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ANALYSIS OF POSITIVE SOLUTIONS FOR A FOURTH-ORDER BEAM EQUATION WITH SIGN-CHANGING GREEN'S FUNCTION

HUIJUAN ZHU AND FANGLEI WANG

Abstract. In this work, we investigate the existence of positive solutions for a fourth-order beam equation

$$\begin{cases} u^{(4)}(x) - \lambda u''(x) = \mu h(x) f(u(x)), \ x \in [0,1], \\ u(0) = u''(0) = u(1) = 0, \delta u''(1) - u'''(\zeta) + \lambda u'(\zeta) = 0, \end{cases}$$

with sign-changing Green's function. Combine a priori estimates of Green's function, we apply the Leray-Schauder fixed point theorem to obtain the existence of positive solutions. Moreover, we give the Ulam-Hyers stability result of the solution for the equation.

1. Introduction

Prior to the early 20th century, it appears that most suspension bridges were constructed primarily on the basis of rules of thumb without any theoretical support. As a result, they were prone to a wide range of oscillations, including torsional and purely vertical oscillations. This led to the general modeling of the beam equations

$$mu_{tt} + EIu_{xxxx} + \delta u_t = -ku^+ + W(x) + \varepsilon f(x,t),$$

where E and I are physical constants associated with the beam, Young's modulus and the second moment of inertia, respectively, see [3, 10, 17].

Suppose that u represents an elastic beam of length $L = 1$, which is fixed at $x = 0$ (i.e., no displacement or rotation) and has no displacement at $x = 1$, but may have a support that can slide or rotate. Along the direction of the beam, a varying load $q(x) = \mu h(x) f(u(x))$ is applied to the beam, the magnitude and direction of this load is determined by the functions $f(u(x))$ and $h(x)$ (see Fig.(a)). Now, let $EI = 1$, then from above remarks we can get the following boundary value problem(BVP)

$$\begin{cases} u^{(4)}(x) = \mu h(x) f(u(x)), \ x \in [0,1], \\ u(0) = u''(0) = u(1) = 0, \\ \delta u''(1) - u'''(\zeta) + \lambda u'(\zeta) = 0. \end{cases}$$

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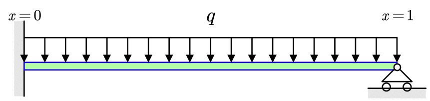
This model may be used to simulate girder and bridge design in structural engineering, especially in girders where nonlinear loading effects are considered, such as in the presence of large deflections, prestressing, dynamic loading, or with specific boundary conditions. Solving this equation can help in designing and predicting the performance of a structure under actual operating conditions, including strength, stability and vibration characteristics.



(a) $q(x) = \mu h(x) f(u(x))$ is arbitrary load form that may change sign, which implies that the wind direction is variable. At $x = 1$, the linear displacement at that point must be zero, allowing for angular displacement, that is, the point is a fixed hinge constraint.

With the application of the beam equation in engineering mechanics, the suitability of solutions for fourth-order boundary value problems has been extensively studied, see [1, 14, 15]. In recent years, the existence of solutions for fourth-order nonlinear boundary value problems has received extensive attention from many authors. For example, Yan et al. in [21] investigated the global structure of positive solutions for fourth-order problem with clamped beam boundary conditions. In [11], Li investigated the existence of positive solutions for fourth-order BVP

$$u^{(4)} + \beta u'' - \delta u = f(t,u), t \in (0,1),$$

under the boundary conditions

$$u(0) = u(1) = u''(0) = u''(1) = 0, \tag{1.1}$$

where $\beta < 2\pi^2$, $\frac{\delta}{\pi^4} + \frac{\beta}{\pi^2} < 1$ and $\delta \geq -\frac{\beta^2}{4}$. In addition, in [24], by the Leray-Schauder's fixed point theorem, Zhang and Chen studied the existence of positive solutions for fourth-order differential equation

$$u^{(4)}(t) + c(t)u(t) = \lambda a(t) f(u),$$

under the boundary conditions (1.1), where $a(t)$ is continuous with $a(t) \neq 0$, and there exists $K > 0$ such that

$$\int_0^1 G(t,s)a^+(s)ds \geq K \int_0^1 G(t,s)a^-(s)ds.$$

The above researches are based on the non-negativity of Green's function. For similar results, please refer to [2, 8, 12, 13, 22, 23] and the references therein.

The existence of positive solutions of differential equations for which the Green's function is nonpositive has also been extensively studied. For example, in [4, 19, 20], in the case where the Green's function is sign-changing, the authors studied the third-order three-point

BVP and they obtained the existence of at least $2m-1$ positive solutions[20], existence and incrementality[19], existence, nonexistence and multiplicity[4] of positive solutions.

On the other hand, some scholars focused on the existence, uniqueness and multiplicity of positive solutions for fourth-order boundary value problems with changing-sign Green's functions (see [7, 18, 25]). In addition, Ma[16] studied the existence and nonexistence of positive solutions for the periodic boundary value problem

$$\begin{cases} u''(t)+a(t)u(t)=\lambda b(t)f(u),\ a.e.\ t\in[0,T],\\ u(0)=u(T),\quad u'(0)=u'(T),\end{cases}$$

where the positive part of Green's function is greater than the negative part. In particular, Gao, Zhang and Ma in [9] considered the existence of positive solutions of the above equation with $a(t)\equiv(\frac{1}{2}+\varepsilon)^2$ $(\varepsilon\in(0,\frac{1}{2}))$ and $T=2\pi$ by the Leray-Schauder's fixed point theorem. In [5], Chen showed the existence of positive solutions for the semi-linear elliptic system. Moreover, Zhang and An[26] used the fixed point index theory, combined with the properties of the positive and negative parts of Green's function, to obtain the existence of positive solutions for second-order differential equation.

Based on the inspiration of the above research, we will study the existence of positive solution for the following fourth-order three-point BVP

$$\begin{cases} u^{(4)}(x)-\lambda u''(x)=\mu h(x)f(u(x)),\ x\in[0,1],\\ u(0)=u''(0)=u(1)=0,\\ \delta u''(1)-u'''(\zeta)+\lambda u'(\zeta)=0,\end{cases} \tag{1.2}$$

where $\lambda>-\pi^2$, $\mu>0$, $\delta\in[0,1)$ and $\zeta\in(0,1)$; h is a Lebesgue integrable function on $[0,1]$; $f:[0,+\infty)\to\mathbb{R}$ is continuous. The proof is based on the Leray-Schauder fixed point theorem. The key of this paper is the calculation of the Green's function. The innovation of this paper is the existence of positive solutions for the equation when the positive part of the Green's function (or the product of the Green's function and the weight function) is greater than the negative part.

Remark 1.1. The third-order derivative of deflection is one of the critical indicators in civil engineering and architecture. The third-order derivative of deflection can be used to predict the deformation and fatigue damage of structures under different loads, and thus guide maintenance and repair work.

The following arrangement of the article is as follows: In the second section, we will calculate the expression of Green's function and consider some of its properties. In the third and fourth sections, we will give the main results (the existence of positive solutions and the Ulam-Hyers stability) of this paper. In the fifth section, we will give practical examples to support our main results.

2. Preliminaries

Firstly, we consider the corresponding linear BVP

$$\left\{\begin{array}{l} u^{(4)}(x)-\lambda u''(x)=w(x),\ x\in[0,1], \\ u(0)=u''(0)=u(1)=0, \\ \delta u''(1)-u'''(\zeta)+\lambda u'(\zeta)=0, \end{array}\right. \tag{2.1}$$

where $\delta\in[0,1)$, $\zeta\in(0,1)$, $w\in L[0,1]$ and $\lambda>-\pi^2$.

Lemma 2.1. Let $\vartheta=\sqrt{|\lambda|}$. The equation (2.1) has the unique solution

$$u(x)=\int_0^1 G(x,y)w(y)dy,$$

where (i) if $\lambda=0$,

$$G(x,y)=\frac{1}{6}\begin{cases} y(3x-y^2)(1-x)+\frac{\delta xy}{1-\delta}(1-x^2), & y\le\min\{x,\zeta\}, \\ x[y(1-y)(3-y)+y^2-x^2]+\frac{\delta xy}{1-\delta}(1-x^2), & x\le y\le\zeta, \\ -y(1-x)(y^2-3x)-\frac{(\delta y-1)x(1-x^2)}{\delta-1}, & \zeta<y\le x, \\ \frac{\delta x(1-y)(1-x^2)}{\delta-1}-x(1-y)^3, & \max\{x,\zeta\}\le y; \end{cases}$$

(ii) if $\lambda>0$,

$$G(x,y)=\begin{cases} \frac{(\delta-1-\delta x)y}{\vartheta^2(\delta-1)}-\frac{\sinh\vartheta y\sinh\vartheta(1-x)}{\vartheta^3\sinh\vartheta}+\frac{y\sinh\vartheta x}{\vartheta^2(\delta-1)\sinh\vartheta}, & y\le\min\{x,\zeta\}, \\ \frac{(\delta-1-\delta y)x}{\vartheta^2(\delta-1)}-\frac{\sinh\vartheta x\sinh\vartheta(1-y)}{\vartheta^3\sinh\vartheta}+\frac{y\sinh\vartheta x}{\vartheta^2(\delta-1)\sinh\vartheta}, & x\le y\le\zeta, \\ \frac{\delta y(1-x)+x-y}{\vartheta^2(\delta-1)}-\frac{\sinh\vartheta y\sinh\vartheta(1-x)}{\vartheta^3\sinh\vartheta}-\frac{(1-y)\sinh\vartheta x}{\vartheta^2(\delta-1)\sinh\vartheta}, & \zeta<y\le x, \\ \frac{\delta x(1-y)}{\vartheta^2(\delta-1)}-\frac{\sinh\vartheta x\sinh\vartheta(1-y)}{\vartheta^3\sinh\vartheta}-\frac{(1-y)\sinh\vartheta x}{\vartheta^2(\delta-1)\sinh\vartheta}, & \max\{x,\zeta\}\le y; \end{cases}$$

