 0$;

(H4) there $u_0,\ v_0 \in P$ and $r \in (0,1)$ such that

$$\begin{aligned} & u_0(s,t) \leqslant r v_0(s,t), \\ & \int_\epsilon^T G(t,\xi) f\left(s,\xi,u_0(s,\xi),\frac{\partial}{\partial s}v_0(s,\xi)\right) d\xi \geqslant u_0(s,t), \\ & \int_\epsilon^T G(t,\xi) f\left(s,\xi,u_0(s,\xi),\frac{\partial}{\partial s}v_0(s,\xi)\right) d\xi \leqslant v_0(s,t) \end{aligned}$$

for $(s,t) \in [a,b] \times [\epsilon, T]$. Then the fractional-boundary-value problem (3.7) has a unique solution $u^* \in [u_0, rv_0]$. Moreover, the sequences

$$\begin{cases} u_{n+1}(s,t) = \int_\epsilon^T G(t,\xi) f\left(s,\xi,u_n(s,\xi),\frac{\partial}{\partial s}v_n(s,\xi)\right) d\xi, \\ v_{n+1}(s,t) = \int_\epsilon^T G(t,\xi) f\left(s,\xi,v_n(s,\xi),\frac{\partial}{\partial s}u_n(s,\xi)\right) d\xi \end{cases} \quad n = 0, 1, \cdots$$

satisfy that $\lim_{n\to\infty} \|u_n - u^*\| = 0$ and $\lim_{n\to\infty} \|v_n - u^*\| = 0$.

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