Existence for a nonlinear integro-differential equation with the Hilfer fractional derivative

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Abstract

This paper is to study the uniqueness of solutions to a new nonlinear Hilfer integro-differential equation with an initial condition and arbitrary numbers of the Riemann-Liouville fractional integral operators. Our investigation is based on an equivalent implicit integral equation in series obtained from Babenko's approach, the multivariate Mittag-Leffler function as well as Banach's contractive principle in a new Banach space. The technique used clearly opens up new directions for studying other types of initial or boundary value problems with different fractional derivatives and variable coefficients. An illustrative example is also provided to demonstrate applications of the key theorem.

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1 Introduction

Let $-\infty < a < b < +\infty$ and $\lambda_i \in R$ for $i = 1, 2, \cdots, m$. We shall consider the following nonlinear integro-differential equation with an initial condition:

$$\begin{cases} D_{a^+}^{\alpha,\beta} u(x) + \sum_{i=1}^{m} \lambda_i \, I_{a^+}^{\beta_i} u(x) = I_{a^+}^{\beta} g(x, u(x)), \quad 0 < \alpha < 1, \ 0 \leq \beta < 1, \ \beta_i \geq \beta, \\ I_{a^+}^{1-\gamma} u(a) = u_a \in R, \quad \gamma = \alpha + \beta - \alpha\beta, \end{cases} \tag{1.1}$$

where $x \in (a, b]$ and $D_{a^+}^{\alpha,\beta}$ is the Hilfer fractional derivative of order α and type β [1, 2], which is an interpolation between the Riemann-Liouville and Caputo fractional derivatives. The operator $I_{a^+}^{\beta_i}$ is the Riemann-Liouville fractional integral of the order β_i, the nonlinear term $g : (a, b] \times R \to R$ is a function satisfying certain conditions. In 2000, Hilfer introduced the Hilfer fractional derivative which combines Caputo and Riemann-Liouville fractional derivatives, and can be used in the theoretical simulation of dielectric relaxation in glass

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forming materials [3, 4]. Sandev et al. [5] derived the existence results of the fractional diffusion equation with the Hilfer fractional derivative which attained in terms of the Mittag Leffler functions. In 2015, Gu and Trujillo [6] studied the existence results of the fractional differential equations with the Hilfer derivative based on noncompact measure method.

Clearly, the parameter γ satisfies

$$0 < \max\{\alpha, \beta\} \leq \gamma < 1, \quad 1 - \gamma < 1 - \beta(1 - \alpha).$$

The two-parameter fractional derivative $D_{a^+}^{\alpha,\beta}$ generates more types of stationary states and gives an extra degree of freedom on the initial condition with applications in physics [3, 4, 7]. In 2012, Furati et al. [8] studied the following nonlinear Hilfer differential equation with an initial condition:

$$\begin{cases} D_{a^+}^{\alpha,\beta} u(x) = g(x, u(x)), \ 0 < \alpha < 1, \ 0 \leq \beta \leq 1, x \in (a, b], \\ I_{a^+}^{1-\gamma} u(a^+) = u_a \in R, \quad \gamma = \alpha + \beta - \alpha\beta. \end{cases}$$

They proved the existence and uniqueness of global solutions in a space of weighted continuous functions using Banach's fixed point theorem. More generally, Wang and Zhang [9] considered the existence of solutions to the following nonlocal initial value problem in 2015:

$$\begin{cases} D_{a^+}^{\alpha,\beta} u(x) = g(x, u(x)), \ 0 < \alpha < 1, \ 0 \leq \beta \leq 1, x \in (a, b], \\ I_{a^+}^{1-\gamma} u(a^+) = \sum_{i=1}^{m} \lambda_i u(\tau_i), \quad \gamma = \alpha + \beta - \alpha\beta, \quad \tau_i \in (a, b]. \end{cases}$$

The remainder of this paper is organized as follows. Section 2 introduces some basic concepts, a Banach space $C_{1-\gamma}[a, b]$ with $\gamma = \alpha + \beta - \alpha\beta, \beta < 1$, a subspace $W_\gamma[a, b] \subset C_{1-\gamma}[a, b]$, the multivariate Mittag-Leffler function and Babenko's approach. In addition, we convert Equation (1.1) to an equivalent implicit integral equation in series using Babenko's technique. Then we obtain sufficient conditions for the uniqueness of solutions with the help of Banach's contractive principle in the Banach space $W_\gamma[a, b]$, and further demonstrate applications of the main result by an example in Section 3. At the end, we summarize the entire work in Section 4.

