# Integral representation formula for linear non-autonomous difference-delay equations

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**Abstract**

This note states and proves an integral representation formula of the "variation-of-constant" type for continuous solutions of linear non-autonomous difference delay systems, in terms of a Lebesgue-Stieltjes integral involving a fundamental solution and the initial data of the system. This gives a precise and correct version of several formulations appearing in the literature, and extends them to the time-varying case. This is of importance for further stability studies of various kinds of delay systems.

*Keywords:* linear systems, difference delay systems, integral representation, Volterra equations.

## 1. Introduction and statement of the result

Transport phenomena commonly occur in practical situations in biology or physics, for example to model the migration of cells in an organism, as well as road traffic networks or high-frequency signals traveling through an electrical circuit.These phenomena are typically modelled by delay equations. Basic properties thereof, such as stability, stabilization and controllability, have been widely considered in the literature [1, 2, 3, 4]. Reference [1] deals with applications to electrical engineering, in particular the stability of microwave circuits, which was the initial motivation of the authors to study time-periodic systems described by equations such as (1.1) below; in fact, the latter constitutes a natural model for the high-frequency limit system of a microwave circuit, whose stability is crucial to the one of the circuit itself (see [5, 6]).

In this note, we consider a linear non-autonomous difference-delay system of the form:

$$y(t) = \sum_{j=1}^{N} D_j(t) y(t-\tau_j), \qquad t > s, \qquad y(s+\theta) = \phi(\theta) \text{ for } -\tau_N \le \theta \le 0, \tag{1.1}$$

where $d$, $N$ are positive integers and the delays $\tau_1 < \cdots < \tau_N$ are strictly positive real numbers, while $D_1(t), \ldots, D_N(t)$ are complex[2] $d \times d$ matrices depending continuously on time $t$, the real number $s \in \mathbb{R}$ is the initial time and the function $\phi : [-\tau_N, 0] \to \mathbb{C}^d$ the *initial data*. By writing $t > s$, we imply that this system is understood in forward time only. To deal with backward systems, one would have to make assumptions on the invertibility of the map $t \mapsto D_{\tau_N}(t)$.

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[2]We treat here the case of complex coefficients (in the matrices $D_j$ as well as in the solutions); real coefficients can be handled in exactly the same way.



3 Jun 2024 09:39:42 PDT
231119-Fueyo Version 2 - Submitted to J. Integr. Eq. Appl.

A *solution* is a map $y : [s-\tau_N, +\infty) \to \mathbb{C}^d$ that satisfies (1.1). Clearly, in order to get a continuous solution, the initial data which is but the restriction of $y$ to the interval $[-\tau_N, 0]$ must satisfy a compatibility condition, namely $\phi$ should belong to the set $C_s$ defined by

$$C_s := \{\phi \in C^0([-\tau_N, 0], \mathbb{C}^d) : \phi(0) = \sum_{j=1}^{N} D_j(s)\phi(-\tau_j)\}, \tag{1.2}$$

where $C^0(E, F)$ indicates the set of continuous maps from $E$ into $F$. Conversely, given $\phi$ in $C_s$, an easy recursion shows that System (1.1) has a unique solution $y$ with initial data $\phi$, and that this solution is continuous.

The goal of this note is to give a precise statement, as well as a detailed proof, of the integral formula, often called representation formula (see *e.g.* [7]), expressing the solution to System (1.1) in terms of the initial condition $\phi \in C_s$ and the so-called *fundamental solution*, that can be viewed as a particular (non-continuous) matrix-valued solution of (1.1). By definition, the fundamental solution is the map $X : \mathbb{R}^2 \to \mathbb{C}^{d\times d}$ satisfying

$$X(t,s) = \begin{cases} 0 \text{ for } t < s, \\ I_d + \sum\limits_{j=1}^{N} D_j(t)X(t-\tau_j, s) \text{ for } t \geq s, \end{cases} \tag{1.3}$$

where $I_d$ denotes the $d \times d$ identity matrix. Arguing inductively, it is easy to check that $X$ uniquely exists and is continuous at each $(t, s)$ such that $t-s \notin \mathcal{F}$, where $\mathcal{F}$ is the positive lattice in $[0, +\infty)$ generated by the numbers $\tau_\ell$:

$$\mathcal{F} := \left\{ \sum_{\ell=1}^{N} n_\ell \, \tau_\ell \,, \; (n_1, \ldots, n_N) \in \mathbb{N}^N \right\} . \tag{1.4}$$

Clearly, $X$ has a bounded jump across each line $t - s = \mathfrak{f}$ for $\mathfrak{f} \in \mathcal{F}$; in fact, a moment's thinking will convince the reader that $X$ has the form

$$X(t,s) \; = \; - \sum_{\mathfrak{f} \in [0, t-s] \cap \mathcal{F}} \mathfrak{C}_{\mathfrak{f}}(t) \, , \quad s \leq t \, , \tag{1.5}$$

where each $\mathfrak{C}_{\mathfrak{f}}(.)$ is continuous[3] $\mathbb{R} \to \mathbb{C}^{d\times d}$.

The announced representation formula is now given by the following theorem, whose proof can be found in Section 4.

**Theorem 1** (representation formula). *For $s \in \mathbb{R}$ and $\phi$ in $C_s$ the solution $y \in C^0([s-\tau_N, +\infty), \mathbb{C}^d)$ to* (1.1) *is given by*

$$y(t) = -\sum_{j=1}^{N} \int_{s^-}^{(s+\tau_j)^-} d_\alpha X(t, \alpha) D_j(\alpha)\phi(\alpha - \tau_j - s), \qquad t \geq s \, , \tag{1.6}$$

*where $X$ was defined in* (1.3).

[3]Additional smoothness assumptions on the maps $D_j(.)$ would transfer to the $\mathfrak{C}_{\mathfrak{f}}$. One can also see that $\mathfrak{C}_{\mathfrak{f}}(t)$ is a finite sum of products of $D_j(t - \mathfrak{f}')$, where $\mathfrak{f}'$ ranges over the elements of $\mathcal{F}$ whose defining integers $n_\ell$ in (1.4) do not exceed those defining $\mathfrak{f}$, the empty product being the identity matrix. A precise expression for $\mathfrak{C}_{\mathfrak{f}}$ can be obtained by reasoning as in [8, Sec. 3.2] or [9, Sec. 4.5], but we will not need it.