(iii) if $-\pi^2<\lambda<0$,

$$G(x,y)=\begin{cases} -\frac{(\delta-1-\delta x)y}{\vartheta^2(\delta-1)}+\frac{\sin\vartheta y\sin\vartheta(1-x)}{\vartheta^3\sin\vartheta}-\frac{y\sin\vartheta x}{\vartheta^2(\delta-1)\sin\vartheta}, & y\le\min\{x,\zeta\}, \\ -\frac{(\delta-1-\delta y)x}{\vartheta^2(\delta-1)}+\frac{\sin\vartheta x\sin\vartheta(1-y)}{\vartheta^3\sin\vartheta}-\frac{y\sin\vartheta x}{\vartheta^2(\delta-1)\sin\vartheta}, & x\le y\le\zeta, \\ -\frac{\delta s(1-x)+x-y}{\vartheta^2(\delta-1)}+\frac{\sin\vartheta y\sin\vartheta(1-x)}{\vartheta^3\sin\vartheta}+\frac{(1-y)\sin\vartheta x}{\vartheta^2(\delta-1)\sin\vartheta}, & \zeta<y\le x, \\ -\frac{\delta x(1-y)}{\vartheta^2(\delta-1)}+\frac{\sin\vartheta x\sin\vartheta(1-y)}{\vartheta^3\sin\vartheta}+\frac{(1-y)\sin\vartheta x}{\vartheta^2(\delta-1)\sin\vartheta}, & \max\{x,\zeta\}\le s. \end{cases}$$

Proof. Integrating the equation $u^{(4)}(x)-\lambda u''(x)=w(x)$ from 0 to x, we get

$$u'''(x)-\lambda u'(x)=\int_0^x w(y)dy+c.$$

Now, combining $u(0)=u''(0)=0$, and integrating the above equation from 0 to x, we can get

$$u''(x)-\lambda u(x)=\int_0^x (x-y)w(y)dy+cx.$$

From $\delta u''(1)-u'''(\zeta)+\lambda u'(\zeta)=0$, it follows that

$$c=\frac{1}{\delta-1}[\delta\int_0^1(1-y)w(y)dy-\int_0^\zeta w(y)dy],$$

which yields that

$$u''(x)-\lambda u(x)=\int_0^x (x-y)w(y)dy-\frac{x}{\delta-1}[\delta\int_0^1(1-y)w(y)dy-\int_0^\zeta w(y)dy].$$

Combining $u(1)=0$ and using the method of comparing coefficients, we obtain that

$$u(x)=\int_0^1 G(x,y)w(y)dy,$$

where Green's function $G(x,y)$ is divided into the following three cases:

Case 1. When $\lambda=0$, if $y\le\zeta$, $y\le x$,

$$\begin{aligned} G(x,y) &= \frac{(1-x)(\delta-1-\delta y)}{\delta-1}\int_0^y \tau^2 d\tau+(1-x)y\int_y^x \tau(1-\frac{\delta\tau}{\delta-1})d\tau \\ &\quad +xy\int_x^1(1-\tau)(1-\frac{\delta\tau}{\delta-1})d\tau \\ &= \frac{(\delta-1-\delta y)(1-x)y^3}{3(\delta-1)}+\frac{(1-x)y}{2}(x^2-y^2)-\frac{\delta(1-x)y}{3(\delta-1)}(x^3-y^3) \\ &\quad +\frac{xy(1-x)^2}{2}-\frac{\delta xy(1-x^2)}{2(\delta-1)}+\frac{\delta xy(1-x^3)}{3(\delta-1)} \\ &= \frac{y}{6}(3x-y^2)(1-x)+\frac{\delta xy}{6(1-\delta)}(1-x^2); \end{aligned}$$

if $y\le\zeta$, $y\ge x$,

$$\begin{aligned} G(x,y) &= \frac{(1-x)(\delta-1-\delta y)}{\delta-1}\int_0^x \tau^2 d\tau+x(1-\frac{\delta y}{\delta-1})\int_x^y(1-\tau)\tau d\tau \\ &\quad +xy\int_y^1(1-\tau)(1-\frac{\delta\tau}{\delta-1})d\tau \\ &= \frac{(\delta-1-\delta y)(1-x)x^3}{3(\delta-1)}+\frac{(\delta-1-\delta y)x}{2(\delta-1)}(y^2-x^2)-\frac{(\delta-1-\delta y)x}{3(\delta-1)}(y^3-x^3) \\ &\quad +\frac{xy(1-y)^2}{2}-\frac{\delta xy(1-y^2)}{2(\delta-1)}+\frac{\delta xy(1-y^3)}{3(\delta-1)} \end{aligned}$$

$$= \frac{x}{6}[y(1-y)(3-y)+y^2-x^2]+\frac{\delta xy}{6(1-\delta)}(1-x^2);$$

if $y>\zeta$, $y\le x$,

$$\begin{aligned}
G(x,y) &= \frac{\delta(1-x)(1-y)}{\delta-1}\int_0^y \tau^2 d\tau+(1-x)\int_y^x \tau\frac{\delta y(1-\tau)+\tau-y}{\delta-1}d\tau\\
&\quad +x\int_x^1(1-\tau)\frac{\delta y(1-\tau)+\tau-y}{\delta-1}d\tau\\
&= \frac{\delta(1-y)(1-x)y^3}{3(\delta-1)}+\frac{(1-x)y}{2}(x^2-y^2)-\frac{(\delta y-1)(1-x)}{3(\delta-1)}(x^3-y^3)+\frac{xy(1-x)^2}{2}\\
&\quad -\frac{(\delta y-1)x}{2(\delta-1)}(1-x^2)+\frac{(\delta y-1)x}{3(\delta-1)}(1-x^3)\\
&= -\frac{y(1-x)(y^2-3x)}{6}-\frac{(\delta y-1)x(1-x^2)}{6(\delta-1)};
\end{aligned}$$

if $y>\zeta$, $y\ge x$,

$$\begin{aligned}
G(x,y) &= \frac{\delta(1-x)(1-y)}{\delta-1}\int_0^x \tau^2 d\tau+\frac{\delta x(1-y)}{\delta-1}\int_x^y(1-\tau)\tau d\tau\\
&\quad +x\int_y^1(1-\tau)\frac{\delta y(1-\tau)+\tau-y}{\delta-1}d\tau\\
&= \frac{\delta(1-y)(1-x)x^3}{3(\delta-1)}+\frac{\delta x(1-y)}{2(\delta-1)}(y^2-x^2)-\frac{\delta x(1-y)}{3(\delta-1)}(y^3-x^3)\\
&\quad +\frac{xy(1-y)^2}{2}-\frac{(\delta y-1)x}{2(\delta-1)}(1-y^2)+\frac{(\delta y-1)x}{3(\delta-1)}(1-y^3)\\
&= \frac{\delta x(1-y)(1-x^2)}{6(\delta-1)}+\frac{x}{6}(y^3-3y^2+3y-1).
\end{aligned}$$

Case 2. When $\lambda>0$, if $y\le\zeta$, $y\le x$,

$$\begin{aligned}
G(x,y) &= \frac{(\delta-1-\delta y)\sinh\vartheta(1-x)}{(\delta-1)\vartheta\sinh\vartheta}\int_0^y \tau\sinh\vartheta\tau d\tau+\frac{y\sinh\vartheta(1-x)}{(\delta-1)\vartheta\sinh\vartheta}\int_y^x(\delta-1-\delta\tau)\sinh\vartheta\tau d\tau\\
&\quad +\frac{y\sinh\vartheta x}{(\delta-1)\vartheta\sinh\vartheta}\int_x^1(\delta-1-\delta\tau)\sinh\vartheta(1-\tau)d\tau\\
&= \frac{(\delta-1-\delta y)\sinh\vartheta(1-x)}{(\delta-1)\vartheta\sinh\vartheta}\Big(\frac{y\cosh\vartheta y}{\vartheta}-\frac{\sinh\vartheta y}{\vartheta^2}\Big)+\frac{y\sinh\vartheta(1-x)}{(\delta-1)\vartheta\sinh\vartheta}\Big(\frac{\delta-1-\delta x}{\vartheta}\cosh\vartheta x\\
&\quad -\frac{\delta-1-\delta y}{\vartheta}\cosh\vartheta y+\frac{\delta\sinh\vartheta x}{\vartheta^2}-\frac{\delta\sinh\vartheta y}{\vartheta^2}\Big)\\
&\quad +\frac{y\sinh\vartheta x}{(\delta-1)\vartheta\sinh\vartheta}\Big(\frac{1}{\vartheta}+\frac{\delta-1-\delta x}{\vartheta}\cosh\vartheta(1-x)-\frac{\delta\sinh\vartheta(1-x)}{\vartheta^2}\Big)\\
&= \frac{(\delta-1-\delta x)y}{\vartheta^2(\delta-1)}-\frac{\sinh\vartheta y\sinh\vartheta(1-x)}{\vartheta^3\sinh\vartheta}+\frac{y\sinh\vartheta x}{\vartheta^2(\delta-1)\sinh\vartheta};
\end{aligned}$$

