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THE GARABEDIAN FUNCTION OF AN ARBITRARY COMPACT SET

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If the outer boundary of the compact plane set E is the union of finitely many disjoint analytic Jordan curves, the Garabedian function of E is a familiar object. J. Garnett and S. Y. Havinson have each asked whether the Garabedian functions of a decreasing sequence of such sets must converge. The present paper shows that they do converge. This fact leads to a natural definition of the Garabedian function of an arbitrary compact plane set. As an intermediate step, an approximate formula is obtained for the analytic capacity of the union of a compact set E and a small disc not intersecting E.

1. Prerequisites and notation. Good introductions to Analytic Capacity are given in [2], pp. 1–26, and [1], Ch. 8; and so we shall give only a brief outline.

$\boldsymbol{C}$ denotes the complex plane. S^2 denotes the extended complex plane with its usual topology. $D(z; r)$ denotes the closed disc with centre z and radius r.

Let E be a compact subset of $\boldsymbol{C}$. $\Omega(E)$ denotes the component of $S^2\backslash E$ containing ∞. The *outer boundary* of E is the boundary $\partial\Omega(E)$ of $\Omega(E)$. The *analytic capacity* of E is:

$$\gamma(E) = \sup\{|g'(\infty)|: g \text{ analytic on } \Omega(E),\ |g| < 1\} .$$

This supremum is attained by a unique function, the *Ahlfors function* of E ([1], p. 197).

$\mathscr{S}$ will denote the class of all compact plane sets whose outer boundary is the union of finitely many pairwise disjoint analytic Jordan curves. Let $E \in \mathscr{S}$, and write $\Omega = \Omega(E)$. The *Hardy space* $H^p(\Omega)$ $(0 < p < \infty)$ is the space of all analytic functions g on Ω such that there exists a harmonic function u on Ω with $|g|^p < u$. If $g \in H^p(\Omega)$ then g has a finite nontangential limit $g(z)$ at almost every point $z \in \partial\Omega$. $H^2(\Omega)$ is a separable Hilbert space with the inner product:

$$(g, h) = \int_{\partial\Omega} g(z)h(z)^* ds \quad (g, h \in H^2(\Omega)) .$$

If $\zeta \in \Omega$ there is a unique function $K(z, \zeta)$ in $H^2(\Omega)$, the *Szegö kernel function*, such that:

$$g(\zeta) = \int_{\partial\Omega} g(z)K(z, \zeta)^* ds \quad (g \in H^2(\Omega)) .$$

$K(z, \zeta)$ is the inner product between the functionals on $H^2(\Omega)$ given by evaluation at z and ζ, so that $K(z, \zeta) = \sum u_n(z)u_n(\zeta)^*$, whenever $\{u_n\}$ is an orthonormal basis for $H^2(\Omega)$. The *Garabedian function* is most easily defined for our purpose as:

$$\psi(z) = \frac{2\pi}{i}\gamma(E)^2K(z, \infty)^2 .$$

See [2], pp. 13-23.

Throughout, E will be a compact plane set, $\Omega = \Omega(E)$, and f will be the Ahlfors function of E. If $E \in \mathscr{S}$, $K(z, \zeta)$ will denote its Szegö kernel function, and ψ its Garabedian function.

We shall use the following results.

1.1. Let $\{E_n\}$ be a decreasing sequence of compact sets with intersection E. Let f_n be the Ahlfors function of E_n. Then $f_n \rightarrow f$ uniformly on compact subsets of Ω, and $\gamma(E_n) \rightarrow \gamma(E)$ ([1], p. 198).

1.2. Let $E \in \mathscr{S}$. Then:
(1) f and ψ are analytic across $\partial\Omega$.
(2) $|f| = 1$ on $\partial\Omega$.
(3) $f(z)\psi(z)dz \geqq 0$ on $\partial\Omega$.
(4) $\psi(\infty) = 1/(2\pi i)$.
(5) $K(\infty, \infty) = 1/(2\pi\gamma(E))$.
([2], pp. 18-23).