3 Jun 2024 09:39:42 PDT
231119-Fueyo Version 2 - Submitted to J. Integr. Eq. Appl.

The integrals $\int_{s^-}^{(s+\tau_j)^-}$ in Equation (1.6) are understood as Lebesgue-Stieltjes integrals on the intervals $[s, s+\tau_j)$. They are well defined because, for fixed $t$, the function $X(t,\cdot)$ is locally of bounded variation. Basic facts regarding Lebesgue-Stieltjes integral and functions of bounded variation are recalled in Section 3.1, for the ease of the reader.

## 2. Further motivation and comments

Representation formulas like (1.6) are fundamental to deal with linear functional dynamical systems. In particular, they offer a base to devise stability criteria *via* a frequency domain approach, using Laplace transforms: in [10], [7] or in the recent manuscript [6] by the authors, exponential stability is investigated this way for difference-delay system and various functional differential equations, either time-invariant or time-varying. In fact, (1.6) entails that the solutions of System (1.1) are exponentially stable if the total variation of the function $X(t,.)$ on $[s, s+\tau_N]$ is bounded by $ce^{-\alpha(t-s)}$ for some strictly positive $c$ and $\alpha$. Hence, a careful study of $X(t,.)$ (for example through an analysis of the $\mathfrak{C}_\mathfrak{f}$ in (1.5)) yields important information on the exponential stability properties of that system. These results, in turn, are relevant to the stability of 1-D hyperbolic PDE's; see *e.g.* [11, Theorem 3.5 and Theorem 3.8] and [12].

Though formulas of the type (1.6) appear at several places in the literature for various classes of linear dynamical systems, the purpose of the present note is to state it carefully and prove it in detail. Our motivation here is twofold. On the one hand, the time-varying case has apparently not been treated. On the other hand, and more importantly perhaps, we found that even for linear *autonomous* difference-delay equations several representation formulas stated in the literature (like [10, Lemma 3.4], [13, Chapter 12, Equation (3.16)] or [7, Chapter 9, Theorem 1.2] that deals with more general Volterra equations) seem to have issues[4], along with their proofs; see Section 3.2 and Footnote 5. For instance, when seeking to establish (1.6) for systems having periodic coefficients, which is a basic ingredient of the proofs in [6], the authors could not even come up with a satisfactory reference in the time-invariant case. The *raison d'être* for the present note is thus to present a result that addresses the non-autonomous case, and at the same time that can be referenced even in the time-invariant setting. Our method of proof is in line with the approach in [7, Chapter 9, Sec. 1], trading (1.3) for a Volterra equation and proving the existence of a resolvent for the latter. In this connection, Lemmata 2 and 3 below are fundamental steps of the proof, that we could not locate in the literature and may be of independent interest.

## 3. Summary on functions with bounded variations, Lebesgue-Stieltjes integral and Volterra integral equations with $B^\infty$ kernel

In this section, we recall definitions and basic facts regarding functions of bounded variation and Lebesgue-Stieltjes integrals, that the reader might want to consult before proceeding with Lemma 2, Lemma 3 and the proof of Theorem 1.

The real and complex fields are denoted by $\mathbb{R}$ and $\mathbb{C}$. For $d$ a strictly positive integer, we write $\|\cdot\|$ for the Euclidean norm on $\mathbb{C}^d$ and $|||\cdot|||$ for the norm of a matrix $M \in \mathbb{C}^{d\times d}$:

$$|||M||| = \sup_{\|x\|=1} \|Mx\|.$$

[4] For instance, the formula in [10, Lemma 3.4], stated for time-invariant systems of the same type as (1.1) (possibly with infinitely many delays) does not agree with (1.6). A check on systems with two delays will convince the reader that it is faulty because, probably due to misprinted indices, the integration is not carried out on the right interval. Still, our formula agrees with the one in [10] in the case of a single delay ($N = 1$)

*3.1. Functions with bounded variations and the Lebesgue-Stieltjes integral*

For $I$ a real interval and $f : I \to \mathbb{R}$ a function, the *total variation* of $f$ on $I$ is defined as

$$W_I(f) := \sup_{\substack{x_0<x_1<\cdots<x_N\\ x_i\in I, N\in\mathbb{N}}} \sum_{i=1}^{N} |f(x_i) - f(x_{i-1})|. \tag{3.1}$$

The space $BV(I)$ of *functions with bounded variation* on $I$ consists of those $f$ such that $W_I(f) < \infty$, endowed with the norm $\|f\|_{BV(I)} = W_I(f) + |f(y_0)|$ where $y_0 \in I$ is arbitrary but fixed. Different $y_0$ give rise to equivalent norms for which $BV(I)$ is a Banach space, and $\|.\|_{BV(I)}$ is stronger than the uniform norm. We let $BV_r(I)$ and $BV_l(I)$ be the closed subspaces of $BV(I)$ comprised of right and left continuous functions, respectively. We write $BV_{loc}(\mathbb{R})$ for the space of functions whose restriction to any bounded interval $I \subset \mathbb{R}$ lies in $BV(I)$. Since $f(x_i)g(x_i) - f(x_{i-1})g(x_{i-1}) = \big(f(x_i) - f(x_{i-1})\big)g(x_i) + f(x_{i-1})\big(g(x_i) - g(x_{i-1})\big)$, we observe from (3.1) that