2 Preliminaries

The Riemann-Liouville fractional integral of the order $s \geq 0$ of function $u(x)$ is defined by [10]

$$(I_{a^+}^s u)(x) = \frac{1}{\Gamma(s)} \int_a^x (x - t)^{s-1} u(t) dt, \quad x > a,$$

and

$$I_{a^+}^0 u(x) = u(x),$$

from [11].

The Riemann-Liouville fractional derivative of order $\alpha \in [n-1, n)$, for $n \in N$, of function $u(x)$ is defined by [2]

$$\left(D_{a^+}^{\alpha} u\right)(x) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dx}\right)^n \int_a^x (x-t)^{n-\alpha-1} u(t) dt, \quad x > a.$$

The Hilfer fractional derivative of order $0 < \alpha < 1$ and $0 \leq \beta \leq 1$ of function $u(x)$ is defined by [3]

$$D_{a^+}^{\alpha,\beta} u(x) = \left(I_{a^+}^{\beta(1-\alpha)} D I_{a^+}^{(1-\beta)(1-\alpha)}\right) u(x),$$

where $D = \dfrac{d}{dx}$.

It follows from [3] that the operator $D_{a^+}^{\alpha,\beta}$ can also be written as

$$D_{a^+}^{\alpha,\beta} = I_{a^+}^{\beta(1-\alpha)} D_{a^+}^{\gamma},$$

where $\alpha \leq \gamma = \alpha + \beta - \alpha\beta$. Clearly, the Riemann-Liouville fractional derivative $D_a^{\alpha} = D_{a^+}^{\alpha,0}$ and the Caputo fractional derivative ${}^C D_a^{\alpha} = D_{a^+}^{\alpha,1}$.

For any $0 \leq \gamma < 1$, we define the Banach space $C_{1-\gamma}[a, b]$ as

$$C_{1-\gamma}[a, b] = \left\{u : (a, b] \to R : (x-a)^{1-\gamma} u(x) \in C[a, b]\right\},$$

with the norm

$$\|u\|_{C_{1-\gamma}} = \max_{x \in [a,b]} |(x-a)^{1-\gamma} u(x)|.$$

Clearly, $C[a, b] \subset C_{1-\gamma}[a, b]$ for any $0 \leq \gamma < 1$. Further, a subspace $C_{1-\gamma}^1[a, b] \subset C_{1-\gamma}[a, b]$ is defined as

$$C_{1-\gamma}^1[a, b] = \left\{u \in C[a, b] : u' \in C_{1-\gamma}[a, b]\right\},$$

with the norm

$$\|u\|_{C_{1-\gamma}^1} = \|u\|_C + \left\|u'\right\|_{C_{1-\gamma}}.$$

Evidently, $C_{1-\gamma}^1[a, b]$ is a Banach space. Finally, the Banach space $W_\gamma[a, b]$ is defined as

$$W_\gamma[a, b] = \left\{u \in C_{1-\gamma}[a, b] : I_{a^+}^{1-\gamma} u \in C_{1-\gamma}^1[a, b]\right\} \subset C_{1-\gamma}[a, b],$$

with the norm

$$\|u\|_{W_\gamma} = \max\left\{\|u\|_{C_{1-\gamma}}, \left\|I_{a^+}^{1-\gamma} u\right\|_C, \left\|D I_{a^+}^{1-\gamma} u\right\|_{C_{1-\gamma}}\right\}.$$

Lemma 1. (see [8]) Let $0 < \alpha < 1$ and $\gamma = \alpha + \beta - \alpha\beta$ with $0 \leq \beta < 1$. If $u \in C_{1-\gamma}[a, b]$ and $I_{a^+}^{1-\beta+\alpha\beta} u \in C_{1-\gamma}^1[a, b]$, then $D_{a^+}^{\alpha,\beta} I_{a^+}^{\alpha} u$ exists in $(a, b]$ and

$$D_{a^+}^{\alpha,\beta} I_{a^+}^{\alpha} u = u,$$

for all $x \in (a, b]$.

The following lemma can be found in [2].