1.3. Let E, F be compact, $\gamma(E) = 0$. Then $\gamma(E \cup F) = \gamma(F)$ (an immediate consequence of [2], Theorem 1.4, pp. 10-11).

1.4. Let E be compact, $0 \notin E$, $E \subset D(0; R)$. Denote by E_* the inversion of E in the unit circle. Then:

$$\gamma(E_*) \geqq \gamma(E)/8R^2$$

(proof similar to [1], Lemma 12.2, p. 229).

Finally we need the following result on Hilbert spaces.

PROPOSITION 1.5. *Let h be a separable Hilbert space, and let $\{u_n\}$ be a sequence of vectors in h whose closed linear span is h. Suppose that the infinite matrix T given by $T_{ij} = (u_j, u_i)$ is bounded and invertible (as an operator on l_2). Then for every bounded linear functional f on h the sequence $\{f(u_i)\}$ is square-summable and:*

$$\| f \|^2 = \sum_{i,j=1}^{\infty} (T^{-1})_{ij} f(u_i) f(u_j)^* .$$

Proof. T is positive, and so is the matrix of a positive operator

$P \in B(l_2)$. P has a positive square root $P^{1/2}$, which is invertible since P is invertible. For $i = 1, 2, 3, \cdots$, write $w_i = P^{1/2}e_i$, where e_i is the vector with 1 in its ith place and 0 elsewhere. Since $P^{1/2}$ is invertible, l_2 is the closed linear span of the w_i. $(w_j, w_i) = (P^{1/2}e_j, P^{1/2}e_i) = (Pe_j, e_i) = T_{ij} = (u_j, u_i)$; so we can define a unitary $J: l_2 \to h$ by $J(w_i) = u_i$ for all i, extended to the whole of l_2 by linearity and continuity. The bounded linear functional J^*f on l_2 is represented by some $s \in l_2$. $(e_i, P^{1/2}s) = (P^{1/2}e_i, s) = (w_i, s) = (J^*f)(w_i) = f(Jw_i) = f(u_i)$. Hence $\{f(u_i)\}$ is square-summable. Also:

$$\begin{aligned}\| f \|^2 &= \| s \|^2 = (P^{-1}(P^{1/2}s), (P^{1/2}s)) \\ &= \sum_{i,j=1}^{\infty} (T^{-1})_{ij}(e_i, P^{1/2}s)(e_j, P^{1/2}s)^* = \sum_{i,j=1}^{\infty} (T^{-1})_{ij} f(u_i) f(u_j)^* .\end{aligned}$$

2. The slope function. The purpose of this section is to establish Theorem 2.2, which gives an expression, up to first order in ε, for the analytic capacity of a set of the form $E \cup D(z; \varepsilon)$, where $E \in \mathscr{S}$ and $z \in \Omega(E)$. This will be extended to arbitrary compact sets E in § 3. First we need a lemma which gives bounds on the Szegö kernel function.

LEMMA 2.1. *Let $E \in \mathscr{S}$, $\zeta \in \Omega(E)$, $\zeta \neq \infty$. Let r, R be the least and greatest distances of points of E from ζ. Then:*

$$\frac{r^2}{16\pi R^2 \gamma(E)} \leqq K(\zeta, \zeta) \leqq \frac{8R^2}{2\pi r^2 \gamma(E)} .$$

Proof. We prove the upper bound: the lower one is similar. We may assume that $\zeta = 0$. Let $g \in H^2(\Omega)$, $\| g \| \leqq 1$. Denote inversion in the unit circle by $_*$. Define g_* on Ω_* by $g_*(z) = g(z_*)^*$. Clearly $g_* \in H^2(\Omega_*)$ and $\| g_* \| \leqq 1/r$. Hence:

$$| g(0) |^2 = | g_*(\infty) |^2 \leqq \frac{\| g_* \|^2}{2\pi\gamma(E_*)} \leqq \frac{1}{2\pi r^2 \gamma(E_*)} \leqq \frac{8R^2}{2\pi r^2 \gamma(E)}$$

by 1.4. So:

$$K(0, 0) = \sup \{| g(0) |^2 : g \in H^2(\Omega), \| g \| \leqq 1\} \leqq \frac{8R^2}{2\pi r^2 \gamma(E)} .$$