$$W_I(fg) \le W_I(f)\sup_{x\in I}|g(x)| + W_I(g)\sup_{x\in I}|f(x)|. \tag{3.2}$$

Each $f \in BV(I)$ has a limit $f(x^-)$ (resp. $f(x^+)$) from the left (resp. right) at every $x \in I$ where the limit applies [14, sec. 1.4]. Hence, one can associate to $f$ a finite signed Borel measure $\nu_f$ on $I$ such that $\nu_f((a,b)) = f(b^-) - f(a^+)$, and if $I$ is bounded on the right (resp. left) and contains its endpoint $b$ (resp. $a$), then $\nu_f(\{b\}) = f(b) - f(b^-)$ (resp. $f(a^+) - f(a)$) [14, ch. 7, pp. 185–189]. Note that different $f$ may generate the same $\nu_f$: for example if $f$ and $f_1$ coincide except at isolated interior points of $I$, then $\nu_f = \nu_{f_1}$. For $g : I \to \mathbb{R}$ a measurable function summable against $\nu_f$, the *Lebesgue-Stieltjes integral* $\int gdf$ is defined as $\int gd\nu_f$, whence the differential element $df$ identifies with $d\nu_f$ [14, ch. 7, pp. 190–191]. This type of integral is useful for it is suggestive of integration by parts, but caution must be used when integrating a function against $df$ over a subinterval $J \subset I$ because the measure $\nu_{(f_{|J})}$ associated to the restricted function $f_{|J}$ needs *not* coincide with the restriction $(\nu_f)_{|J}$ of the measure $\nu_f$ to $J$. More precisely, if the lower bound $a$ (resp. the upper bound $b$) of $J$ belongs to $J$ and lies interior to $I$, then the two measures may differ by the weight they put on $\{a\}$ (resp. $\{b\}$), and they agree only when $f$ is left (resp. right) continuous at $a$ (resp. $b$). By $\int_J gdf$, we always mean that we integrate $g$ against $\nu_{(f_{|J})}$ and *not* against $(\nu_f)_{|J}$. As in the main formula (1.6), we often trade the notation $\int_J gdf$ for one of the form $\int_{a^\pm}^{b^\pm} gdf$, where the interval of integration $J$ is encoded in the bounds we put on the integral sign: a lower bound $a^-$ (resp. $a^+$) means that $J$ contains (resp. does not contain) its lower bound $a$, while an upper bound $b^+$ (resp. $b^-$) means that $J$ contains (resp. does not contain) its upper bound $b$. Then, the previous word of caution applies to additive rules: for example, when splitting $\int_{a^\pm}^{b^\pm} gdf$ into $\int_{a^\pm}^{c^\pm} gdf + \int_{c^\pm}^{b^\pm} gdf$ where $c \in (a,b)$, we must use $c^+$ (resp. $c^-$) if $f$ is right (resp. left) continuous at $c$.

To a finite, signed or complex Borel measure $\mu$ on $I$, one can associate its *total variation measure* $|\mu|$, defined on a Borel set $B \subset I$ by $|\mu|(B) = \sup_{\mathcal{P}} \sum_{E\in\mathcal{P}} |\mu(E)|$ where $\mathcal{P}$ ranges over all partitions of $B$ into Borel sets, see [15, sec. 6.1]; its total mass $|\mu|(I)$ is called the total variation of $\mu$, denoted as $\|\mu\|$. Thus, the total variation is defined both for functions of bounded variation and for measures, with different meanings. When $f \in BV(I)$ is monotonic then $W_I(f) = \|\nu_f\|$, but in general it only holds that $\|\nu_f\| \le 2W_I(f)$; this follows from the Jordan decomposition of $f$ as a difference of two increasing functions, each of which has variation at most $W_I(f)$ on $I$ [14, Thm. 1.4.1]. In any case, it holds that $|\int gdf| \le \int |g|d|\nu_f| \le 2W_I(f)\sup_I |g|$.

The previous notations and definitions also apply to vector and matrix-valued functions $BV$-functions, replacing absolute values in (3.1) by Euclidean and matricial norms, respectively.

*3.2. Volterra integral equations with kernels of type $B^\infty$*

*Volterra equations for functions of a single variable have been studied extensively, see e.g.* [16, 17]. *However, the specific assumption that the kernel has bounded variation seems to have been treated somewhat tangentially. On the one hand, it is subsumed in the measure-valued case presented in* [17, Ch. 10], *but no convenient criterion is given there for the existence of a resolvent kernel. On the other hand,* [7, Ch. 9, Sec. 1] *sketches the main arguments needed to handle kernels of bounded variation, but the exposition has issues*[5].

We define a *Stieltjes-Volterra kernel of type* $B^\infty$ on $[a,b]\times[a,b]$ as a measurable function $\kappa:[a,b]\times[a,b]\to\mathbb{R}^{d\times d}$, with $\kappa(t,\tau)=0$ for $\tau\geq t$, such that the partial maps $\kappa(t,.)$ lie in $BV_l([a,b])$ and $\|\kappa(t,.)\|_{BV([a,b])}$ is uniformly bounded with respect to $t\in[a,b]$. In addition, we require that $\lim_{\tau\to t^-}W_{[\tau,t)}(\kappa(t,.))=0$ uniformly with respect to $t$; *i.e.*, to every $\varepsilon>0$, there exists $\eta>0$ such that $W_{[\tau,t)}(\kappa(t,.))<\varepsilon$ as soon as $0<t-\tau<\eta$. Note that $W_{[\tau,t)}(\kappa(t,.))\to 0$ for fixed $t$ as $\tau\to t^-$ since $\kappa(t,.)$ has bounded variation on $[a,b]$, by the very definition (3.1); so, the assumption really is that the convergence is uniform with respect to $t$. We endow the space $\mathcal{K}_{[a,b]}$ of such kernels with the norm $\|\kappa\|_{[a,b]}:=\sup_{t\in[a,b]}\|\kappa(t,.)\|_{BV([a,b])}$. If $\kappa_k$ is a Cauchy sequence in $\mathcal{K}_{[a,b]}$, then $\kappa_k$ converges uniformly on $[a,b]\times[a,b]$ to a $\mathbb{R}^{d\times d}$-valued function $\kappa$ because

$$|||\kappa_k(t,\tau)-\kappa_l(t,\tau)|||=|||(\kappa_k(t,\tau)-\kappa_l(t,\tau))-(\kappa_k(t,t)-\kappa_l(t,t))|||\leq\|\kappa_k(t,.)-\kappa_l(t,.)\|_{BV([a,b])}.$$

Moreover, if $m$ is so large that $\|\kappa_k-\kappa_l\|_{[a,b]}<\varepsilon$ for $k,l\geq m$ and $\eta>0$ so small that $W_{[\tau,t)}(\kappa_m)<\varepsilon$ when $t-\tau<\eta$, we get that $W_{[\tau,t)}(\kappa_l)\leq W_{[\tau,t)}(\kappa_m)+W_{[\tau,t)}(\kappa_m-\kappa_l)<2\varepsilon$, and letting $l\to\infty$ we get from [14, thm. 1.3.5] that $W_{[\tau,t)}(\kappa)\leq 2\varepsilon$. The same reference implies that $\|\kappa\|_{[a,b]}\leq\sup_k\|\kappa_k\|_{[a,b]}$, so that $\kappa\in\mathcal{K}_{[a,b]}$. Finally, writing that $W_{[a,b]}(\kappa_k(t,.)-\kappa_l(t,.))<\varepsilon$ for each $t$ when $k,l\geq m$ and passing to the limit as $l\to\infty$, we see that $\lim_k\|\kappa_k-\kappa\|_{[a,b]}=0$ whence $\mathcal{K}_{[a,b]}$ is a Banach space. Note that a Stieltjes-Volterra kernel $\kappa$ of type $B^\infty$ is necessarily bounded with $\sup_{[a,b]\times[a,b]}|||\kappa(t,\tau)|||\leq\|\kappa\|_{[a,b]}$.