Lemma 2. (see [2]) Let $0 < t < 1$ and $0 \le s < 1$. If $u \in C_s[a,b]$ and $I_{a^+}^{1-t} u \in C_s^1[a,b]$, then

$$I_{a^+}^t D_{a^+}^t u(x) = u(x) - \frac{I_{a^+}^{1-t} u(a)}{\Gamma(t)} (x-a)^{t-1},$$

for all $x \in (a,b]$.

It follows from $t = \gamma$ and $s = 1-\gamma$ that

$$I_{a^+}^\gamma D_{a^+}^\gamma u(x) = u(x) - \frac{I_{a^+}^{1-\gamma} u(a)}{\Gamma(\gamma)} (x-a)^{\gamma-1},$$

for all $x \in (a,b]$ and $u \in W_\gamma[a,b]$.

The multivariate Mittag-Leffler function [12] is defined as follows

$$E_{(\alpha_1,\cdots,\alpha_m),\beta}(z_1,\cdots,z_m) = \sum_{k=0}^{\infty} \sum_{k_1+\cdots+k_m=k} \binom{k}{k_1,\cdots,k_m} \frac{z_1^{k_1}\cdots z_m^{k_m}}{\Gamma(\alpha_1 k_1 + \cdots + \alpha_m k_m + \beta)},$$

where $\alpha_i, \beta > 0$ for $i = 1, 2, \cdots, m$ and

$$\binom{k}{k_1,\cdots,k_m} = \frac{k!}{k_1!\cdots k_m!}.$$

3 Main results

One of the powerful methods for solving differential equations with initial conditions, as well as integral equations is Babenko's approach [13]. This method is generally the same as the Laplace transform, while dealing with equations with constant coefficients. However, this technique can be applied for differential and integral equations with continuous variable coefficients and boundary value problems [14, 15]. In the following, to show the applications of this approach, we will deduce an implicit integral equation which is equivalent to Equation (1.1) in the space $W_\gamma[a,b]$.

Theorem 3. *Let $g : (a,b] \times R \to R$ be a continuous and bounded function. Then Equation (1.1) is equivalent to the following implicit integral equation*

$$\begin{aligned} u(x) \;=\; & \frac{u_a}{\Gamma(\gamma)} \sum_{k=0}^{\infty} (-1)^k \sum_{k_1+k_2+\cdots+k_m=k} \binom{k}{k_1,k_2,\cdots,k_m} \lambda_1^{k_1}\lambda_2^{k_2}\cdots\lambda_m^{k_m} \\ & \times I_{a^+}^{(\alpha+\beta_1)k_1+\cdots+(\alpha+\beta_m)k_m} (x-a)^{\gamma-1} \\ & + \sum_{k=0}^{\infty} (-1)^k \sum_{k_1+k_2+\cdots+k_m=k} \binom{k}{k_1,k_2,\cdots,k_m} \lambda_1^{k_1}\lambda_2^{k_2}\cdots\lambda_m^{k_m} \\ & \times I_{a^+}^{\alpha+\beta+(\alpha+\beta_1)k_1+\cdots+(\alpha+\beta_m)k_m} g(x,u(x)), \end{aligned} \tag{3.1}$$

in the space $W_\gamma[a,b]$.

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Proof. Clearly for $\gamma = \alpha + \beta - \alpha\beta$ with $\beta < 1$,

$$I_{a^+}^{\alpha} D_{a^+}^{\alpha,\beta} u(x) = I_{a^+}^{\alpha} I_{a^+}^{\beta(1-\alpha)} D_{a^+}^{\gamma} u(x) = I_{a^+}^{\gamma} D_{a^+}^{\gamma} u(x) = u(x) - \frac{I_{a^+}^{1-\gamma} u(a)}{\Gamma(\gamma)} (x-a)^{\gamma-1},$$

for all $x \in (a, b]$ and $u \in W_\gamma[a, b]$.