if $y \leq \zeta$, $y \geq x$,

$$\begin{aligned} G(x,y) &= \frac{(\delta-1-\delta y)\sinh\vartheta(1-x)}{(\delta-1)\vartheta\sinh\vartheta}\int_0^x \tau\sinh\vartheta\tau d\tau + \frac{(\delta-1-\delta y)\sinh\vartheta x}{(\delta-1)\vartheta\sinh\vartheta}\int_x^y \tau\sinh\vartheta(1-\tau)d\tau \\ &\quad + \frac{y\sinh\vartheta x}{(\delta-1)\vartheta\sinh\vartheta}\int_y^1 (\delta-1-\delta\tau)\sinh\vartheta(1-\tau)d\tau \\ &= \frac{(\delta-1-\delta y)\sinh\vartheta(1-x)}{(\delta-1)\vartheta\sinh\vartheta}\Big(\frac{x\cosh\vartheta x}{\vartheta} - \frac{\sinh\vartheta x}{\vartheta^2}\Big) + \frac{(\delta-1-\delta y)\sinh\vartheta x}{(\delta-1)\vartheta\sinh\vartheta}\times \\ &\quad \Big(\frac{x}{\vartheta}\cosh\vartheta(1-x) - \frac{y}{\vartheta}\cosh\vartheta(1-y) - \frac{1}{\vartheta^2}(\sinh\vartheta(1-y) - \sinh\vartheta(1-x))\Big) \\ &\quad + \frac{y\sinh\vartheta x}{(\delta-1)\vartheta\sinh\vartheta}\Big(\frac{1}{\vartheta} + \frac{\delta-1-\delta y}{\vartheta}\cosh\vartheta(1-y) - \frac{\delta\sinh\vartheta(1-y)}{\vartheta^2}\Big) \\ &= \frac{(\delta-1-\delta y)x}{\vartheta^2(\delta-1)} - \frac{\sinh\vartheta x\sinh\vartheta(1-y)}{\vartheta^3\sinh\vartheta} + \frac{y\sinh\vartheta x}{\vartheta^2(\delta-1)\sinh\vartheta}; \end{aligned}$$

if $y > \zeta$, $y \leq x$,

$$\begin{aligned} G(x,y) &= \frac{\delta(1-y)\sinh\vartheta(1-x)}{(\delta-1)\vartheta\sinh\vartheta}\int_0^y \tau\sinh\vartheta\tau d\tau \\ &\quad + \frac{\sinh\vartheta(1-x)}{(\delta-1)\vartheta\sinh\vartheta}\int_y^x [\delta y(1-\tau)+\tau-y]\sinh\vartheta\tau d\tau \\ &\quad + \frac{\sinh\vartheta x}{(\delta-1)\vartheta\sinh\vartheta}\int_x^1 [\delta y(1-\tau)+\tau-y]\sinh\vartheta(1-\tau)d\tau \\ &= \frac{\delta(1-y)\sinh\vartheta(1-x)}{(\delta-1)\vartheta\sinh\vartheta}\Big(\frac{y\cosh\vartheta y}{\vartheta} - \frac{\sinh\vartheta y}{\vartheta^2}\Big) + \frac{\sinh\vartheta(1-x)}{(\delta-1)\vartheta\sinh\vartheta}\times \\ &\quad \Big(\frac{\delta y(1-x)+x-y}{\vartheta}\cosh\vartheta x - \frac{\delta y(1-y)}{\vartheta}\cosh\vartheta y - \frac{1-\delta y}{\vartheta^2}(\sinh\vartheta x - \sinh\vartheta y)\Big) \\ &\quad + \frac{\sinh\vartheta x}{(\delta-1)\vartheta\sinh\vartheta}\Big(-\frac{1-y}{\vartheta} + \frac{\delta y(1-x)+x-y}{\vartheta}\cosh\vartheta(1-x) - \frac{1-\delta y}{\vartheta^2}\sinh\vartheta(1-x)\Big) \\ &= \frac{\delta y(1-x)+x-y}{\vartheta^2(\delta-1)} - \frac{\sinh\vartheta y\sinh\vartheta(1-x)}{\vartheta^3\sinh\vartheta} - \frac{(1-y)\sinh\vartheta x}{\vartheta^2(\delta-1)\sinh\vartheta}; \end{aligned}$$

if $y > \zeta$, $y \geq x$,

$$\begin{aligned} G(x,y) &= \frac{\delta(1-y)\sinh\vartheta(1-x)}{(\delta-1)\vartheta\sinh\vartheta}\int_0^x \tau\sinh\vartheta\tau d\tau + \frac{\delta(1-y)\sinh\vartheta x}{(\delta-1)\vartheta\sinh\vartheta}\int_x^y \tau\sinh\vartheta(1-\tau)d\tau \\ &\quad + \frac{\sinh\vartheta x}{(\delta-1)\vartheta\sinh\vartheta}\int_y^1 [\delta y(1-\tau)+\tau-y]\sinh\vartheta(1-\tau)d\tau \\ &= \frac{\delta(1-y)\sinh\vartheta(1-x)}{(\delta-1)\vartheta\sinh\vartheta}\Big(\frac{x\cosh\vartheta x}{\vartheta} - \frac{\sinh\vartheta x}{\vartheta^2}\Big) + \frac{\delta(1-y)\sinh\vartheta x}{(\delta-1)\vartheta\sinh\vartheta}\Big(-\frac{y}{\vartheta}\cosh\vartheta(1-y) \\ &\quad + \frac{x}{\vartheta}\cosh\vartheta(1-x) - \frac{\sinh\vartheta(1-y)}{\vartheta^2} + \frac{\sinh\vartheta(1-x)}{\vartheta^2}\Big) \end{aligned}$$

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\begin{aligned}
&+\frac{\sinh\vartheta x}{(\delta-1)\vartheta\sinh\vartheta}\Big(-\frac{1-y}{\vartheta}+\frac{\delta y(1-y)}{\vartheta}\cosh\vartheta(1-y)+\frac{1-\delta x}{\vartheta^2}\sinh\vartheta(1-y)\Big)\\
=&\ \frac{\delta x(1-y)}{\vartheta^2(\delta-1)}-\frac{\sinh\vartheta x\sinh\vartheta(1-y)}{\vartheta^3\sinh\vartheta}-\frac{(1-y)\sinh\vartheta x}{\vartheta^2(\delta-1)\sinh\vartheta}.
\end{aligned}
$$

Case 3. When $-\pi^2<\lambda<0$, if $y\le\zeta$, $y\le x$,

$$
\begin{aligned}
G(x,y)=&\ \frac{(\delta-1-\delta y)\sin\vartheta(1-x)}{(\delta-1)\vartheta\sin\vartheta}\int_0^y\tau\sin\vartheta\tau d\tau+\frac{y\sin\vartheta(1-x)}{(\delta-1)\vartheta\sin\vartheta}\int_y^x(\delta-1-\delta\tau)\sin\vartheta\tau d\tau\\
&+\frac{y\sin\vartheta x}{(\delta-1)\vartheta\sin\vartheta}\int_x^1(\delta-1-\delta\tau)\sin\vartheta(1-\tau)d\tau\\
=&\ \frac{(\delta-1-\delta y)\sin\vartheta(1-x)}{(\delta-1)\vartheta\sin\vartheta}\Big(-\frac{y\cos\vartheta y}{\vartheta}+\frac{\sin\vartheta y}{\vartheta^2}\Big)+\frac{y\sin\vartheta(1-x)}{(\delta-1)\vartheta\sin\vartheta}\Big(-\frac{\delta-1-\delta x}{\vartheta}\cos\vartheta x\\
&+\frac{\delta-1-\delta y}{\vartheta}\cos\vartheta y-\frac{\delta\sin\vartheta x}{\vartheta^2}+\frac{\delta\sin\vartheta y}{\vartheta^2}\Big)\\
&+\frac{y\sin\vartheta x}{(\delta-1)\vartheta\sin\vartheta}\Big(-\frac{1}{\vartheta}-\frac{\delta-1-\delta x}{\vartheta}\cos\vartheta(1-x)+\frac{\delta\sin\vartheta(1-x)}{\vartheta^2}\Big)\\
=&\ -\frac{(\delta-1-\delta x)y}{\vartheta^2(\delta-1)}+\frac{\sin\vartheta y\sin\vartheta(1-x)}{\vartheta^3\sin\vartheta}-\frac{y\sin\vartheta x}{\vartheta^2(\delta-1)\sin\vartheta};
\end{aligned}
$$

if $y\le\zeta$, $y\ge x$,

$$
\begin{aligned}
G(x,y)=&\ \frac{(\delta-1-\delta y)\sin\vartheta(1-x)}{(\delta-1)\vartheta\sin\vartheta}\int_0^x\tau\sin\vartheta\tau d\tau+\frac{(\delta-1-\delta y)\sin\vartheta x}{(\delta-1)\vartheta\sin\vartheta}\int_x^y\tau\sin\vartheta(1-\tau)d\tau\\
&+\frac{y\sin\vartheta x}{(\delta-1)\vartheta\sin\vartheta}\int_y^1(\delta-1-\delta\tau)\sin\vartheta(1-\tau)d\tau\\
=&\ \frac{(\delta-1-\delta y)\sin\vartheta(1-x)}{(\delta-1)\vartheta\sin\vartheta}\Big(-\frac{x\cos\vartheta x}{\vartheta}+\frac{\sin\vartheta x}{\vartheta^2}\Big)+\frac{(\delta-1-\delta y)\sin\vartheta x}{(\delta-1)\vartheta\sin\vartheta}\Big(\frac{y}{\vartheta}\cos\vartheta(1-y)\\
&-\frac{x}{\vartheta}\cos\vartheta(1-x)+\frac{1}{\vartheta^2}(\sin\vartheta(1-y)-\sin\vartheta(1-x))\Big)\\
&+\frac{y\sin\vartheta x}{(\delta-1)\vartheta\sin\vartheta}\Big(-\frac{1}{\vartheta}-\frac{\delta-1-\delta y}{\vartheta}\cos\vartheta(1-y)+\frac{\delta\sin\vartheta(1-y)}{\vartheta^2}\Big)\\
=&\ -\frac{(\delta-1-\delta y)x}{\vartheta^2(\delta-1)}+\frac{\sin\vartheta x\sin\vartheta(1-y)}{\vartheta^3\sin\vartheta}-\frac{y\sin\vartheta x}{\vartheta^2(\delta-1)\sin\vartheta};
\end{aligned}
$$

if $y>\zeta$, $y\le x$,

$$
\begin{aligned}
G(x,y)=&\ \frac{\delta(1-y)\sin\vartheta(1-x)}{(\delta-1)\vartheta\sin\vartheta}\int_0^y\tau\sin\vartheta\tau d\tau+\frac{\sin\vartheta(1-x)}{(\delta-1)\vartheta\sin\vartheta}\int_y^x[\delta y(1-\tau)+\tau-y]\sin\vartheta\tau d\tau\\
&+\frac{\sin\vartheta x}{(\delta-1)\vartheta\sin\vartheta}\int_x^1[\delta y(1-\tau)+\tau-y]\sin\vartheta(1-\tau)d\tau\\
=&\ \frac{\delta(1-y)\sin\vartheta(1-x)}{(\delta-1)\vartheta\sin\vartheta}\Big(-\frac{y\cos\vartheta y}{\vartheta}+\frac{\sin\vartheta y}{\vartheta^2}\Big)+\frac{\sin\vartheta(1-x)}{(\delta-1)\vartheta\sin\vartheta}\times
\end{aligned}
$$