There is a simpler bound for the Garabedian function: for, in the above notation:

$$\left| \psi(\zeta) - \frac{1}{2\pi i} \right| = \left| \frac{1}{2\pi i} \int_{\partial\Omega} \frac{\psi(z) dz}{z - \zeta} \right| \leqq \frac{1}{2\pi r} \int_{\partial\Omega} | \psi | ds = \frac{\gamma(E)}{2\pi r} .$$

THEOREM 2.2. *Let $E \in \mathscr{S}$. There is a positive real-valued func-*

tion $a_E(\zeta)$, the slope function of E, defined on Ω, with the property that for all $\zeta \in \Omega$:

$$\gamma(E \cup D(\zeta; \varepsilon)) = \gamma(E) + \varepsilon a_E(\zeta) + O(\varepsilon^2) .$$

$a_E(\zeta)$ is given explicitly by:

$$a_E(\zeta) = 2\pi |\psi(\zeta)| \{1 - |f(\zeta)|^2\} .$$

The bound in the error term depends only on $\gamma(E)$ and on the ratio of the greatest and least distances of points of E from ζ.

Proof. We may suppose that $\zeta = 0$. Let r, R be the least and greatest distances of points of E from 0. Note that $E \subset D(0; R)$, so that $\gamma(E) \leqq R$. We shall prove the theorem by showing that:

$$\varepsilon < 10^{-5}(r/R)^5\gamma(E) \Longrightarrow \qquad |\gamma(E \cup D(0; \varepsilon)) - \gamma(E) - \varepsilon a_E(0)| \leqq 10^9(R/r)^{10}\gamma(E)^{-1}\varepsilon^2 .$$

Fix $\varepsilon < 10^{-5}(r/R)^5\gamma(E)$. Since $r < R$ and $\gamma(E) \leqq R$, we have $\varepsilon < 10^{-5}r$; so $D(0; \varepsilon)$ does not meet E. Write $E_1 = E \cup D(0; \varepsilon)$, $\Omega_1 = \Omega(E_1)$, $H^2 = H^2(\Omega)$, $H_1^2 = H^2(\Omega_1)$, $\gamma = \gamma(E)$, $\gamma_1 = \gamma(E_1)$. Choose an orthonormal basis $\{u_n\}$ for H^2. Now we can use the Cauchy integral to express any element of H_1^2 as the sum of an element of H^2 and an element of $H^2(S^2 \backslash D(0; \varepsilon))$. The latter space is the closed linear span of $\{z^{-n}: n \geqq 0\}$. It follows that if, for $n \geqq 1$, v_n is any function analytic on $\bar{\Omega}$ except for a pole of order n at 0, then H_1^2 is the closed linear span of $\{u_n\} \cup \{v_n\}$. To be specific, we shall put:

$$v_n(z) = \frac{\varepsilon^{n-1/2}}{\sqrt{(2\pi)}\, K(0, 0)} \frac{K(z, 0)}{z^n} .$$

1.2 (5) says that $1/(2\pi\gamma_1)$ is the square of the norm of evaluation at ∞ in H_1^2. Our proof consists of calculating this by applying Proposition 1.5 to $\{u_n\} \cup \{v_n\}$.

We shall calculate various bounds now, so as not to break continuity later. Throughout, "$\| \ \|$" and "norm" will refer to the norm of an element of a Hilbert space, or the norm of an infinite matrix considered as a bounded operator on l_2; and "$\| \ \|_\infty$" will denote the supremum of the absolute value of a function on the set $D(0; \varepsilon)$.