Applying the operator $I_{a^+}^{\alpha}$ to both sides of Equation

$$D_{a^+}^{\alpha,\beta} u(x) + \sum_{i=1}^{m} \lambda_i \, I_{a^+}^{\beta_i} u(x) = I_{a^+}^{\beta} g(x, u(x)),$$

we come to

$$u(x) - \frac{u_a}{\Gamma(\gamma)} (x-a)^{\gamma-1} + \sum_{i=1}^{m} \lambda_i \, I_{a^+}^{\alpha+\beta_i} u(x) = I_{a^+}^{\alpha+\beta} g(x, u(x)),$$

using the initial condition

$$I_{a^+}^{1-\gamma} u(a) = u_a.$$

This implies that

$$\left(1 + \sum_{i=1}^{m} \lambda_i \, I_{a^+}^{\alpha+\beta_i}\right) u(x) = \frac{u_a}{\Gamma(\gamma)} (x-a)^{\gamma-1} + I_{a^+}^{\alpha+\beta} g(x, u(x)).$$

Treating the factor $\left(1 + \sum_{i=1}^{m} \lambda_i \, I_{a^+}^{\alpha+\beta_i}\right)$ as a variable, we derive that by Babenko's approach

$$\begin{aligned}
u(x) &= \left(1 + \sum_{i=1}^{m} \lambda_i \, I_{a^+}^{\alpha+\beta_i}\right)^{-1} \left(\frac{u_a}{\Gamma(\gamma)} (x-a)^{\gamma-1} + I_{a^+}^{\alpha+\beta} g(x, u(x))\right) \\
&= \sum_{k=0}^{\infty} (-1)^k \left(\sum_{i=1}^{m} \lambda_i \, I_{a^+}^{\alpha+\beta_i}\right)^k \left(\frac{u_a}{\Gamma(\gamma)} (x-a)^{\gamma-1} + I_{a^+}^{\alpha+\beta} g(x, u(x))\right) \\
&= \sum_{k=0}^{\infty} (-1)^k \sum_{k_1+k_2+\cdots+k_m=k} \binom{k}{k_1, k_2, \cdots, k_m} \lambda_1^{k_1} \cdots \lambda_m^{k_m} I_{a^+}^{(\alpha+\beta_1)k_1+\cdots+(\alpha+\beta_m)k_m} \times \\
&\quad \left(\frac{u_a}{\Gamma(\gamma)} (x-a)^{\gamma-1} + I_{a^+}^{\alpha+\beta} g(x, u(x))\right) \\
&= \frac{u_a}{\Gamma(\gamma)} \sum_{k=0}^{\infty} (-1)^k \sum_{k_1+k_2+\cdots+k_m=k} \binom{k}{k_1, k_2, \cdots, k_m} \lambda_1^{k_1} \lambda_2^{k_2} \cdots \lambda_m^{k_m} \\
&\quad \times I_{a^+}^{(\alpha+\beta_1)k_1+\cdots+(\alpha+\beta_m)k_m} (x-a)^{\gamma-1} \\
&\quad + \sum_{k=0}^{\infty} (-1)^k \sum_{k_1+k_2+\cdots+k_m=k} \binom{k}{k_1, k_2, \cdots, k_m} \lambda_1^{k_1} \lambda_2^{k_2} \cdots \lambda_m^{k_m} \\
&\quad \times I_{a^+}^{\alpha+\beta+(\alpha+\beta_1)k_1+\cdots+(\alpha+\beta_m)k_m} g(x, u(x)).
\end{aligned}$$

Next, we are going to show that $u \in W_\gamma[a,b]$. Clearly,

$$\begin{aligned}
& I_{a^+}^{(\alpha+\beta_1)k_1+\cdots+(\alpha+\beta_m)k_m}(x-a)^{\gamma-1} \\
& = \frac{\Gamma(\gamma)}{\Gamma((\alpha+\beta_1)k_1+\cdots+(\alpha+\beta_m)k_m+\gamma)}(x-a)^{(\alpha+\beta_1)k_1+\cdots+(\alpha+\beta_m)k_m+\gamma-1},
\end{aligned}$$