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$$\begin{aligned}
&\Big(-\frac{\delta y(1-x)+x-y}{\vartheta}\cos\vartheta x+\frac{\delta y(1-y)}{\vartheta}\cos\vartheta y+\frac{1-\delta y}{\vartheta^2}(\sin\vartheta x-\sin\vartheta y)\Big)\\
&+\frac{\sin\vartheta x}{(\delta-1)\vartheta\sin\vartheta}\Big(\frac{1-y}{\vartheta}-\frac{\delta y(1-x)+x-y}{\vartheta}\cos\vartheta(1-x)-\frac{1-\delta y}{\vartheta^2}\sin\vartheta(1-x)\Big)\\
=\ & -\frac{\delta y(1-x)+x-y}{\vartheta^2(\delta-1)}+\frac{\sin\vartheta y\sin\vartheta(1-x)}{\vartheta^3\sin\vartheta}+\frac{(1-y)\sin\vartheta x}{\vartheta^2(\delta-1)\sin\vartheta};
\end{aligned}$$

if $y>\zeta$, $y\geq x$,

$$\begin{aligned}
G(x,y)\ =\ & \frac{\delta(1-y)\sin\vartheta(1-x)}{(\delta-1)\vartheta\sin\vartheta}\int_0^x\tau\sin\vartheta\tau d\tau+\frac{\delta(1-y)\sin\vartheta x}{(\delta-1)\vartheta\sin\vartheta}\int_x^y\tau\sin\vartheta(1-\tau)d\tau\\
&+\frac{\sin\vartheta x}{(\delta-1)\vartheta\sin\vartheta}\int_y^1[\delta y(1-\tau)+\tau-y]\sin\vartheta(1-\tau)d\tau\\
=\ & \frac{\delta(1-y)\sin\vartheta(1-x)}{(\delta-1)\vartheta\sin\vartheta}\Big(-\frac{x\cos\vartheta x}{\vartheta}+\frac{\sin\vartheta x}{\vartheta^2}\Big)+\frac{\delta(1-y)\sin\vartheta x}{(\delta-1)\vartheta\sin\vartheta}\Big(\frac{y}{\vartheta}\cos\vartheta(1-y)\\
&-\frac{x}{\vartheta}\cos\vartheta(1-x)+\frac{\sin\vartheta(1-y)}{\vartheta^2}-\frac{\sin\vartheta(1-x)}{\vartheta^2}\Big)\\
&+\frac{\sin\vartheta x}{(\delta-1)\vartheta\sin\vartheta}\Big(\frac{1-y}{\vartheta}-\frac{\delta y(1-y)}{\vartheta}\cos\vartheta(1-y)-\frac{1-\delta y}{\vartheta^2}\sin\vartheta(1-y)\Big)\\
=\ & -\frac{\delta x(1-y)}{\vartheta^2(\delta-1)}+\frac{\sin\vartheta x\sin\vartheta(1-y)}{\vartheta^3\sin\vartheta}+\frac{(1-y)\sin\vartheta y}{\vartheta^2(\delta-1)\sin\vartheta}.
\end{aligned}$$

Therefore, we complete the proof. □

Lemma 2.2. Green's function $G(x,y)$ satisfies:

$$\begin{cases} G(x,y)\geq 0 \ \text{ for } \ 0\leq y\leq\zeta,\\ G(x,y)\leq 0 \ \text{ for } \ \zeta<y\leq 1.\end{cases}$$

Proof. From Lemma 2.1, when $\lambda=0$, if $0\leq y\leq\zeta$,

$$G(x,y)=\begin{cases}(1-x)(1-\frac{\delta y}{\delta-1})\int_0^y\tau^2d\tau+(1-x)y\int_y^x\tau(1-\frac{\delta\tau}{\delta-1})d\tau\\
\qquad +xy\int_x^1(1-\tau)(1-\frac{\delta\tau}{\delta-1})d\tau, \qquad 0\leq y\leq x\leq 1\\
(1-x)(1-\frac{\delta y}{\delta-1})\int_0^x\tau^2d\tau+x(1-\frac{\delta y}{\delta-1})\int_x^y(1-\tau)\tau d\tau\\
\qquad +xy\int_y^1(1-\tau)(1-\frac{\delta\tau}{\delta-1})d\tau, \qquad 0\leq x\leq y\leq 1,\end{cases}$$

from $\delta\in[0,1)$, we get $1-\frac{\delta\tau}{\delta-1}\geq 0$ and $1-\frac{\delta y}{\delta-1}\geq 0$, which means $G(x,y)\geq 0$ for $y\leq\zeta$.

If $y > \zeta$,

$$G(x,y) = \frac{1}{\delta - 1}\begin{cases} \delta(1-x)(1-y)\int_0^y \tau^2 d\tau + (1-x)\int_y^x \tau(\delta y(1-\tau)+\tau - y)d\tau \\ \qquad +x\int_x^1 (1-\tau)(\delta y(1-\tau)+\tau-y)d\tau, \qquad 0\le y\le x\le 1, \\ \delta(1-x)(1-y)\int_0^x \tau^2 d\tau + \delta x(1-y)\int_x^y (1-\tau)\tau d\tau \\ \qquad +x\int_y^1 (1-\tau)(\delta y(1-\tau)+\tau-y)d\tau, \qquad 0\le x\le y\le 1, \end{cases}$$

from $\delta \in [0,1)$, we get $\frac{1}{\delta-1} < 0$, which means $G(x,y) \le 0$ for $y > \zeta$.

When $\lambda > 0$, if $y \le \zeta$,

$$G(x,y) = \frac{1}{\delta - 1}\begin{cases} \frac{(\delta-1-\delta y)\sinh\vartheta(1-x)}{\vartheta\sinh\vartheta}\int_0^y \tau\sinh\vartheta\tau d\tau + \frac{y\sinh\vartheta(1-x)}{\vartheta\sinh\vartheta}\int_y^x (\delta-1-\delta\tau)\sinh\vartheta\tau d\tau \\ \qquad +\frac{y\sinh\vartheta x}{\vartheta\sinh\vartheta}\int_x^1 (\delta-1-\delta\tau)\sinh\vartheta(1-\tau)d\tau, \qquad 0\le y\le x\le 1, \\ \frac{(\delta-1-\delta y)\sinh\vartheta(1-x)}{\vartheta\sinh\vartheta}\int_0^x \tau\sinh\vartheta\tau d\tau + \frac{(\delta-1-\delta y)\sinh\vartheta x}{\vartheta\sinh\vartheta}\int_x^y \tau\sinh\vartheta(1-\tau)d\tau \\ \qquad +\frac{y\sinh\vartheta x}{\vartheta\sinh\vartheta}\int_y^1 (\delta-1-\delta\tau)\sinh\vartheta(1-\tau)d\tau, \qquad 0\le x\le y\le 1, \end{cases}$$

from $\delta \in [0,1)$, we get $\frac{1}{\delta-1} < 0$, $\delta - 1 - \delta y \le 0$ and $\delta - 1 - \delta\tau \le 0$, which means $G(x,y) \ge 0$ for $y \le \zeta$.

If $s > \zeta$,

$$G(x,y) = \frac{1}{\delta - 1}\begin{cases} \frac{\delta(1-y)\sinh\vartheta(1-x)}{\vartheta\sinh\vartheta}\int_0^y \tau\sinh\vartheta\tau d\tau + \frac{\sinh\vartheta(1-x)}{\vartheta\sinh\vartheta}\int_y^x [\delta y(1-\tau)+\tau-y]\sinh\vartheta\tau d\tau \\ \qquad +\frac{\sinh\vartheta x}{\vartheta\sinh\vartheta}\int_x^1 [\delta y(1-\tau)+\tau-y]\sinh\vartheta(1-\tau)d\tau, \qquad 0\le y\le x\le 1, \\ \frac{\delta(1-y)\sinh\vartheta(1-x)}{\vartheta\sinh\vartheta}\int_0^x \tau\sinh\vartheta\tau d\tau + \frac{\delta(1-y)\sinh\vartheta x}{\vartheta\sinh\vartheta}\int_x^y \tau\sinh\vartheta(1-\tau)d\tau \\ \qquad +\frac{\sinh\vartheta x}{\vartheta\sinh\vartheta}\int_y^1 [\delta y(1-\tau)+\tau-y]\sinh\vartheta(1-\tau)d\tau, \qquad 0\le x\le y\le 1, \end{cases}$$

from $\delta \in [0,1)$, we get $\frac{1}{\delta-1} \le 0$ and $\delta y(1-\tau)+\tau-y \ge 0$ for $\tau \ge y$, which means $G(x,y) \le 0$ for $y > \zeta$.