Let $z_0 \in \boldsymbol{C}$, $|z_0| \leqq \varepsilon$. For $n \geqq 1$:

$$u_n(z_0) = \frac{1}{2\pi i} \int_{|z|=r/2} \frac{u_n(z)dz}{z - z_0} .$$

Hence:

$$\| u_n \|_\infty \leqq \frac{1}{2\pi(r/2 - \varepsilon)} \int_{|z|=r/2} | u_n | ds ,$$

so that by Schwarz's inequality:

$$\| u_n \|_\infty^2 \leqq \frac{\pi r}{4\pi^2(r/2 - \varepsilon)^2} \int_{|z|=r/2} | u_n |^2 ds .$$

Summing over n and using Lemma 2.1 gives:

$$\begin{aligned} \sum \| u_n \|_\infty^2 &\leqq \frac{r}{4\pi(r/2 - \varepsilon)^2} \int_{|z|=r/2} K(z, z) ds \\ &\leqq \frac{r \cdot \pi r}{4\pi(r/2 - \varepsilon)^2} \frac{8(R + r/2)^2}{2\pi(r/2)^2 \gamma} \leqq 3R^2 r^{-2} \gamma^{-1} \end{aligned} \tag{1}$$

since $\varepsilon < 10^{-5} r$ and $r < R$. Analogous computation gives:

$$\sum \| u_n' \|_\infty^2 \leqq 50 R^2 r^{-4} \gamma^{-1} . \tag{2}$$

In particular:

$$\sum | u_n'(0) |^2 \leqq 50 R^2 r^{-4} \gamma^{-1} . \tag{3}$$

Next we want a bound for $\| d^k/dz^k\, K(z, 0) \|_\infty$. Let $z_0 \in C$, $| z_0 | \leqq \varepsilon$. Then for $k \geqq 1$ and for all $s < r$:

$$\begin{aligned} \left| \frac{d^k}{dz^k} K(z, 0) |_{z=z_0} \right| &= \left| \frac{k!}{2\pi i} \int_{|z|=s} \frac{K(z, 0) dz}{(z - z_0)^{k+1}} \right| \\ &\leqq \frac{k!}{2\pi} \frac{2\pi s}{(s - \varepsilon)^{k+1}} \frac{8R(R + s)}{2\pi r(r - s)\gamma} . \end{aligned}$$

(Here we have estimated $| K(z, 0) |$ by $K(z, z)^{1/2} K(0, 0)^{1/2}$ and then used Lemma 2.1.) In particular, putting $s = kr/(k + 1)$:

$$\begin{aligned} \left| \frac{d^k}{dz^k} K(z, 0) |_{z=z_0} \right| &\leqq \frac{k!}{2\pi} \frac{8R \cdot 2R}{r(r - (k + 1)\varepsilon/k)^{k+1}\gamma} \frac{(k + 1)^{k+1}}{k^k} \\ &\leqq \frac{(k + 1)!}{2\pi} \frac{16R^2}{r(r - 2\varepsilon)^{k+1}\gamma} e \leqq \frac{7R^2(k + 1)!}{r(r - 2\varepsilon)^{k+1}\gamma} . \end{aligned}$$

This holds also for $k = 0$ by Lemma 2.1. Hence for $k = 0, 1, 2, \cdots$:

$$\left\| \frac{d^k}{dz^k} K(z, 0) \right\|_\infty \leqq \frac{7R^2(k + 1)!}{r(r - 2\varepsilon)^{k+1}\gamma} . \tag{4}$$

We need one more estimate. Since $\varepsilon < 10^{-5} r$, (4) gives:

$$\left\| \frac{dK(z, 0)}{dz} \right\|_\infty \leqq \frac{15R^2}{r^3 \gamma} .$$

Hence, using Lemma 2.1 and the fact that $\varepsilon < 10^{-5} R^{-4} r^5$, we have:

$$(5) \qquad \| K(z, 0) \|_\infty \leqq K(0, 0) + \varepsilon \left\| \frac{dK(z, 0)}{dz} \right\|_\infty \leqq 1.01K(0, 0) .$$