A *resolvent* for the Stieltjes-Volterra kernel $\kappa$ on $[a,b]\times[a,b]$ is a Stieltjes-Volterra kernel $\rho$ on $[a,b]\times[a,b]$ satisfying

$$\rho(t,\beta)=-\kappa(t,\beta)+\int_{\beta^-}^{t^-}d\kappa(t,\tau)\rho(\tau,\beta),\qquad a\leq t,\beta\leq b. \tag{3.3}$$

We stress that $d\kappa(t,\tau)$ under the integral sign is here the differential with respect to $\tau$ of a matrix-valued measure, and that $d\kappa(t,\tau)\rho(\tau,\beta)$ is a matrix product resulting in a matrix whose entries are sums of ordinary Lebesgue-Stieltjes integrands. Because matrices do not commute, a (generally) different result would be obtained upon writing $\int_{\beta^-}^{t^-}\rho(\tau,\beta)d\kappa(t,\tau)$. In all cases, the repeated variable under the integral sign ($\tau$ in the present case) is a dummy one.

The following two lemmata, proved in Section 4, are the technical core of this paper.

**Lemma 2.** *If $\kappa$ is a Stieltjes-Volterra kernel of type $B^\infty$ on $[a,b]\times[a,b]$, a resolvent for $\kappa$ uniquely exists.*

**Lemma 3.** *Let $\kappa$ be a Stieltjes-Volterra kernel of type $B^\infty$ on $[a,b]\times[a,b]$ and $\rho$ its resolvent. For each $\mathbb{R}^d$-valued function $g\in BV_r([a,b])$, the unique bounded measurable solution to the equation*

$$y(t)=\int_{a^-}^{t^-}d\kappa(t,\tau)y(\tau)+g(t),\qquad a\leq t\leq b, \tag{3.4}$$

[5] For example, the equation for $\widetilde{\rho}(t,s)$ stated at the top of page 258 of that reference is not right.



3 Jun 2024 09:39:42 PDT
231119-Fueyo Version 2 - Submitted to J. Integr. Eq. Appl.

*is given by*

$$y(t) = g(t) - \int_{a^-}^{t^-} d\rho(t,\alpha)g(\alpha), \qquad a \le t \le b. \tag{3.5}$$

The importance of Lemmata 2 and 3 stems from the fact that to prove our main result at the end of Section 4, we will frame the difference delay equation (1.1) into a Stieltjes-Volterra integral equation with $B^\infty$ kernel.

## 4. Proofs

The proofs of Lemmata 2 and 3 are given before the one of Theorem 1, which relies on them.

*Proof of Lemma 2.* Pick $r > 0$ to be adjusted later, and for $\Psi \in \mathcal{K}_{[a,b]}$ let us define

$$F_r(\Psi)(t,\beta) := \int_{\beta^-}^{t^-} e^{-r(t-\tau)} d\kappa(t,\tau)\Psi(\tau,\beta), \qquad a \le \beta, t \le b.$$

Then, $F_r(\Psi)(t,\beta) = 0$ for $\beta \ge t$, and for $a \le \beta_1 < \beta_2 < t$ we have that

$$F_r(\Psi)(t,\beta_2) - F_r(\Psi)(t,\beta_1) = \int_{\beta_2^-}^{t^-} e^{-r(t-\tau)} d\kappa(t,\tau) \left(\Psi(\tau,\beta_2) - \Psi(\tau,\beta_1)\right) - \int_{\beta_1^-}^{\beta_2^-} e^{-r(t-\tau)} d\kappa(t,\tau)\Psi(\tau,\beta_1),$$

where we used that $\kappa(t,.)$ is left continuous to assign the lower (resp. upper) bound $\beta_2^-$ to the first (resp. second) integral in the above right hand side. Now, the first integral goes to 0 as $\beta_1 \to \beta_2$ by dominated convergence, as $\Psi(t,.)$ is left-continuous; the second integral also goes to 0, because $|\nu_{\kappa(t,.)}|([\beta_1,\beta_2)) \to 0$ when $\beta_1 \to \beta_2$, since $\cap_{\beta_1 \in [a,\beta_2)} [\beta_1,\beta_2) = \varnothing$. Altogether, we see that $F_r(\Psi)(t,.)$ is left-continuous. Moreover, for $[c,d] \subset [a,t)$ and $c = \beta_0 < \beta_1 < \cdots < \beta_N = d$, one has:

$$\begin{aligned}
&\sum_{i=1}^{N} |||F_r(\Psi)(t,\beta_i) - F_r(\Psi)(t,\beta_{i-1})||| \\
&\quad \le \sum_{i=1}^{N} \left|\left|\left| \int_{\beta_i^-}^{t^-} e^{-r(t-\tau)}\, d\kappa(t,\tau) \left(\Psi(\tau,\beta_i) - \Psi(\tau,\beta_{i-1})\right) \right|\right|\right| + \sum_{i=1}^{N} \left|\left|\left| \int_{\beta_{i-1}^-}^{\beta_i^-} e^{-r(t-\tau)} d\kappa(t,\tau)\Psi(\tau,\beta_{i-1}) \right|\right|\right| \\
&\quad \le \sum_{i=1}^{N} \int_{\beta_i^-}^{t^-} e^{-r(t-\tau)}\, d|\nu_{\kappa(t,.)}|\; |||(\Psi(\tau,\beta_i) - \Psi(\tau,\beta_{i-1}))||| + \sum_{i=1}^{N} \int_{\beta_{i-1}^-}^{\beta_i^-} e^{-r(t-\tau)}\, d|\nu_{\kappa(t,.)}|\; |||\Psi(\tau,\beta_{i-1})||| \\
&\quad \le \int_{d^-}^{t^-} d|\nu_{\kappa(t,.)}| \sum_{i=1}^{N} |||(\Psi(\tau,\beta_i) - \Psi(\tau,\beta_{i-1}))||| \\
&\qquad + e^{-r(t-d)} \int_{c^-}^{d^-} d|\nu_{\kappa(t,.)}| \sum_{i=1}^{N} |||(\Psi(\tau,\beta_i) - \Psi(\tau,\beta_{i-1}))||| + \sup_{[a,t]\times[a,t]} |||\Psi||| \int_{c^-}^{d^-} e^{-r(t-\tau)} d|\nu_{\kappa(t,.)}| \\
&\quad \le 2\, W_{[d,t)}(\kappa(t,.)) \sup_{\tau\in[d,t)} W_{[c,d]}(\Psi(\tau,.)) \\
&\qquad + 2\, e^{-r(t-d)} W_{[c,d)}(\kappa(t,.)) \sup_{\tau\in[c,d)} W_{[c,d]}(\Psi(\tau,.)) + 2\, e^{-r(t-d)} \sup_{[a,t]\times[a,t]} |||\Psi|||\; W_{[c,d)}(\kappa(t,.)).
\end{aligned}$$