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When $-\pi^2 < \lambda < 0$, if $y \leq \zeta$,

$$G(x,y) = \frac{1}{\delta - 1}\begin{cases} \frac{(\delta-1-\delta y)\sin\vartheta(1-x)}{\vartheta\sin\vartheta}\int_0^y \tau\sin\vartheta\tau d\tau + \frac{y\sin\vartheta(1-x)}{\vartheta\sin\vartheta}\int_y^x(\delta-1-\delta\tau)\sin\vartheta\tau d\tau \\ \quad + \frac{y\sin\vartheta x}{\vartheta\sin\vartheta}\int_x^1(\delta-1-\delta\tau)\sin\vartheta(1-\tau)d\tau, \qquad 0 \leq y \leq x \leq 1 \\ \frac{(\delta-1-\delta y)\sin\vartheta(1-x)}{\vartheta\sin\vartheta}\int_0^x \tau\sin\vartheta\tau d\tau + \frac{(\delta-1-\delta y)\sin\vartheta x}{\vartheta\sin\vartheta}\int_x^y \tau\sin\vartheta(1-\tau)d\tau \\ \quad + \frac{y\sin\vartheta x}{\vartheta\sin\vartheta}\int_y^1(\delta-1-\delta\tau)\sin\vartheta(1-\tau)d\tau, \qquad 0 \leq x \leq y \leq 1, \end{cases}$$

from $\delta \in [0,1)$, we get $\frac{\delta-1-\delta y}{\delta-1} \geq 0$ and $\frac{\delta-1-\delta\tau}{\delta-1} \geq 0$, which means $G(x,y) \geq 0$ for $y \leq \zeta$.

If $s > \zeta$,

$$G(x,y) = \frac{1}{\delta - 1}\begin{cases} \frac{\delta(1-y)\sin\vartheta(1-x)}{\vartheta\sin\vartheta}\int_0^y \tau\sin\vartheta\tau d\tau + \frac{\sin\vartheta(1-x)}{\vartheta\sin\vartheta}\int_y^x[\delta y(1-\tau)+\tau-y]\sin\vartheta\tau d\tau \\ \quad + \frac{\sin\vartheta x}{\vartheta\sin\vartheta}\int_x^1[\delta y(1-\tau)+\tau-y]\sin\vartheta(1-\tau)d\tau, \qquad 0 \leq y \leq x \leq 1, \\ \frac{\delta(1-y)\sin\vartheta(1-x)}{\vartheta\sin\vartheta}\int_0^x \tau\sin\vartheta\tau d\tau + \frac{\delta(1-y)\sin\vartheta x}{\vartheta\sin\vartheta}\int_x^y \tau\sin\vartheta(1-\tau)d\tau \\ \quad + \frac{\sin\vartheta x}{\vartheta\sin\vartheta}\int_y^1[\delta y(1-\tau)+\tau-y]\sin\vartheta(1-\tau)d\tau, \qquad 0 \leq x \leq y \leq 1, \end{cases}$$

from $\delta \in [0,1)$, we get $\frac{1}{\delta-1} \leq 0$ and $\delta y(1-\tau)+\tau-y \geq 0$ for $\tau \geq y$, which means $G(x,y) \leq 0$ for $y > \zeta$. □

For the general function $\varphi(x)$, we define the positive and negative parts of $\varphi(x)$ by

$$\varphi^+(x) = \begin{cases} \varphi(x), & \varphi(x) \geq 0, \\ 0, & \varphi(x) < 0, \end{cases} \qquad \varphi^-(x) = \begin{cases} -\varphi(x), & \varphi(x) \leq 0, \\ 0, & \varphi(x) > 0. \end{cases}$$

3. Existence of positive solution

In this section, we will use the properties of Green's function and Leray-Schauder fixed point theorem to study the existence of positive solutions of BVP (1.2).

Before proving the main result, we give the following Lemmas.

Lemma 3.1. (see [6]) Let $T : E \to E$ be a completely continuous operator, where E is a Banach space. Suppose that there exists a constant $K > 0$, such that each solution $(x,\nu) \in E \times [0,1]$ of

$$x = \nu Tx, \nu \in [0,1], x \in E$$

satisfies $\|x\|_E \leq K$. Then T has a fixed point.

Theorem 3.1. Suppose that:
$(H1)$ h is a Lebesgue integrable function on $[0,1]$ and $h(x)$ maybe change sign;
$(H2)$ there exists a constant $L > 1$ such that $\int_0^1 (G(x,y)h(y))^+ dy \geq L\int_0^1 (G(x,y)h(y))^- dy$;

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$(H3)$ $f:[0,+\infty)\to\mathbb{R}$ is a continuous function with $f(0)>0$.

Then there exists μ_0 such that for $\mu\in(0,\mu_0)$, the problem (1.2) has a positive solution

$$u(x)=\mu\int_0^1 G(x,y)h(y)f(u(y))dy.$$

Before proving Theorem 3.1, we need the following lemma:

Lemma 3.2. Assume that $(H1)$ and $(H3)$ hold. Let $r\in(0,1)$, then there exists $\bar{\mu}>0$ such that for $\mu\in(0,\bar{\mu})$, $T^+u=\mu\int_0^1(G(x,y)h(y))^+f(u(y))dy$ has a fixed point u^* with $\|u^*\|\to 0$ as $\mu\to 0$, satisfying

$$u^*(x)\geq \mu r f(0)g(x) \ \text{ for } \ x\in[0,1],$$

where $g(x)=\int_0^1(G(x,y)h(y))^+dy$.

Proof. Firstly, one can readily observe $T^+:E\to E$ is completely continuous. According to the continuity of f, let $f(x)\geq rf(0)$ hold, for $0\leq x\leq\gamma$, where $\gamma>0$. Since

$$u(x)=\mu\int_0^1(G(x,y)h(y))^+f(u(y))dy,$$

we can get

$$u(x)\geq \mu r f(0)\int_0^1(G(x,y)h(y))^+dy=\mu r f(0)g(x).$$

Suppose that

$$\mu<\frac{\gamma}{2\|g\|\Lambda(\gamma)}, \text{ where } \Lambda(x)=\max_{s\in[0,x]}f(s),$$

then for $0\leq x\leq\gamma$,

$$\frac{\Lambda(\gamma)}{\gamma}<\frac{1}{2\mu\|g\|} \text{ and } \frac{\Lambda(x)}{x}\geq\frac{rf(0)}{x}\to+\infty \text{ as } x\to 0.$$

We know from the continuity of f that there exists $\ell_\mu\in(0,\gamma)$ such that

$$\frac{\Lambda(\ell_\mu)}{\ell_\mu}=\frac{1}{2\mu\|g\|}.$$

Let $u\in E$ and $\theta\in(0,1)$ such that $u=\theta T^+u=\theta\mu\int_0^1(G(x,y)h(y))^+f(u(y))dy$. Then

$$\frac{\Lambda(\|u\|)}{\|u\|}\geq\frac{1}{\mu\|g\|},$$

which means that $\|u\|\neq\ell_\mu$. Note that $\ell_\mu\to 0$ as $\mu\to 0$. We know from Lemma 3.1 that T^+ has a fixed point u^* with $\|u^*\|\leq\ell_\mu<\gamma$. Consequently, $u^*(x)\geq\mu rf(0)g(x)$ for $x\in[0,1]$. □

Proof of Theorem 3.1. Let $\psi(x)=\int_0^1(G(x,y)h(y))^-dy$. By $(H2)$ and Lemma 3.1, there exist two numbers $a_1,a_2\in(0,1)$ such that

$$\psi(x)|f(y)|\leq a_2 g(x)f(0), \ \text{ for } \ y\in[0,a_1],\ x\in[0,1]. \tag{3.1}$$

In fact,

$$\psi(x)|f(y)|=\int_0^1(G(x,y)h(y))^-dy\cdot|f(y)|\leq\frac{1}{L}g(x)|f(y)|\leq a_2 g(x)f(0),$$

where a_2 satisfying $\frac{1}{L}|f(y)| \le a_2 f(0)$ for $y \in [0,a_1]$.

Fix $r \in (a_2,1)$ and let $\mu_0 > 0$ such that

$$\|u^*\| + \mu r f(0)\|g\| \le a_1 \quad \text{for } \mu < \mu_0, \tag{3.2}$$

where u^* is given by Lemma 3.2, and for $x,y \in [-a_1,a_1]$,

$$|f(x) - f(y)| \le \frac{r-a_2}{2} f(0), \tag{3.3}$$

with $|x-y| \le \mu_0 r f(0)\|g\|$.