We shall imagine the basis $\{u_n\} \cup \{v_n\}$ to be partitioned into three sections. The first section consists of all the u_n, the second section consists of v_1 alone, and the third section consists of $v_2, v_3, v_4, \cdots$. The corresponding matrix T of inner products will be in block form:

$$(6) \qquad T = I + M , \quad M = \begin{bmatrix} A & B^H & C^H \\ B & D & E^H \\ C & E & F \end{bmatrix} .$$

Next we calculate the inner products. Denote inner products in H_1^2 by (,). By a statement of the form "$X = Y$ with error Z" we shall mean $|X - Y| \leqq Z$, or $\| X - Y \| \leqq Z$, according to context.

$$(u_n, u_m) = \int_{\partial \Omega_1} u_n u_m^* ds = \int_{\partial \Omega} u_n u_m^* ds + \int_{|z|=\varepsilon} u_n u_m^* ds = \delta_{mn} + 2\pi\varepsilon u_m(0)^* u_n(0)$$
$$+ \varepsilon \int_0^{2\pi} [u_m(0)^*(u_n(\varepsilon e^{i\theta}) - u_n(0)) + (u_m(\varepsilon e^{i\theta})^* - u_m(0)^*) u_n(\varepsilon e^{i\theta})] d\theta .$$

$$|(u_n, u_m) - \delta_{mn} - 2\pi\varepsilon u_m(0)^* u_n(0)| \leqq 2\pi\varepsilon^2(\| u_m \|_\infty \| u_n' \|_\infty + \| u_m' \|_\infty \| u_n \|_\infty) .$$

Now the matrix $[2\pi\varepsilon u_m(0)^* u_n(0)]$ has norm $2\pi\varepsilon(\sum |u_m(0)|^2 |u_n(0)|^2)^{1/2} = 2\pi\varepsilon K(0, 0) \leqq 8R^2 r^{-2} \gamma^{-1} \varepsilon$ by Lemma 2.1. The norm of the matrix $[2\pi\varepsilon^2(\| u_m \|_\infty \| u_n' \|_\infty + \| u_m' \|_\infty \| u_n \|_\infty)]$ is at most $4\pi\varepsilon^2(\sum \| u_m \|_\infty^2 \| u_n' \|_\infty^2)^{1/2} \leqq 200R^2 r^{-3} \gamma^{-1} \varepsilon^2$ by (1) and (2). So (see the format (6)):

$$(7) \qquad A = [2\pi\varepsilon u_m(0)^* u_n(0)] \text{ with error } 200R^2 r^{-3} \gamma^{-1} \varepsilon^2 .$$

Also, $\| A \| \leqq 8R^2 r^{-2} \gamma^{-1} \varepsilon + 200R^2 r^{-3} \gamma^{-1} \varepsilon^2 \leqq 9(R/r)^2 \gamma^{-1} \varepsilon \leqq 9(R/r)^5 \gamma^{-1} \varepsilon$ since $\varepsilon < 10^{-5} r$ and $r < R$. In fact the cruder bound $\| A \| \leqq 2500(R/r)^5 \gamma^{-1} \varepsilon$ will be sufficient. Observe that, since $\varepsilon < 10^{-5}(r/R)^5 \gamma$, we have also $\| A \| \leqq 1/40$.