When $d = t$, the same inequality holds but then $W_{[d,t)}(\kappa(t,.))$ is zero. Setting $c = a$ and $d = t$, we get from the above majorization that $W_{[a,t]}(F_r(\Psi)(t,.)) \le 4\|\kappa\|_{[a,b]}\|\Psi\|_{[a,b]}$, and since $F_r(\Psi)(t,\tau) =$



3 Jun 2024 09:39:42 PDT
231119-Fueyo Version 2 - Submitted to J. Integr. Eq. Appl.

0 for $\tau \geq t$ we deduce that $W_{[a,b]}(F_r(\Psi)(t,.)) = W_{[a,t]}(F_r(\Psi)(t,.))$ is bounded, uniformly with respect to $t$. Next, if we fix $\varepsilon > 0$ and pick $\eta > 0$ so small that $W_{[\tau,t)}(\kappa(t,.)) \leq \varepsilon$ as soon as $t - \tau \leq \eta$ (this is possible because $\kappa \in \mathcal{K}_{[a,b]}$), the same estimate yields

$$W_{[c,t)}(F_r(\Psi)(t,.)) \leq 4W_{[c,t)}(\kappa(t,.))\|\Psi\|_{[a,b]} \leq 4\,\varepsilon\,\|\Psi\|_{[a,b]}, \qquad t - c \leq \eta. \tag{4.1}$$

Altogether, we just showed that $F_r(\Psi) \in \mathcal{K}_{[a,b]}$. Moreover, if we take $r$ so large that $e^{-r\eta} < \varepsilon$, then either $t - a \leq \eta$ and then (4.1) with $c = a$ gives us $W_{[a,t)}(F_r(\Psi)(t,.) \leq 4\varepsilon\|\Psi\|_{[a,b]}$, or else $t - \eta > a$ in which case (4.1) with $c = t - \eta$, together with our initial estimate when $c = a$ and $d = t - \eta$, team up to produce:

$$\begin{aligned}
W_{[a,t)}(F_r(\Psi))(t,.) &= W_{[a,t-\eta]}(F_r(\Psi)(t,.)) + W_{[t-\eta,t)}(F_r(\Psi)(t,.)) \\
&\leq 2\varepsilon \sup_{\tau\in[t-\eta,t)} W_{[a,t-\eta]}(\Psi(\tau,.)) + 2\varepsilon W_{[a,t-\eta)}(\kappa(t,.)) \sup_{\tau\in[a,t-\tau)} W_{[a,t-\eta]}(\Psi(\tau,.)) \\
&\quad + 2\varepsilon \sup_{[a,t]\times[a,t]} |||\Psi||| W_{[a,t-\eta)}(\kappa(t,.)) + 4\varepsilon\|\Psi\|_{[a,b]} \\
&\leq 2\,\varepsilon\,\|\Psi\|_{[a,b]}\,\left(3 + 2\|\kappa\|_{[a,b]}\right).
\end{aligned} \tag{4.2}$$

Consequently, as $W_{[a,t)}(F_r(\Psi)(t,.)) = W_{[a,t]}(F_r(\Psi)(t,.))$ by the left continuity of $F_r(\Psi)(t,.)$, we can ensure upon choosing $r$ sufficiently large that the operator $F_r : \mathcal{K}_{[a,b]} \to \mathcal{K}_{[a,b]}$ has arbitrary small norm. Hereafter, we fix $r$ so that $|||F_r||| < \lambda < 1$.

Now, let $\widetilde{\rho}_0 = 0$ and define inductively:

$$\widetilde{\rho}_{k+1}(t,\beta) = -e^{-rt}\kappa(t,\beta) + F_r(\widetilde{\rho}_k)(t,\beta).$$

Clearly $(t,\beta) \mapsto e^{-rt}\kappa(t,\beta)$ lies in $\mathcal{K}_{[a,b]}$, so that $\widetilde{\rho}_k \in \mathcal{K}_{[a,b]}$ for all $k$. Moreover, we get from what precedes that $\|\widetilde{\rho}_{k+1} - \widetilde{\rho}_k\|_{[a,b]} \leq \lambda\|\widetilde{\rho}_k - \widetilde{\rho}_{k-1}\|_{[a,b]}$. Thus, by the shrinking lemma, $\widetilde{\rho}_k$ converges in $\mathcal{K}_{[a,b]}$ to the unique $\widetilde{\rho} \in \mathcal{K}_{[a,b]}$ such that

$$\begin{aligned}
\widetilde{\rho}(t,\beta) &= -e^{-rt}\kappa(t,\beta) + F_r(\widetilde{\rho})(t,\beta) \\
&= -e^{-rt}\kappa(t,\beta) + \int_{\beta^-}^{t^-} e^{-r(t-\tau)} d\kappa(t,\tau)\widetilde{\rho}(\tau,\beta), \quad a \leq t,\beta \leq b.
\end{aligned} \tag{4.3}$$

Putting $\rho(t,\beta) := e^{rt}\widetilde{\rho}(t,\beta)$, one can see that $\rho$ lies in $\mathcal{K}_{[a,b]}$ if and only if $\widetilde{\rho}$ does, and that (4.3) is equivalent to (3.3). This achieves the proof. □

*Proof Lemma 3.* By Lemma 2 we know that $\kappa$ has a unique resolvent, say $\rho$. Define $y$ through (3.5) so that $y(a) = g(a)$, by inspection. Since $g \in BV_r([a,b])$ and $\rho(t,\cdot)$, $k(t,.)$ lie in $BV_l([a,b])$, an integration by parts [14, thm. 7.5.9][6] using (3.3) along with Fubini's theorem and the relations

[6] Reference [14] restricts to integration by parts over open intervals only, and we do the same at the cost of a slightly lengthier computation.