Now, assume that $\mu < \mu_0$. We seek a solution u of (1.2) in the form of $u^* + v^*$. Thus, v^* satisfies

$$v^*(x) = \mu \int_0^1 G(x,y)h(y)f(u^*+v^*)dy - \mu \int_0^1 (G(x,y)h(y))^+ f(u^*)dy, \ x \in [0,1].$$

For each $\omega \in E$, let $v = A\omega$ be the solution of

$$v^*(x) = \mu \int_0^1 G(x,y)h(y)f(u^*+\omega)dy - \mu \int_0^1 (G(x,y)h(y))^+ f(u^*)dy.$$

Then $A : E \to E$ is completely continuous. Let $v \in E$ and $\theta \in (0,1)$ such that $v = \theta A v$. Thus,

$$v(x) = \theta\mu \int_0^1 G(x,y)h(y)f(u^*+v)dy - \theta\mu \int_0^1 (G(x,y)h(y))^+ f(u^*)dy.$$

Now, we show that $\|v\| \ne \mu r f(0)\|g\|$. Suppose on the contrary that $\|v\| = \mu r f(0)\|g\|$. Then, we can obtain from (3.2) and (3.3) that

$$\|u^*+v\| \le \|u^*\| + \|v\| \le a_1$$

and

$$|f(u^*+v) - f(u^*)| \le \frac{r-a_2}{2} f(0),$$

which together with (3.1) implies that

$$\begin{aligned} |v(x)| &= |\theta\mu \int_0^1 G(x,y)h(y)f(u^*+v)dy - \theta\mu \int_0^1 (G(x,y)h(y))^+ f(u^*)dy| \\ &\le |\mu \int_0^1 (G(x,y)h(y))^+ (f(u^*+v) - f(u^*))dy| + |\mu \int_0^1 (G(x,y)h(y))^- f(u^*)dy| \\ &\le \frac{(r-a_2)\mu}{2} f(0)g(x) + \mu a_2 f(0)g(x), \end{aligned}$$

i.e.

$$|v(x)| \le \frac{(r+a_2)\mu}{2} f(0)g(x) \ \text{ for } x \in [0,1]. \tag{3.4}$$

In particular, $\|v\| \le \frac{(r+a_2)\mu}{2} f(0)\|g\| \le \mu r f(0)\|g\|$, which is a contradiction. Thus, A has a fixed point v^* with $\|v^*\| \le \mu r f(0)\|g\|$, by the Leray-Schauder fixed point theorem. Moreover, v^* satisfies (3.4) and we know from Lemma 3.2 that

$$u(x) \ge u^*(x) - |v^*(x)| \ge \mu r f(0)g(x) - \frac{(r+a_2)\mu}{2} f(0)g(x) = \frac{(r-a_2)\mu}{2} f(0)g(x),$$

which means that u is a positive solution of (1.2). □

Corollary 3.1. Suppose that $(H3)$ hold. Replace $(H1)$ and $(H2)$ respectively by:
$(H1')$ h is a Lebesgue integrable function on $[0,1]$ with $h \succ 0$ if $h \geq 0$ for a.e. $t \in [0,1]$;
$(H2')$ there exists a constant $\varepsilon > 0$ such that $\int_0^1 G^+(x,y)dy \geq (1+\varepsilon)\int_0^1 G^-(x,y)dy$.
Then there exists μ^* such that for $\mu \in (0,\mu^*)$, the problem (1.2) has a positive solution.

4. Stability analysis

In this section, we discuss the stability of solution (perturbation) u for the beam equation (1.2).

Definition 4.1. Equation (1.2) is said to be Ulam-Hyers stable, if there exists a real number $M > 0$ such that, for $\forall \varepsilon > 0$ and each solution $v \in C^4[0,1]$ of the inequality

$$|v^{(4)}(x) - \lambda v''(x) - \mu h(x) f(v(x))| \leq \varepsilon, \quad x \in [0,1], \tag{4.1}$$

with boundary conditions $v(0) = v''(0) = v(1) = 0$ and $\alpha v''(1) - v'''(\eta) + \lambda v'(\eta) = 0$, then there exists a solution $u \in C^4[0,1]$ of problem (1.2) such that

$$|u(x) - v(x)| \leq M\varepsilon, \ x \in [0,1].$$

Theorem 4.1. Assume that $(H1)$ and $(H3)$ hold. In addition, there exists a real number $K > 0$ such that

$$|f(u) - f(v)| \leq K|u - v|, \forall\, u, v \in C[0,1].$$

Then the problem (1.2) is Ulam-Hyers stable if $\mu K G^* \|h\|_{L^1} < 1$, where $G^* = \max_{x\in[0,1]} |G(x,y)|$.

Proof. Let $\mathrm{L}v = v^{(4)}(x) - \lambda v''(x)$, then by Lemma 2.1, it yields that L is an invertible operator. For convenience, let L^{-1} be the inverse operator of L. Furthermore, for any $h(x) \in C[0,1]$, by Lemma 2.2, we have

$$|\mathrm{L}^{-1}[h(x)]| \;=\; |\int_0^1 G(x,y)h(y)dy| \leq G^* \|h\|_{L^1},$$

which implies that $\|\mathrm{L}^{-1}\| \leq G^*$.

So, if $v \in C^4[0,1]$ is the solution of (4.1), we have

$$\begin{aligned} & |v(x) - \mu \int_0^1 G(x,y)h(y)f(v(y))dy| \\ = \;& |v - \mu \mathrm{L}^{-1}[h(x)f(v(x))]| \\ = \;& |\mathrm{L}^{-1}(\mathrm{L}v - \mu h(x)f(v(x)))| \\ \leq \;& \|\mathrm{L}^{-1}\| \cdot \|\mathrm{L}v - \mu h(x)f(v(x))\| \\ \leq \;& G^*\varepsilon. \end{aligned}$$

From Theorem 3.1, it concludes that the problem(1.2) has a solution $u(x)$ satisfying

$$u(x) = \mu \int_0^1 G(x,y)h(y)f(u(y))dy.$$

Then for $t \in [0,1]$,

$$\begin{aligned} |v(x)-u(x)| &= |v(x)-\int_0^1 G(x,y)h(y)f(u(y))dy| \\ &\le |v(x)-\int_0^1 G(x,y)h(y)f(v(y))dy| + |\int_0^1 G(x,y)h(y)(f(u(y))-f(v(y)))dy| \\ &\le G^*\varepsilon + \mu K\int_0^1 |G(x,y)h(y)(u(y)-v(y))|dy \\ &\le G^*\varepsilon + \mu K G^*\int_0^1 |h(y)||u(y)-v(y)|dy \\ &\le G^*\varepsilon + \mu K G^*\|h\|_{L^1}|u(x)-v(x)|, \end{aligned}$$

which yields

$$|v(x)-u(x)| \le \frac{G^*\varepsilon}{1-\mu K G^*\|h\|_{L^1}} = M\varepsilon,\ x\in[0,1].$$

Therefore, the problem (1.2) is Ulam-Hyers stable. □

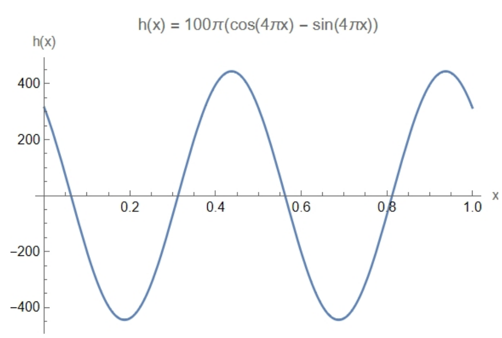
5. Examples

In this part, we will give examples to illustrate the rationality of condition and the existence of positive solutions.

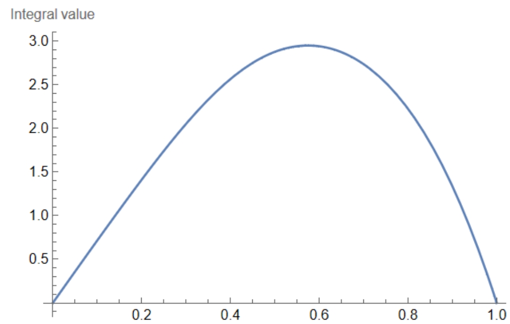
Example 5.1. If $\lambda = 0$, we consider the following simply supported beam

$$\begin{cases} u^{(4)}(x) = 100\pi\mu(\cos(4\pi x)-\sin(4\pi x)),\ x\in[0,1], \\ u(0)=u''(0)=u(1)=0,\ \frac{1}{2}u''(1)-u'''(\frac{1}{2})=0. \end{cases}$$

Let $f(u)=1$, $\delta=\zeta=\frac{1}{2}$ and $h(x)=100\pi(\cos(4\pi x)-\sin(4\pi x))$, where $h(x)$ is sign-changing (see Fig. (b)). Then the function image of the integral $\int_0^1 G(x,y)h(y)dy$ is shown in Fig. (c).



(b) $h(x)$ is the gust load, which is variable sign and corresponds to a change in wind direction.



(c) Integral of $G(x,y)h(y)$ over s from 0 to 1. Deflection map when $\mu = 1$, i.e., image of the solution $u(x)$.

Remark 5.1. Since the solution $u(x) = \mu \int_0^1 G(x,y)h(y)dy$ of the equation in Example 5.1, $u(x) \geq 0$ can be obtained from Fig. (c).

Example 5.2. If $\lambda = 0$, the following BVP is taken into consideration:

$$\begin{cases} u^{(4)}(x) = \mu(\sin u + 1), \ t \in [0,1], \\ u(0) = u''(0) = u(1) = 0, \frac{1}{2}u''(1) - u'''(\frac{1}{2}) = 0. \end{cases}$$

Let $f(u) = \sin u + 1$, $\delta = \zeta = \frac{1}{2}$ and $h(x) = 1$, then for $x \in [0, \frac{1}{2}]$,

$$\begin{aligned} & \int_0^1 G^+(x,y)dy - (1+\varepsilon)\int_0^1 G^-(x,y)dy \\ = & \frac{1}{6}\Bigg[\int_0^x \Big(s(3x-y^2)(1-x) + xy(1-x^2)\Big)dy \\ & +x\int_x^{\frac{1}{2}} \Big(y(1-y)(3-y) + y^2 - x^2 + y(1-x^2)\Big)dy \\ & +(1+\varepsilon)x\int_{\frac{1}{2}}^1 \Big(-(1-y)(1-x^2) + y^3 - 3y^2 + 3y - 1\Big)dy\Bigg] \\ = & \frac{x}{24}\Big[x^3 + (\frac{\varepsilon}{2} - 1)x^2 + 1 - \frac{9\varepsilon}{16}\Big]. \end{aligned}$$

Let $\beta(x) = t^3 + (\frac{\varepsilon}{2} - 1)x^2 + 1 - \frac{9\varepsilon}{16}$, $x \in [0, \frac{1}{2}]$, then

$$\beta'(x) = (3x + \varepsilon - 2)x, \ \ \beta(0) = 1 - \frac{9\varepsilon}{16} \ \text{ and } \ \beta(1) = 1 - \frac{\varepsilon}{16}.$$

If $\frac{2-\varepsilon}{3} \geq \frac{1}{2}$, i.e. $\varepsilon \leq \frac{1}{2}$, then $\beta(x)$ is increasing on $[0, \frac{1}{2}]$, so $\beta(x) \geq \beta(0) = 1 - \frac{9\varepsilon}{16} > 0$.