The mth element of B is:

$$(u_m, v_1) = \frac{\varepsilon^{1/2}}{\sqrt{(2\pi)}\, K(0, 0)} \int_{\partial \Omega} \frac{K(z, 0)^* u_m ds}{z^*}$$
$$+ \frac{\varepsilon^{1/2}}{\sqrt{(2\pi)}\, K(0, 0) \varepsilon i} \int_{|z|=\varepsilon} K(z, 0)^* u_m(z) dz .$$

Now the second term on the right-hand side is:

$$\frac{\varepsilon^{-1/2}}{\sqrt{(2\pi)}\, K(0, 0) i} \int_{|z|=\varepsilon} (K(z, 0)^* - K(0, 0)^*) u_m(z) dz$$

by Cauchy's theorem, and is therefore bounded in magnitude by $(2\pi e^{3/2}/(\sqrt{(2\pi)} K(0, 0)))\,(15R^2/r^3\gamma)\, \| u_m \|_\infty$ by (4). So:

$$(8)\qquad \begin{aligned} B &= \left[\frac{\varepsilon^{1/2}}{\sqrt{(2\pi)}\,K(0,0)}\int_{\partial\Omega}\frac{K(z,0)^* u_m ds}{z^*}\right] \\ &\text{with error } \frac{2\pi\varepsilon^{3/2}}{\sqrt{(2\pi)}\,K(0,0)}\frac{15R^2}{r^3\gamma}(\textstyle\sum \| u_m \|_\infty^2)^{1/2} \\ &\leqq 66K(0,0)^{-1}R^3r^{-4}\gamma^{-3/2}\varepsilon^{3/2} \end{aligned}$$

by (1). The norm of the matrix in the square brackets is at most:

$$\begin{aligned} &\frac{\varepsilon^{1/2}}{\sqrt{(2\pi)}\,K(0,0)}\left(\int_{\partial\Omega}\frac{|K(z,0)|^2 ds}{|z|^2}\right)^{1/2} \\ &\leqq \frac{\varepsilon^{1/2}}{\sqrt{(2\pi)}\,K(0,0)r}\left(\int_{\partial\Omega}|K(z,0)|^2 ds\right)^{1/2} \leqq \frac{\varepsilon^{1/2}}{\sqrt{(2\pi)}\,K(0,0)^{1/2}r}\,. \end{aligned}$$

Hence, using Lemma 2.1 and the fact that $\varepsilon < 10^{-5}(r/R)^4\gamma$, (8) gives $\| B \| \leqq 3Rr^{-2}\gamma^{1/2}\varepsilon^{1/2}$. The cruder bounds $\| B \| \leqq 1/40$ and $\| B \|^2 \leqq 2500(R/r)^5\gamma^{-1}\varepsilon$ will suffice. Also, using (8) and the estimates calculated in the last few lines, we have:

$$(9)\qquad \begin{aligned} B^H B &= \left[\frac{\varepsilon}{2\pi K(0,0)^2}\int_{\partial\Omega}\frac{K(z,0)u_m^* ds}{z}\int_{\partial\Omega}\frac{K(z,0)^* u_n ds}{z^*}\right] \\ &\text{with error } 20000R^6r^{-8}\varepsilon^2\,. \end{aligned}$$

The elements of C are, for $m \geqq 1$, $n \geqq 2$:

$$\begin{aligned} (u_m, v_n) &= \frac{\varepsilon^{n-1/2}}{\sqrt{(2\pi)}\,K(0,0)}\int_{\partial\Omega}\frac{K(z,0)^* u_m ds}{(z^*)^n} \\ &\quad + \frac{\varepsilon^{-n+1/2}}{\sqrt{(2\pi)}\,K(0,0)i}\int_{|z|=\varepsilon} K(z,0)^* u_m(z) z^{n-1} dz\,. \end{aligned}$$

Call the first and second terms of the above expression P_{mn} and Q_{mn} respectively. Then:

$$\begin{aligned} \| P \| &\leqq \frac{1}{\sqrt{(2\pi)}\,K(0,0)}\left(\sum_{n=2}^{\infty}\int_{\partial\Omega}\frac{|K(z,0)|^2 ds}{|z|^{2n}}\varepsilon^{2n-1}\right)^{1/2} \\ &\leqq \frac{1}{\sqrt{(2\pi)}\,K(0,0)}\left(\sum_{n=2}^{\infty}\frac{\varepsilon^{2n-1}}{r^{2n}}K(0,0)\right)^{1/2} \\ &\leqq 3Rr^{-3}\gamma^{1/2}\varepsilon^{3/2} \leqq (R/r)^5\gamma^{-1}\varepsilon\,. \end{aligned}$$