$\kappa(t,\alpha)=\rho(t,\alpha)=0$ for $\alpha\geq t$ gives us:

$$\begin{aligned}
\textstyle\int_{a^-}^{t^-} d\kappa(t,\alpha)y(\alpha) &= (\kappa(t,a^+)-\kappa(t,a))y(a)+\textstyle\int_{a^+}^{t^-} d\kappa(t,\alpha)y(\alpha)\\
&= (\kappa(t,a^+)-\kappa(t,a))g(a)+\textstyle\int_{a^+}^{t^-} d\kappa(t,\alpha)g(\alpha)-\int_{a^+}^{t^-} d\kappa(t,\alpha)\int_{a^-}^{\alpha^-} d\rho(\alpha,\beta)g(\beta)\\
&= (\kappa(t,a^+)-\kappa(t,a))g(a)+\textstyle\int_{a^+}^{t^-} d\kappa(t,\alpha)g(\alpha)-\int_{a^+}^{t^-} d\kappa(t,\alpha)\int_{a^+}^{\alpha^-} d\rho(\alpha,\beta)g(\beta)\\
&\qquad -\textstyle\int_{a^+}^{t^-} d\kappa(t,\alpha)(\rho(\alpha,a^+)-\rho(\alpha,a))g(a)\\
&= (\kappa(t,a^+)-\kappa(t,a))g(a)+\textstyle\int_{a^+}^{t^-} d\kappa(t,\alpha)g(\alpha)+\int_{a^+}^{t^-} d\kappa(t,\alpha)\int_{a^+}^{\alpha^-} \rho(\alpha,\beta)dg(\beta)\\
&\qquad -\textstyle\int_{a^+}^{t^-} d\kappa(t,\alpha)\left[\rho(\alpha,\beta)g(\beta)\right]_{\beta=a^+}^{\beta=\alpha^-}-\int_{a^+}^{t^-} d\kappa(t,\alpha)(\rho(\alpha,a^+)-\rho(\alpha,a))g(a)\\
&= (\kappa(t,a^+)-\kappa(t,a))g(a)+\textstyle\int_{a^+}^{t^-} d\kappa(t,\alpha)g(\alpha)+\int_{a^+}^{t^-}\left(\int_{\beta^-}^{t^-} d\kappa(t,\alpha)\rho(\alpha,\beta)\right)dg(\beta)\\
&\qquad +\textstyle\int_{a^+}^{t^-} d\kappa(t,\alpha)\rho(\alpha,a^+)g(a)-\int_{a^+}^{t^-} d\kappa(t,\alpha)(\rho(\alpha,a^+)-\rho(\alpha,a))g(a)\\
&= (\kappa(t,a^+)-\kappa(t,a))g(a)+\textstyle\int_{a^+}^{t^-} d\kappa(t,\alpha)g(\alpha)\\
&\qquad +\textstyle\int_{a^+}^{t^-}\big(\rho(t,\beta)+\kappa(t,\beta)\big)dg(\beta)+\int_{a^+}^{t^-} d\kappa(t,\alpha)\rho(\alpha,a)g(a)\\
&= (\kappa(t,a^+)-\kappa(t,a))g(a)+\left[\kappa(t,\alpha)g(\alpha)\right]_{\alpha=a^+}^{\alpha=t^-}+\textstyle\int_{a^+}^{t^-}\rho(t,\beta)dg(\beta)+\int_{a^+}^{t^-} d\kappa(t,\alpha)\rho(\alpha,a)g(a)\\
&= -\kappa(t,a)g(a)+\left[\rho(t,\beta)g(\beta)\right]_{\beta=a^+}^{\beta=t^-}-\textstyle\int_{a^+}^{t^-} d\rho(t,\beta)g(\beta)+\int_{a^-}^{t^-} d\kappa(t,\alpha)\rho(\alpha,a)g(a)\\
&= -\kappa(t,a)g(a)-\rho(t,a^+)g(a)-\textstyle\int_{a^+}^{t^-} d\rho(t,\beta)g(\beta)+\big(\kappa(t,a)+\rho(t,a)\big)g(a)\\
&= -\textstyle\int_{a^-}^{t^-} d\rho(t,\beta)g(\beta)=y(t)-g(t).
\end{aligned}$$

Thus, $y$ is a solution to (3.4). Clearly, it is measurable, and it is also bounded since $\|\rho(t,.)\|_{BV([a,b])}$ is bounded independently of $t$ and $g$ is bounded. If $\widetilde{y}$ is another solution to (3.4) then $\widetilde{y}(a)=y(a)=g(a)$ by inspection, so that $z:=y-\widetilde{y}$ is a bounded measurable solution to the homogeneous equation:

$$z(t)=\int_{a^+}^{t^-} d\kappa(t,\tau)z(\tau),\qquad a\leq t\leq b.$$

Pick $r>0$ to be adjusted momentarily, and set $\widetilde{z}(t):=e^{-rt}z(t)$ so that

$$\widetilde{z}(t)=\int_{a^+}^{t^-} e^{-r(t-\tau)}d\kappa(t,\tau)\widetilde{z}(\tau). \tag{4.4}$$

Let $\eta>0$ be so small that $W_{[\tau,t)}(\kappa(t,.))\leq 1/4$ as soon as $t-\tau\leq\eta$; this is possible because $\kappa\in\mathcal{K}_{[a,b]}$. Then, it follows from (4.4) that for $t-\eta>a$:

$$\begin{aligned}
|\widetilde{z}(t)| &\leq \left|\int_{a^+}^{(t-\eta)^+} e^{-r(t-\tau)^+}d\kappa(t,\tau)\widetilde{z}(\tau)\right|+\left|\int_{(t-\eta)^+}^{t^-} e^{-r(t-\tau)^-}d\kappa(t,\tau)\widetilde{z}(\tau)\right|\\
&\leq 2e^{-r\eta}W_{(a,t-\eta]}(\kappa(t,\cdot))\sup_{(a,t-\eta]}|\widetilde{z}|+\frac{1}{2}\sup_{(t-\eta,t)}|\widetilde{z}|\leq\sup_{(a,t)}|\widetilde{z}|\big(2e^{-r\eta}\|\kappa\|_{[a,b]}+\frac{1}{2}\big),
\end{aligned}$$

while for $t-\eta\leq a$ we simply get $|\widetilde{z}(t)|\leq\sup_{(a,t)}|\widetilde{z}|/2$. Hence, choosing $r$ large enough, we may assume that $|\widetilde{z}(t)|\leq\lambda\sup_{(a,t)}|\widetilde{z}|$ for some $\lambda<1$ and all $t\in[a,b]$. Thus, if we choose $\lambda'\in(\lambda,1)$ and $t_0\in(a,b]$, we can find $t_1\in(a,t_0)$ such that $|\widetilde{z}(t_1)|\geq(1/\lambda')|\widetilde{z}(t_0)|$, and proceeding inductively we construct a sequence $(t_n)$ in $(a,t_0]$ with $|\widetilde{z}(t_n)|\geq(1/\lambda')^n|\widetilde{z}(t_0)|$. If we had $|\widetilde{z}(t_0)|>0$, this would contradict the boundedness of $\widetilde{z}$, therefore $\widetilde{z}\equiv 0$ on $(a,b]$, whence $z\equiv 0$ so that $y=\widetilde{y}$. □