On the other hand, for $x \in (\frac{1}{2}, 1]$,

$$\int_0^1 G^+(x,y)dy - (1+\varepsilon)\int_0^1 G^-(x,y)dy$$

$$\begin{aligned} &= \frac{1}{6}\left[\int_0^{\frac{1}{2}}\Big(y(3x-y^2)(1-x)+xy(1-x^2)\Big)dy\right.\\ &\quad+\int_{\frac{1}{2}}^{x}\Big(-y(1-x)(y^2-3x)+(y-2)x(1-x^2)\Big)dy\\ &\quad\left.+(1+\varepsilon)x\int_x^1\Big(-(1-y)(1-x^2)+y^3-3y^2+3y-1\Big)dy\right]\\ &= \frac{x}{24}\Big[x^3-x^2+1-\varepsilon(x^4-6x^2+8x-3)\Big]. \end{aligned}$$

Let $\tilde{\beta}(x)=x^3-x^2+1-\varepsilon(x^4-6x^2+8x-3)$, $x\in(\frac{1}{2},1]$, then

$$\tilde{\beta}'(x)=3x^2-2x+\varepsilon(4x^3-12x+8), \quad \text{and} \quad \tilde{\beta}''(x)=12\varepsilon x^2+6x-2(1+6\varepsilon).$$

If $\tilde{\beta}''(\frac{1}{2})\geq 0$ and $\tilde{\beta}'(\frac{1}{2})\geq 0$, then $\tilde{\beta}(x)$ is increasing on $(\frac{1}{2},1]$, i.e. $\tilde{\beta}(x)\geq\tilde{\beta}(\frac{1}{2})=\frac{7(2-\varepsilon)}{16}$. Since $\tilde{\beta}''(\frac{1}{2})=1-9\varepsilon$ and $\tilde{\beta}'(\frac{1}{2})=\frac{10\varepsilon-1}{4}$, so when $\frac{1}{10}\leq\varepsilon\leq\frac{1}{9}$, $\tilde{\beta}(x)\geq 0$ is established. Therefore, there exists $\varepsilon\in[\frac{1}{10},\frac{1}{9}]$ such that

$$\int_0^1 G^+(x,y)dy \geq (1+\varepsilon)\int_0^1 G^-(x,y)dy \quad \text{for} \quad x\in[0,1],$$

which means that the equation has a positive solution.

Example 5.3. If $\lambda>0$, the following BVP is taken into consideration:

$$\begin{cases} u^{(4)}(x)-\lambda u''(x)=\mu(\sin u+1), \ x\in[0,1],\\ u(0)=u''(0)=u(1)=0, \frac{1}{2}u''(1)-u'''(\frac{1}{2})+\lambda u'(\frac{1}{2})=0. \end{cases}$$

Let $f(u)=\sin u+1$, $\delta=\zeta=\frac{1}{2}$ and $h(x)=1$, then for $x\in[0,\frac{1}{2}]$,

$$\begin{aligned} &\int_0^1 G^+(x,y)dy-(1+\varepsilon)\int_0^1 G^-(x,y)dy\\ =\ &\int_0^x\Big(\frac{(1+x)y}{\vartheta^2}-\frac{\sinh\vartheta y\sinh\vartheta(1-x)}{\vartheta^3\sinh\vartheta}-\frac{2y\sinh\vartheta x}{\vartheta^2\sinh\vartheta}\Big)dy\\ &+\int_x^{\frac{1}{2}}\Big(\frac{(1+y)x}{\vartheta^2}-\frac{\sinh\vartheta x\sinh\vartheta(1-y)}{\vartheta^3\sinh\vartheta}-\frac{2y\sinh\vartheta x}{\vartheta^2\sinh\vartheta}\Big)dy\\ &+(1+\varepsilon)\int_{\frac{1}{2}}^1\Big(-\frac{x(1-y)}{\vartheta^2}-\frac{\sinh\vartheta x\sinh\vartheta(1-y)}{\vartheta^3\sinh\vartheta}+\frac{2(1-y)\sinh\vartheta x}{\vartheta^2\sinh\vartheta}\Big)dy\\ =\ &-\frac{1}{\vartheta^4}-\frac{x^2}{2\vartheta^2}+\frac{x}{2\vartheta^2}+\frac{\sinh\vartheta x+\sinh\vartheta(1-x)}{\vartheta^4\sinh\vartheta}+\varepsilon\Big(-\frac{x}{8\vartheta^2}+\frac{\sinh\vartheta x}{\vartheta^4\sinh\vartheta}\\ &-\frac{\sinh\vartheta x\cosh\frac{\vartheta}{2}}{\vartheta^4\sinh\vartheta}+\frac{\sinh\vartheta x}{4\vartheta^2\sinh\vartheta}\Big), \end{aligned}$$

for $x\in(\frac{1}{2},1]$,

$$\int_0^1 G^+(x,y)dy-(1+\varepsilon)\int_0^1 G^-(x,y)dy$$

$$\begin{aligned}
&= \int_0^x \Big(\frac{(1+x)y}{\vartheta^2} - \frac{\sinh\vartheta y \sinh\vartheta(1-x)}{\vartheta^3\sinh\vartheta} - \frac{2y\sinh\vartheta x}{\vartheta^2\sinh\vartheta}\Big)dy \\
&\quad + \int_{\frac{1}{2}}^x \Big(-\frac{(1-x)y+2(x-y)}{\vartheta^2} - \frac{\sinh\vartheta y \sinh\vartheta(1-x)}{\vartheta^3\sinh\vartheta} + \frac{2(1-y)\sinh\vartheta x}{\vartheta^2\sinh\vartheta}\Big)dy \\
&\quad +(1+\varepsilon)\int_x^1 \Big(-\frac{x(1-y)}{\vartheta^2} - \frac{\sinh\vartheta x \sinh\vartheta(1-y)}{\vartheta^3\sinh\vartheta} + \frac{2(1-y)\sinh\vartheta x}{\vartheta^2\sinh\vartheta}\Big)dy \\
&= -\frac{1}{\vartheta^4} - \frac{x^2}{2\vartheta^2} + \frac{x}{2\vartheta^2} + \frac{\sinh\vartheta x + \sinh\vartheta(1-x)}{\vartheta^4\sinh\vartheta} + \varepsilon\Big(-\frac{x(1-x)^2}{2\vartheta^2} + \frac{\sinh\vartheta x}{\vartheta^4\sinh\vartheta} \\
&\quad -\frac{\sinh\vartheta x\cosh\vartheta(1-x)}{\vartheta^4\sinh\vartheta} + \frac{(1-x)^2\sinh\vartheta x}{4\vartheta^2\sinh\vartheta}\Big).
\end{aligned}$$

Let

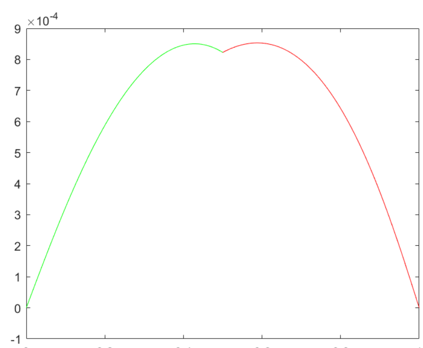
$$\begin{aligned}
g(x) &= -\frac{1}{\vartheta^4} - \frac{x^2}{2\vartheta^2} + \frac{x}{2\vartheta^2} + \frac{\sinh\vartheta x + \sinh\vartheta(1-x)}{\vartheta^4\sinh\vartheta} + \varepsilon\Big(-\frac{x}{8\vartheta^2} + \frac{\sinh\vartheta x}{\vartheta^4\sinh\vartheta} \\
&\quad -\frac{\sinh\vartheta x\cosh\frac{\vartheta}{2}}{\vartheta^4\sinh\vartheta} + \frac{\sinh\vartheta x}{4\vartheta^2\sinh\vartheta}\Big), x\in[0,\frac{1}{2}]
\end{aligned}$$

and

$$\begin{aligned}
m(x) &= -\frac{1}{\vartheta^4} - \frac{x^2}{2\vartheta^2} + \frac{x}{2\vartheta^2} + \frac{\sinh\vartheta x + \sinh\vartheta(1-x)}{\vartheta^4\sinh\vartheta} + \varepsilon\Big(-\frac{x(1-x)^2}{2\vartheta^2} + \frac{\sinh\vartheta x}{\vartheta^4\sinh\vartheta} \\
&\quad -\frac{\sinh\vartheta x\cosh\vartheta(1-x)}{\vartheta^4\sinh\vartheta} + \frac{(1-x)^2\sinh\vartheta x}{4\vartheta^2\sinh\vartheta}\Big), x\in(\frac{1}{2},1].
\end{aligned}$$

Then choose $\vartheta = 10$, $\varepsilon = \frac{1}{2}$, we can see from the Fig. (d) that $g(x) \geq 0$ for $x \in [0,\frac{1}{2}]$ and $m(x) \geq 0$ for $x \in (\frac{1}{2},1]$, namely

$$\int_0^1 G^+(x,y)dy \geq (1+\varepsilon)\int_0^1 G^-(x,y)dy \quad \text{when} \quad \lambda > 0.$$



(d) $g(x)$ is green and $m(x)$ is red.