and

$$\begin{aligned}
& \max_{x\in[a,b]} |(x-a)^{1-\gamma}u(x)| \leq |u_a| \sum_{k=0}^{\infty} \sum_{k_1+k_2+\cdots+k_m=k} \binom{k}{k_1, k_2, \cdots, k_m} |\lambda_1|^{k_1}\cdots|\lambda_m|^{k_m} \\
& \times \frac{(b-a)^{(\alpha+\beta_1)k_1+\cdots+(\alpha+\beta_m)k_m}}{\Gamma((\alpha+\beta_1)k_1+\cdots+(\alpha+\beta_m)k_m+\gamma)} \\
& + (b-a)^{1+\alpha\beta} \sum_{k=0}^{\infty} \sum_{k_1+k_2+\cdots+k_m=k} \binom{k}{k_1, k_2, \cdots, k_m} |\lambda_1|^{k_1}\cdots|\lambda_m|^{k_m} \\
& \times \frac{(b-a)^{(\alpha+\beta_1)k_1+\cdots+(\alpha+\beta_m)k_m}}{\Gamma((\alpha+\beta_1)k_1+\cdots+(\alpha+\beta_m)k_m+\alpha+\beta+1)} \sup_{x\in(a,b]} |g(x,u(x))| \\
& = |u_a| E_{(\alpha+\beta_1,\cdots,\alpha+\beta_m),\gamma}\left(|\lambda_1|(b-a)^{\alpha+\beta_1}, \cdots, |\lambda_m|(b-a)^{\alpha+\beta_m}\right) \\
& \quad + (b-a)^{1+\alpha\beta} E_{(\alpha+\beta_1,\cdots,\alpha+\beta_m),\,\alpha+\beta+1}\left(|\lambda_1|(b-a)^{\alpha+\beta_1}, \cdots, |\lambda_m|(b-a)^{\alpha+\beta_m}\right) \\
& \quad \times \sup_{x\in(a,b]} |g(x,u(x))| < +\infty.
\end{aligned}$$

Using

$$I_{a^+}^{1-\gamma}(x-a)^{\gamma-1} = \Gamma(\gamma),$$

we get

$$\begin{aligned}
& \left\| I_{a^+}^{1-\gamma} u \right\|_C \leq |u_a| \sum_{k=0}^{\infty} \sum_{k_1+k_2+\cdots+k_m=k} \binom{k}{k_1, k_2, \cdots, k_m} |\lambda_1|^{k_1}\cdots|\lambda_m|^{k_m} \\
& \times \frac{(b-a)^{(\alpha+\beta_1)k_1+\cdots+(\alpha+\beta_m)k_m}}{\Gamma((\alpha+\beta_1)k_1+\cdots+(\alpha+\beta_m)k_m+1)} \\
& + \sum_{k=0}^{\infty} \sum_{k_1+k_2+\cdots+k_m=k} \binom{k}{k_1, k_2, \cdots, k_m} |\lambda_1|^{k_1}\cdots|\lambda_m|^{k_m} \\
& \times \frac{(b-a)^{1+\alpha\beta+(\alpha+\beta_1)k_1+\cdots+(\alpha+\beta_m)k_m}}{\Gamma((\alpha+\beta_1)k_1+\cdots+(\alpha+\beta_m)k_m+2+\alpha\beta)} \sup_{x\in(a,b]} |g(x,u(x))| \\
= \;& |u_a| E_{(\alpha+\beta_1,\cdots,\alpha+\beta_m),\,1}\left(|\lambda_1|(b-a)^{\alpha+\beta_1}, \cdots, |\lambda_m|(b-a)^{\alpha+\beta_m}\right) \\
& + (b-a)^{1+\alpha\beta} E_{(\alpha+\beta_1,\cdots,\alpha+\beta_m),\,2+\alpha\beta}\left(|\lambda_1|(b-a)^{\alpha+\beta_1}, \cdots, |\lambda_m|(b-a)^{\alpha+\beta_m}\right) \\
& \times \sup_{x\in(a,b]} |g(x,u(x))| < +\infty.
\end{aligned}$$

Finally, we consider the norm

$$\begin{aligned}
\left\| D I_{a^+}^{1-\gamma} u \right\|_{C_{1-\gamma}} &= \max_{x\in[a,b]} \left| (x-a)^{1-\gamma} D I_{a^+}^{1-\gamma} u \right| \\
&\leq |u_a| \sum_{k=1}^{\infty} \sum_{k_1+k_2+\cdots+k_m=k} \binom{k}{k_1, k_2, \cdots, k_m} |\lambda_1|^{k_1} \cdots |\lambda_m|^{k_m} \\
&\times \frac{(b-a)^{(\alpha+\beta_1)k_1+\cdots+(\alpha+\beta_m)k_m-\gamma}}{\Gamma((\alpha+\beta_1)k_1+\cdots+(\alpha+\beta_m)k_m)} \\
&+ (b-a)^{1-\gamma+\alpha\beta} \sum_{k=0}^{\infty} \sum_{k_1+k_2+\cdots+k_m=k} \binom{k}{k_1, k_2, \cdots, k_m} |\lambda_1|^{k_1} \cdots |\lambda_m|^{k_m} \\
&\times \frac{(b-a)^{(\alpha+\beta_1)k_1+\cdots+(\alpha+\beta_m)k_m}}{\Gamma((\alpha+\beta_1)k_1+\cdots+(\alpha+\beta_m)k_m+\alpha\beta+1)} < +\infty,
\end{aligned}$$

by noting that

$$(\alpha+\beta_1)k_1+\cdots+(\alpha+\beta_m)k_m-\gamma \geq 0,$$

for $k_1+\cdots+k_m = k \geq 1$, since $\beta_i \geq \beta$ for all $i = 1, 2, \cdots, m$. In summary, $u \in W_\gamma[a, b]$.