*Proof of Theorem 1.* It follows from (1.3) that $\alpha\mapsto X(t,\alpha)$ lies in $BV_{loc}(\mathbb{R})$ for all $t$ and satisfies

$$d_\alpha X(t,\alpha)=\sum_{j=1}^{N}D_j(t)\,d_\alpha X(t-\tau_j,\alpha)\qquad\text{on}\quad[s,s+\tau_j),\quad 1\leq j\leq N. \tag{4.5}$$



3 Jun 2024 09:39:42 PDT
231119-Fueyo Version 2 - Submitted to J. Integr. Eq. Appl.

Note, since $\alpha \mapsto X(t,\alpha)$ is left continuous (by (1.3) again) while $[s, s+\tau_j)$ is open on the right, that $d_\alpha X(t,\alpha)$ and $d_\alpha X(t-\tau_j,\alpha)$ in (4.5) coincide with (the differential of) the restrictions to $[s, s+\tau_j)$ of the (matrix-valued) measures $\nu_{X(t,\cdot)_{|[s,s+\tau_N)}}$ and $\nu_{X(t-\tau_j,\cdot)_{|[s,s+\tau_N)}}$, provided that $t \geq s+\tau_N$.

Now, substituting (4.5) in (1.6) formally yields (1.1) for $t \geq s+\tau_N$. Hence, by uniqueness of a solution $y$ to (1.1) satisfying $y(s+\theta) = \phi(\theta)$ for $\theta \in [-\tau_N, 0]$, it is enough to check (1.6) when $s \leq t < s+\tau_N$. For this, we adopt the point of view of reference [7], which is to construe delay systems as Stieltjes-Volterra equations upon representing delays by measures. More precisely, we can rewrite (1.1) as a Lebesgue-Stieltjes integral:

$$y(t) = \int_{-\tau_N^-}^{0^-} d\mu(t,\theta) y(t+\theta), \qquad t \geq s, \tag{4.6}$$

with

$$\mu(t,\theta) = \sum_{j=1}^{N} D_j(t) \mathfrak{H}(\theta + \tau_j), \tag{4.7}$$

where $y(\tau)$ is understood to be $\phi(\tau - s)$ when $s - \tau_N \leq \tau \leq s$ and $\mathfrak{H}(\tau)$ is the Heaviside function which is zero for $\tau \leq 0$ and 1 for $\tau > 0$, so that the associated measure on an interval of the form $[0,a]$ or $[0,a)$ is a Dirac delta at 0. Note that $\mathfrak{H}(0) = 0$, which is not the usual convention, but if we defined $\mathfrak{H}$ so that $\mathfrak{H}(0) = 1$ then expanding (4.6) using (4.7) would not give us back (1.1) for the term $D_N(t)y(t-\tau_N)$ would be missing. Observe also, since $\tau_j > 0$ for all $j$, that the minus sign in the upper bound of the integral in (4.6) is immaterial and could be traded for a plus. For $s \leq t \leq s+\tau_N$, singling out the initial data in (4.6) yields

$$y(t) = \int_{(s-t)^-}^{0^-} d\mu(t,\theta) y(t+\theta) + f(t) \quad \text{with} \quad f(t) := \int_{-\tau_N^-}^{(s-t)^-} d\mu(t,\theta)\phi(t+\theta-s), \tag{4.8}$$

where we took into account, when separating the integrals, that $\theta \mapsto \mu(t,\theta)$ is left continuous, while the integral over the empty interval is understood to be zero. It will be convenient to study (4.8) for $t \in [s, s+\tau_N]$, even though in the end the values of $y(t)$ only matter to us for $t \in [s, s+\tau_N)$. Define

$$k(t,\tau) := \begin{cases} \mu(t, \tau - t) - \sum_{j=1}^{N} D_j(t) & \text{for } \tau \in [s,t], \\ 0 & \text{for } \tau > t, \end{cases} \qquad t, \tau \in [s, s+\tau_N]. \tag{4.9}$$

Note that $k(t,\tau) = 0$ when $t - \tau < \tau_1$, and $d_\tau k(t,\tau) = d_\tau \mu(t, \tau - t)$ on $[s, s+\tau_N]$ for fixed $t$. Hence, (4.8) becomes

$$y(t) = \int_{s^-}^{t^-} dk(t,\tau) y(\tau) + f(t), \qquad s \leq t \leq s+\tau_N. \tag{4.10}$$

Now, (4.10) is the Stieltjes-Volterra equation we shall work with.

It suffices to prove (1.6) for $t \in [s, s+\tau_N)$ under the additional assumption that $\phi$ and the $D_j$, which are continuous by hypothesis, also have bounded and locally bounded variation, on $[-\tau_N, 0]$ and $\mathbb{R}$ respectively. Indeed, functions of bounded variation are dense in $C_s$ (say, because $C^1$-functions are), and if $\{\phi_k\}$ converges uniformly to $\phi$ in $C_s$ while $\{D_{j,k}\}$ converges uniformly to $D_j$ in $[s, s+\tau_N]$ as $k \to \infty$, then the solution to (1.1) with initial condition $\phi_k$ and coefficients $D_{j,k}$ converges uniformly on $[s, s+\tau_N]$ to the solution with initial condition $\phi$ and coefficients $D_j$,



3 Jun 2024 09:39:42 PDT
231119-Fueyo Version 2 - Submitted to J. Integr. Eq. Appl.

as is obvious by inspection. Hence, we shall assume without loss of generality that $\phi$ has bounded variation and the $D_j$ have locally bounded variation. Then, since (4.7) and (4.8) imply that

$$f(t) = \sum_{\tau_\ell \in (t-s,\tau_N]} D_\ell(t)\phi(t-s-\tau_\ell), \tag{4.11}$$

it is clear from (4.11) and (3.2) that $f$ is in $BV_r([s, s+\tau_N])$.