We estimate the integral in the expression for Q_{mn} as follows. Replace $K(z, 0)$ by $K(z, 0)$ minus its Taylor expansion about 0 as far as the term in z^{n-1}. By Cauchy's theorem, these added terms do not affect the integral. By Taylor's theorem, $K(z, 0)$ minus its Taylor expansion is bounded on $|z| = \varepsilon$ by $(\varepsilon^n/n!)\| d^n/dz^n K(z,0) \|_\infty$, and (4) gives an estimate for that. This procedure gives:

$$\begin{aligned}\|Q\| &\leqq \frac{14\pi R^2}{r\sqrt{(2\pi)}\,K(0,0)\gamma}\Big(\sum_{n=2}^{\infty}\frac{\varepsilon^{2n+1}(n+1)^2}{(r-2\varepsilon)^{2n+2}}\Big)^{1/2}(\sum\|u_m\|_\infty^2)^{1/2}\\ &\leqq \frac{14\pi R^2}{r\sqrt{(2\pi)}\,\gamma}\,\frac{16\pi R^2\gamma}{r^2}\,\frac{4\varepsilon^{5/2}}{r^3}\,\frac{\sqrt{3}\,R}{\sqrt{\gamma}\,r}\\ &\leqq 6000R^5r^{-7}\gamma^{-1/2}\varepsilon^{5/2} \leqq (R/r)^5\gamma^{-1}\varepsilon\,.\end{aligned}$$

Hence $\|C\| \leqq \|P\| + \|Q\| \leqq 2(R/r)^5\gamma^{-1}\varepsilon$. Once again we shall need only $\|C\| \leqq 2500(R/r)^5\gamma^{-1}\varepsilon \leqq 1/40$.

It is convenient to deal with $\begin{bmatrix} D & E^H \\ E & F\end{bmatrix}$ as a single matrix. Its (m, n)th element (see (6)) is, for $m \geqq 1$, $n \geqq 1$:

$$\frac{\varepsilon^{m+n-1}}{2\pi K(0,0)^2}\int_{\partial\Omega}\frac{|K(z,0)|^2ds}{(z^*)^m z^n} + \Big(\frac{\varepsilon^{m+n-1}}{2\pi K(0,0)^2}\int_{|z|=\varepsilon}\frac{|K(z,0)|^2ds}{(z^*)^m z^n} - \delta_{mn}\Big)\,.$$

Denote by G_{mn}, H_{mn} respectively the first term and the bracketed term of the above expression. We have:

$$|G_{mn}| \leqq \frac{\varepsilon^{m+n-1}}{2\pi K(0,0)^2}\,\frac{1}{r^{m+n}}\int_{\partial\Omega}|K(z,0)|^2ds = \frac{\varepsilon^{m+n-1}}{2\pi K(0,0)r^{m+n}}\,.$$

Hence $\|G\| \leqq (1/(2\pi\varepsilon K(0,0)))\sum_{n=1}^{\infty}\varepsilon^{2n}/r^{2n} \leqq 9R^2r^{-4}\gamma\varepsilon \leqq 9(R/r)^5\gamma^{-1}\varepsilon$. H is trickier to deal with. We have:

$$\begin{aligned}H_{nn} &= \frac{1}{2\pi K(0,0)^2\varepsilon}\int_{|z|=\varepsilon}|K(z,0)|^2ds - 1\\ &= \frac{1}{2\pi i K(0,0)^2}\int_{|z|=\varepsilon}K(z,0)(K(z,0)^* - K(0,0)^*)z^{-1}dz\,.\end{aligned}$$