Conversely, if u is given by Equation (3.1) in the space $W_\gamma[a, b]$ then

$$I_{a^+}^{1-\gamma} u(a) = u_a.$$

Indeed,

$$\begin{aligned}
u(x) &= \frac{u_a}{\Gamma(\gamma)}(x-a)^{\gamma-1} + \frac{u_a}{\Gamma(\gamma)} \sum_{k=1}^{\infty} (-1)^k \sum_{k_1+k_2+\cdots+k_m=k} \binom{k}{k_1, k_2, \cdots, k_m} \lambda_1^{k_1} \lambda_2^{k_2} \cdots \lambda_m^{k_m} \\
&\times I_{a^+}^{(\alpha+\beta_1)k_1+\cdots+(\alpha+\beta_m)k_m} (x-a)^{\gamma-1} \\
&+ \sum_{k=0}^{\infty} (-1)^k \sum_{k_1+k_2+\cdots+k_m=k} \binom{k}{k_1, k_2, \cdots, k_m} \lambda_1^{k_1} \lambda_2^{k_2} \cdots \lambda_m^{k_m} \\
&\times I_{a^+}^{\alpha+\beta+(\alpha+\beta_1)k_1+\cdots+(\alpha+\beta_m)k_m} g(x, u(x)) = \frac{u_a}{\Gamma(\gamma)}(x-a)^{\gamma-1} + u_1(x).
\end{aligned}$$

Using

$$I_{a^+}^{1-\gamma} (x-a)^{\gamma-1} = \Gamma(\gamma),$$

and noting that

$$I_{a^+}^{1-\gamma} u_1(x) \Big|_{x=a} = 0,$$

we have $I_{a^+}^{1-\gamma} u(a) = u_a$.

Furthermore, applying the operator $D_{a^+}^{\alpha,\beta}$ to

$$\left(1 + \sum_{i=1}^{m} \lambda_i I_{a^+}^{\alpha+\beta_i}\right) u(x) = \frac{u_a}{\Gamma(\gamma)}(x-a)^{\gamma-1} + I_{a^+}^{\alpha+\beta} g(x, u(x)),$$

which is equivalent to Equation (3.1), we obtain

$$D_{a^+}^{\alpha,\beta} u(x) + \sum_{i=1}^{m} \lambda_i \, D_{a^+}^{\alpha,\beta} I_{a^+}^{\alpha+\beta_i} u(x) = \frac{u_a}{\Gamma(\gamma)} D_{a^+}^{\alpha,\beta} (x-a)^{\gamma-1} + D_{a^+}^{\alpha,\beta} I_{a^+}^{\alpha+\beta} g(x, u(x)).$$

Clearly,

$$D_{a^+}^{\alpha,\beta} (x-a)^{\gamma-1} = I_{a^+}^{\beta(1-\alpha)} (D_{a^+}^{\gamma} (x-a)^{\gamma-1}) = I_{a^+}^{\beta(1-\alpha)} 0 = 0,$$

for $0 < \gamma < 1$, and

$$D_{a^+}^{\alpha,\beta} I_{a^+}^{\alpha+\beta_i} u = D_{a^+}^{\alpha,\beta} I_{a^+}^{\alpha} I_{a^+}^{\beta_i} u = I_{a^+}^{\beta_i} u,$$

by Lemma 1 due to the fact that

$$I_{a^+}^{\beta_i} u \in C_{1-\gamma}[a,b], \quad \text{and} \;\; I_{a^+}^{1+\beta_i-\beta+\alpha\beta} u \in C_{1-\gamma}^1[a,b].$$

Similarly,

$$D_{a^+}^{\alpha,\beta} I_{a^+}^{\alpha+\beta} g(x, u(x)) = I_{a^+}^{\beta} g(x, u(x)).$$

Hence, u satisfies Equation (1.1). This completes the proof of Theorem 3. □

We are now ready to present the following theorem about the uniqueness of solutions to Equation (1.1).