As $\mathfrak{H}$ is left continuous and the $D_j$ are bounded on $[s, s+\tau_N]$, it is easy to check that $k(t,\tau)$ defined in (4.9) is a Stieltjes-Volterra kernel of type $B^\infty$ on $[s, s+\tau_N] \times [s, s+\tau_N]$. Let $\rho$ denote the resolvent of this Stieltjes-Volterra kernel; it exists thanks to Lemma 2. As $f$ defined in (4.8) lies in $BV_r([s, s+\tau_N])$, the solution $y$ to (4.10) is given, in view of Lemma 3, by

$$y(t) = f(t) - \int_{s^-}^{t^-} d\rho(t,\alpha)f(\alpha), \qquad s \le t \le s+\tau_N. \tag{4.12}$$

Since $\rho(t,\alpha) = 0$ when $\alpha \ge t$, the integral $\int_{s^-}^{t^-}$ can be replaced by $\int_{s^-}^{(s+\tau_N)^+}$ in (4.12). Thus, putting $\delta_t$ for the Dirac delta distribution at $t$ and $\widetilde{X}(t,\alpha) := \mathcal{H}(t-\alpha)I_d + \rho(t,\alpha)$ with $\mathcal{H}(\tau)$ the "standard" Heaviside function which is 0 for $\tau < 0$ and 1 for $\tau \ge 0$, we deduce from (4.11) and (4.12), since $d_\alpha\mathcal{H}(t-\alpha) = -\delta_t$ on $[s, s+\tau_N]$ for $s \le t < s+\tau_N$, that

$$y(t) = -\int_{s^-}^{(s+\tau_N)^+} d\widetilde{X}(t,\alpha)f(\alpha) = -\int_{s^-}^{(s+\tau_N)^+} d\widetilde{X}(t,\alpha)\left(\sum_{\tau_\ell \in (\alpha-s,\tau_N]} D_\ell(\alpha)\phi(\alpha-s-\tau_\ell)\right), \quad s \le t < s+\tau_N.$$

Rearranging, we get that

$$y(t) = -\sum_{j=1}^{N} \int_{s^-}^{(s+\tau_j)^-} d\widetilde{X}(t,\alpha)D_j(\alpha)\phi(\alpha-s-\tau_j), \qquad s \le t < s+\tau_N,$$

which is what we want (namely: formula (1.6) for $s \le t < s+\tau_N$) if only we can show that $\widetilde{X}(t,\alpha)$ coincides with $X(t,\alpha)$ when $\alpha \in [s, s+\tau_j)$ for each $j$ and every $t \in [s, s+\tau_N)$; here, $X(t,\alpha)$ is defined by (1.3) where we set $s = \alpha$.

For this, we first observe that $X(t,\alpha) = \widetilde{X}(t,\alpha) = 0$ when $\alpha > t$ and that $X(t,t) = \widetilde{X}(t,t) = I_d$. Hence, we need only consider the case $\alpha \in [s,t)$ with $s < t < s+\tau_N$. For $s \le \alpha < t$, we get that

$$-k(t,\alpha) = k(t,t^-) - k(t,\alpha) = \int_{\alpha^-}^{t^-} dk(t,\tau).$$

Thus, (3.3) (where $\kappa = k$) in concert with the definition of $\widetilde{X}(t,\alpha)$ imply that

$$\begin{aligned}\widetilde{X}(t,\alpha) &= I_d\,\mathcal{H}(t-\alpha) - k(t,\alpha) + \int_{\alpha^-}^{t^-} dk(t,\tau)\rho(\tau,\alpha) \;=\; I_d + \int_{\alpha^-}^{t^-} dk(t,\tau)\big(I_d + \rho(\tau,\alpha)\big) \\ &= I_d + \int_{\alpha^-}^{t^-} dk(t,\tau)\widetilde{X}(\tau,\alpha).\end{aligned}$$



**3 Jun 2024 09:39:42 PDT**
**231119-Fueyo Version 2 - Submitted to J. Integr. Eq. Appl.**

Now, on $[\alpha, t)$, we compute from (4.7) and (4.9) that $d_\tau k(t,\tau) = \sum_{t-\tau_j \geq \alpha} D_j(t)\delta_{t-\tau_j}$ and hence, since $\widetilde{X}(t-\tau_j, \alpha) = 0$ when $\alpha > t - \tau_j$, the previous equation becomes:

$$\widetilde{X}(t,\alpha) = I_d + \sum_{j=1}^{N} D_j(t)\widetilde{X}(t-\tau_j,\alpha) \qquad \text{for } s \leq \alpha < t \text{ and } s \leq t < s + \tau_N. \tag{4.13}$$

Comparing (4.13) and (1.3), we see that $\widetilde{X}(t,\alpha)$ and $X(t,\alpha)$ coincide on $[s, s+\tau_N) \times [s, s+\tau_N)$, thereby ending the proof. □

## 5. Concluding remarks

We derived in this note a representation formula for linear non-autonomous difference-delay equations.

Note that a very different representation formula for the solutions of the same class of systems appears in [8, Section 3.2]; it gives an explicit combinatorial formula for the solutions in terms of sums of products of matrices whose number of terms and factors increases with time. In the present representation formula, this combinatorial complexity is somehow encoded by an integral formulation that stems out of interpreting System (1.1) as a Volterra integral equation, which provides a powerful tool to derive estimates.

As pointed out at the beginning of the paper, the present representation formula is meant in the first place to help stability proofs; it plays a crucial role, for instance, in the derivation of a necessary and sufficient condition for exponential stability of periodic systems of the form (1.1), see the manuscript [6]. Let us further stress that the stability of more general classes of systems, like periodic differential delay systems "of neutral type"; *i.e.*, of the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(y(t) - \sum_{j=1}^{N} D_j(t)y(t-\tau_j)\right) = B_0(t)y(t) + \sum_{j=1}^{N} B_j(t)y(t-\tau_j) \tag{5.1}$$

(for some periodic matrices $B_0, \dots, B_N$), relies on the stability of (1.1) through perturbation arguments described in [7] in the time-invariant case. It would be very interesting to extend the stability results from [6] to such differential systems of neutral type, and to this effect we feel that a representation formula should be extremely useful.

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