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Example 5.4. If $-\pi^2<\lambda<0$, the following BVP is taken into consideration:

$$\begin{cases} u^{(4)}(x)-\lambda u''(x)=\mu(\sin u+1),\ x\in[0,1],\\ u(0)=u''(0)=u(1)=0, \frac{1}{2}u''(1)-u'''(\frac{1}{2})+\lambda u'(\frac{1}{2})=0.\end{cases}$$

Let $f(u)=\sin u+1$, $\delta=\zeta=\frac{1}{2}$ and $h(x)=1$, then for $x\in[0,\frac{1}{2}]$,

$$\begin{aligned}
&\int_0^1 G^+(x,y)dy-(1+\varepsilon)\int_0^1 G^-(x,y)dy\\
=\ &\int_0^x\Big(-\frac{(1+x)y}{\vartheta^2}+\frac{\sin\vartheta y\sin\vartheta(1-x)}{\vartheta^3\sin\vartheta}+\frac{2y\sin\vartheta x}{\vartheta^2\sin\vartheta}\Big)dy\\
&+\int_x^{\frac{1}{2}}\Big(-\frac{(1+y)x}{\vartheta^2}+\frac{\sin\vartheta x\sin\vartheta(1-y)}{\vartheta^3\sin\vartheta}+\frac{2y\sin\vartheta x}{\vartheta^2\sin\vartheta}\Big)dy\\
&+(1+\varepsilon)\int_{\frac{1}{2}}^1\Big(\frac{x(1-y)}{\vartheta^2}+\frac{\sin\vartheta x\sin\vartheta(1-y)}{\vartheta^3\sin\vartheta}-\frac{2(1-y)\sin\vartheta x}{\vartheta^2\sin\vartheta}\Big)dy\\
=\ &-\frac{1}{\vartheta^4}+\frac{x^2}{2\vartheta^2}-\frac{x}{2\vartheta^2}+\frac{\sin\vartheta x+\sin\vartheta(1-x)}{\vartheta^4\sin\vartheta}+\varepsilon\Big(\frac{x}{8\vartheta^2}+\frac{\sin\vartheta x}{\vartheta^4\sin\vartheta}\\
&-\frac{\sin\vartheta x\cos\frac{\vartheta}{2}}{\vartheta^4\sin\vartheta}+\frac{\sin\vartheta x}{4\vartheta^2\sin\vartheta}\Big).
\end{aligned}$$

Let

$$\begin{aligned}
\psi(x)\ =\ &-\frac{1}{\vartheta^4}+\frac{x^2}{2\vartheta^2}-\frac{x}{2\vartheta^2}+\frac{\sin\vartheta x+\sin\vartheta(1-x)}{\vartheta^4\sin\vartheta}\\
&+\varepsilon\Big(\frac{x}{8\vartheta^2}+\frac{\sin\vartheta x}{\vartheta^4\sin\vartheta}-\frac{\sin\vartheta x\cos\frac{\vartheta}{2}}{\vartheta^4\sin\vartheta}+\frac{\sin\vartheta x}{4\vartheta^2\sin\vartheta}\Big),\ x\in[0,\frac{1}{2}],
\end{aligned}$$

then

$$\psi'(x)=\frac{x}{\vartheta^2}-\frac{1}{2\vartheta^2}+\frac{\cos\vartheta x-\cos\vartheta(1-x)}{\vartheta^3\sin\vartheta}+\varepsilon(\frac{1}{8\vartheta^2}+\frac{\cos\vartheta x(1-\cos\frac{\vartheta}{2})}{\vartheta^3\sin\vartheta}+\frac{\cos\vartheta x}{4\vartheta\sin\vartheta}),$$

$$\psi''(x)=\frac{1}{\vartheta^2}-\frac{\sin\vartheta x+\sin\vartheta(1-x)}{\vartheta^2\sin\vartheta}-\varepsilon(\frac{\sin\vartheta x(1-\cos\frac{\vartheta}{2})}{\vartheta^2\sin\vartheta}+\frac{\sin\vartheta x}{4\sin\vartheta}),$$

and

$$\psi'''(x)=-\frac{\cos\vartheta x-\cos\vartheta(1-x)}{\vartheta\sin\vartheta}-\varepsilon(\frac{\cos\vartheta x(1-\cos\frac{\vartheta}{2})}{\vartheta\sin\vartheta}+\frac{\vartheta\cos\vartheta x}{4\sin\vartheta})\le 0,$$

so $\psi''(x)\ge\psi''(\frac{1}{2})$ and $\psi''(0)>0$.

If $\psi''(\frac{1}{2})\ge 0$, then

$$\psi'(x)\ge\psi'(0)=\frac{1-\cos\vartheta}{\vartheta^3\sin\vartheta}-\frac{1}{2\vartheta^2}+\varepsilon(\frac{1-\cos\frac{\vartheta}{2}}{\vartheta^3\sin\vartheta}+\frac{1}{4\vartheta\sin\vartheta})\ge 0.$$

So $\psi(x)$ is increasing on $[0,\frac{1}{2}]$, namely $\psi(x)\ge\psi(0)=0$.

If $\psi''(\frac{1}{2}) \leq 0$, then there exists $\theta_0 \in (0, \frac{1}{2})$ such that $\psi'(x)$ is increasing on $[0, \theta_0]$, and decreasing on $(\theta_0, \frac{1}{2}]$. Since

$$\psi'(\frac{1}{2}) = \frac{\varepsilon}{16\vartheta^3 \sin\vartheta}[\vartheta \sin\vartheta + 16\cos\frac{\vartheta}{2}(1-\cos\frac{\vartheta}{2}) + 4\vartheta^2\cos\frac{\vartheta}{2}] > 0,$$

and $\psi'(0) > 0$, we have $\psi'(x) > 0$ for $x \in [0, \frac{1}{2}]$, namely $\psi(x) \geq \psi(0) = 0$. Therefore there exists $\varepsilon > 0$ such that

$$\int_0^1 G^+(x,y)dy \geq (1+\varepsilon)\int_0^1 G^-(x,y)dy \quad \text{for} \quad x \in [0, \frac{1}{2}].$$

On the other hand, for $x \in (\frac{1}{2}, 1]$,

$$\begin{aligned}
&\int_0^1 G^+(x,y)dy - (1+\varepsilon)\int_0^1 G^-(x,y)dy \\
= &\int_0^{\frac{1}{2}} \Big(-\frac{(1+x)y}{\vartheta^2} + \frac{\sin\vartheta y \sin\vartheta(1-x)}{\vartheta^3 \sin\vartheta} + \frac{2y\sin\vartheta x}{\vartheta^2 \sin\vartheta}\Big)dy \\
&+\int_{\frac{1}{2}}^x \Big(\frac{y(1-x)+2(x-y)}{\vartheta^2} + \frac{\sin\vartheta y \sin\vartheta(1-x)}{\vartheta^3\sin\vartheta} - \frac{2(1-y)\sin\vartheta x}{\vartheta^2\sin\vartheta}\Big)dy \\
&+(1+\varepsilon)\int_x^1 \Big(\frac{x(1-y)}{\vartheta^2} + \frac{\sin\vartheta x \sin\vartheta(1-y)}{\vartheta^3\sin\vartheta} - \frac{2(1-y)\sin\vartheta x}{\vartheta^2\sin\vartheta}\Big)dy \\
= &-\frac{1}{\vartheta^4} + \frac{x^2}{2\vartheta^2} - \frac{x}{2\vartheta^2} + \frac{\sin\vartheta x + \sin\vartheta(1-x)}{\vartheta^4\sin\vartheta} + \varepsilon\Big(\frac{x(1-x)^2}{2\vartheta^2} + \frac{\sin\vartheta x}{\vartheta^4\sin\vartheta} \\
&-\frac{\sin\vartheta x\cos\vartheta(1-x)}{\vartheta^4\sin\vartheta} - \frac{(1-x)^2\sin\vartheta x}{\vartheta^2\sin\vartheta}\Big).
\end{aligned}$$

Let

$$\begin{aligned}
p(x) = &-\frac{1}{\vartheta^4} + \frac{x^2}{2\vartheta^2} - \frac{x}{2\vartheta^2} + \frac{\sin\vartheta x + \sin\vartheta(1-x)}{\vartheta^4\sin\vartheta} + \varepsilon\Big(\frac{x(1-x)^2}{2\vartheta^2} \\
&+\frac{\sin\vartheta x}{\vartheta^4\sin\vartheta} - \frac{\sin\vartheta x\cos\vartheta(1-x)}{\vartheta^4\sin\vartheta} - \frac{(1-x)^2\sin\vartheta x}{\vartheta^2\sin\vartheta}\Big), \quad x \in (\frac{1}{2}, 1].
\end{aligned}$$

Taking $\vartheta = 3$, $\varepsilon = \frac{1}{2}$, we can see from the Fig. (e) that $p(x) \geq 0$ for $x \in (\frac{1}{2}, 1]$.Therefore

$$\int_0^1 G^+(x,y)dy \geq (1+\varepsilon)\int_0^1 G^-(x,y)dy \text{ when } -\pi^2 < \lambda < 0.$$

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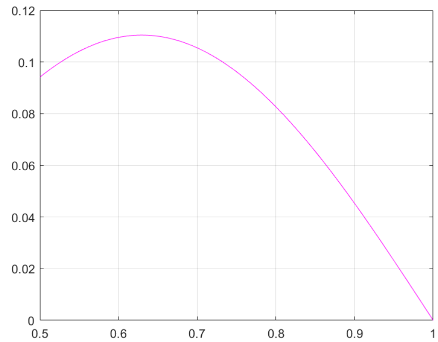
7. Data availibility

The authors confirm that no datas were generated or analyzed during the current study.



(e) Graph of $p(x)$.

8. Declarations

Conflict of interest We declare that they have no conflict of interest.
Ethical approval This submitted paper is original, not published, or submitted elsewhere in any form or language, partially, or fully.

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