Theorem 4. *Let $g : (a,b] \times R \to R$ be a continuous and bounded function satisfying*

$$|g(x, y_1) - g(x, y_2)| \leq L|y_1 - y_2|, \quad y_1, y_2 \in R,$$

for a nonnegative constant L. Furthermore, assume

$$q = L(b-a)^{\alpha+\beta}\Gamma(\gamma) E_{(\alpha+\beta_1,\cdots,\alpha+\beta_m),\, 2(\alpha+\beta)-\alpha\beta} \left(|\lambda_1|(b-a)^{\alpha+\beta_1}, \cdots, |\lambda_m|(b-a)^{\alpha+\beta_m} \right) < 1.$$

Then Equation (1.1) has a unique solution in the space $W_\gamma[a,b]$.

Proof. For $u \in C_{1-\gamma}[a,b]$, we define a nonlinear mapping T over the space $C_{1-\gamma}[a,b]$ by

$$\begin{aligned} (Tu)(x) \;\; = \;\; & \frac{u_a}{\Gamma(\gamma)} \sum_{k=0}^{\infty} (-1)^k \sum_{k_1+k_2+\cdots+k_m=k} \binom{k}{k_1, k_2, \cdots, k_m} \lambda_1^{k_1} \lambda_2^{k_2} \cdots \lambda_m^{k_m} \\ & \times I_{a^+}^{(\alpha+\beta_1)k_1+\cdots+(\alpha+\beta_m)k_m} (x-a)^{\gamma-1} \\ & + \sum_{k=0}^{\infty} (-1)^k \sum_{k_1+k_2+\cdots+k_m=k} \binom{k}{k_1, k_2, \cdots, k_m} \lambda_1^{k_1} \lambda_2^{k_2} \cdots \lambda_m^{k_m} \\ & \times I_{a^+}^{\alpha+\beta+(\alpha+\beta_1)k_1+\cdots+(\alpha+\beta_m)k_m} g(x, u(x)). \end{aligned}$$

It follows from the proof of Theorem 3 that $Tu \in C_{1-\gamma}[a,b]$. We further show that T is contractive. In fact, we get for $u, v \in\in C_{1-\gamma}[a,b]$

$$\begin{aligned}
&\|Tu - Tv\|_{C_{1-\gamma}} \\
&= \max_{x\in[a,b]} \Bigg| (x-a)^{1-\gamma} \sum_{k=0}^{\infty} (-1)^k \sum_{k_1+k_2+\cdots+k_m=k} \binom{k}{k_1, k_2, \cdots, k_m} \lambda_1^{k_1} \cdots \lambda_m^{k_m} \\
&\quad \times I_{a^+}^{\alpha+\beta+(\alpha+\beta_1)k_1+\cdots+(\alpha+\beta_m)k_m} (g(x,u) - g(x,v)) \Bigg| \\
&\leq L(b-a)^{\alpha+\beta}\Gamma(\gamma) \sum_{k=0}^{\infty} \sum_{k_1+k_2+\cdots+k_m=k} \binom{k}{k_1, k_2, \cdots, k_m} |\lambda_1|^{k_1} \cdots |\lambda_m|^{k_m} \\
&\quad \times \frac{(b-a)^{(\alpha+\beta_1)k_1+\cdots+(\alpha+\beta_m)k_m}}{\Gamma(2(\alpha+\beta) - \alpha\beta + (\alpha+\beta_1)k_1 + \cdots + (\alpha+\beta_m)k_m)} \|u-v\|_{C_{1-\gamma}} \\
&= q\,\|u-v\|_{C_{1-\gamma}},
\end{aligned}$$

using

$$g(x,u) - g(x,v) = (x-a)^{\gamma-1}[(x-a)^{1-\gamma}(g(x,u) - g(x,v))],$$

and

$$\begin{aligned}
&I_{a^+}^{\alpha+\beta+(\alpha+\beta_1)k_1+\cdots+(\alpha+\beta_m)k_m} (x-a)^{\gamma-1} \\
&= \frac{\Gamma(\gamma)}{\Gamma(\alpha+\beta+\gamma+(\alpha+\beta_1)k_1+\cdots+(\alpha+\beta_m)k_m)} \\
&\quad \times (x-a)^{\alpha+\beta+(\alpha+\beta_1)k_1+\cdots+(\alpha+\beta_m)k_m+\gamma-1} \\
&= \frac{\Gamma(\gamma)}{\Gamma(2(\alpha+\beta)-\alpha\beta+(\alpha+\beta_1)k_1+\cdots+(\alpha+\beta_m)k_m)} \\
&\quad \times (x-a)^{\alpha+\beta+(\alpha+\beta_1)k_1+\cdots+(\alpha+\beta_m)k_m+\gamma-1}.
\end{aligned}$$

Since $q < 1$, the mapping T has a unique fixed point in the space $C_{1-\gamma}[a,b]$. Moreover, if $u \in W_\gamma[a,b]$ then $Tu \in W_\gamma[a,b]$. Therefore, T has a unique fixed point in $W_\gamma[a,b]$ in the sense of the topology in $C_{1-\gamma}[a,b]$. This implies that Equation (1.1) has a unique solution in $W_\gamma[a,b]$. This completes the proof of Theorem 4. $\square$

Example 5. *The following nonlinear Hilfer integro-differential equation with an initial condition:*

$$\begin{cases} D_{0^+}^{0.6,0.4}u(x) + 2\,I_{0^+}^{0.4}u(x) - 2\,I_{0^+}^{0.5}u(x) = \dfrac{1}{23} I_{0^+}^{0.4} \sin u(x), \\ I_{0^+}^{0.24}u(0) = -4, \end{cases}$$

has a unique solution in the space $W_{0.76}[0,1]$.

Proof. Clearly, $g(x,u(x)) = \dfrac{1}{23}\sin u(x)$ and

$$|g(x,u(x)) - g(x,v(x))| \leq \frac{1}{23}|u(x) - v(x)|,$$

and

$$
\begin{aligned}
q &= \frac{1}{23}\Gamma(0.76)E_{(1,1.1),\,1.76}(2,2) \\
&= \frac{1}{23}\Gamma(0.76)\sum_{k=0}^{\infty}\sum_{k_1+k_2=k}\binom{k}{k_1,k_2}\frac{2^{k_1}2^{k_2}}{\Gamma(k_1+1.1k_2+1.76)}.
\end{aligned}
$$

Applying

$$
\sum_{k_1+k_2=k}\binom{k}{k_1,k_2} = 2^k,
$$

and

$$
\frac{2^{k_1}2^{k_2}}{\Gamma(k_1+1.1k_2+1.76)} \le \frac{2^k}{\Gamma(k+1.76)},
$$

we imply that

$$
q \le \frac{1}{23}\Gamma(0.76)\sum_{k=0}^{\infty}\frac{4^k}{\Gamma(k+1.76)} \approx \frac{22.8418}{23} < 1,
$$

since

$$
\Gamma(0.76)\sum_{k=0}^{\infty}\frac{4^k}{\Gamma(k+1.76)} \approx 22.8418,
$$

by the online calculator on the site https://www.wolframalpha.com/. From Theorem 4, the above equation has a unique solution in the space $W_{0.76}[0,1]$. This completes the proof. □

Remark 6. From the proof of Theorem 3, we imply that

$$
u(x) = \frac{u_a}{\Gamma(\gamma)}(x-a)^{\gamma-1} - \sum_{i=1}^{m}\lambda_i\, I_{a^+}^{\alpha+\beta_i}u(x) + I_{a^+}^{\alpha+\beta}g(x,u(x)),
$$

which can be used in finding an approximate solution by the following recursion:

$$
u_n(x) = \frac{u_a}{\Gamma(\gamma)}(x-a)^{\gamma-1} - \sum_{i=1}^{m}\lambda_i\, I_{a^+}^{\alpha+\beta_i}u_{n-1}(x) + I_{a^+}^{\alpha+\beta}g(x,u_{n-1}(x)),
$$

for $n = 1, 2, \cdots$, and an initial function $u_0(x)$.

4 Conclusion

We have investigated the uniqueness of solutions to the new nonlinear Hilfer integro-differential equation with an initial condition, based on its equivalent implicit integral equation in series derived from Babenko's approach, the multivariate Mittag-Leffler function as well as Banach's contractive principle, with an illustrative example demonstrating applications of the key theorem at the end.

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The author declares no competing interests.

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Data Availability

No data were used to support this study.

Availability of data and materials

Not applicable.

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