Lemma 2.1, (4) with $k = 1$, and (5) now give $|H_{nn}| \leqq 800R^4r^{-5}\varepsilon$. If $m > n$, then:

$$H_{mn} = \frac{\varepsilon^{n-m}}{2\pi i K(0,0)^2}\int_{|z|=\varepsilon}K(z,0)K(z,0)^*z^{m-n-1}dz\,.$$

As before, we may replace the second occurrence of $K(z, 0)$ in the integral by $K(z, 0)$ minus its Taylor expansion, this time as far as the term in z^{m-n-1}. Then by (5), Lemma 2.1, and (4) with $k = m - n$:

$$\begin{aligned}|H_{mn}| &\leqq \varepsilon^{m-n}1.01\frac{16\pi R^2\gamma}{r^2}\,\frac{7R^2(m-n+1)}{r(r-2\varepsilon)^{m-n+1}\gamma}\\ &\leqq 400R^4r^{-5}\varepsilon(|m-n|+1)(1/99998)^{|m-n|-1}\end{aligned}$$

since $\varepsilon < 10^{-5}r$. This holds similarly for $m < n$. Combining the cases $m = n$, $m > n$, and $m < n$, we see that:

$$\begin{aligned}\|H\| &\leqq \frac{\varepsilon R^4}{r^5}\Big(800 + 2\times 400\Big(2 + \frac{3}{99998} + \frac{4}{(99998)^2} + \cdots\Big)\Big)\\ &\leqq 2401R^4r^{-5}\varepsilon \leqq 2401(R/r)^5\gamma^{-1}\varepsilon\,.\end{aligned}$$

So $\begin{bmatrix} D & E^H \\ E & F \end{bmatrix}$ has norm at most $\|G\| + \|H\| \leqq 2500(R/r)^5\gamma^{-1}\varepsilon$. Hence each of $\|D\|, \|E\|, \|F\| \leqq 2500(R/r)^5\gamma^{-1}\varepsilon \leqq 1/40$.

To summarise: we have shown that:

$$\begin{aligned} &\|A\|, \|B\|^2, \|C\|, \|D\|, \|E\|, \|F\| \leqq 2500(R/r)^5\gamma^{-1}\varepsilon\,; \\ &\|A\|, \|B\|, \|C\|, \|D\|, \|E\|, \|F\| \leqq 1/40\,. \end{aligned} \tag{10}$$

In particular we have verified that M is a bounded matrix: indeed that $\|M\| \leqq 3/40 < 1$. Thus $T = I + M$ is invertible, and Proposition 1.5 applies.

Our next step is to calculate the top left-hand block of the inverse of T. Since $T^{-1} = I - M + M^2 - M^3 + \cdots$, this top left-hand block is:

$$\begin{aligned} S = I& \\ &- A \\ &+ A^2 + B^H B + C^H C \\ &- A^3 - AB^H B - AC^H C - B^H BA - B^H DB - B^H E^H C - C^H CA \\ &\quad - C^H EB - C^H FC \\ &+ \cdots\,. \end{aligned}$$

The row of this expression containing products of degree n ($n \geqq 4$) consists of 3^{n-1} terms. Each of these terms has norm at most $(2500)^2(R/r)^{10}\gamma^{-2}\varepsilon^2(1/40)^{n-4}$ by (10). Hence $S = I - A + B^H B$ with error:

$$\begin{aligned} &\frac{(2500)^2\varepsilon^2 R^{10}}{\gamma^2 r^{10}}\Big(1 + 1 + \frac{1}{40} + 1 + \frac{1}{40} + 1 + 1 + \frac{1}{40} + \frac{1}{40} + \frac{1}{40} + \frac{1}{40} \\ &+ 27\Big(1 + \frac{3}{40} + \Big(\frac{3}{40}\Big)^2 + \cdots\Big)\Big) \leqq 3.10^8(R/r)^{10}\gamma^{-2}\varepsilon^2\,. \end{aligned}$$

Using (7) and (9), we have:

$$S = \left[\delta_{mn} - 2\pi\varepsilon u_m(0)^* u_n(0) + \frac{\varepsilon}{2\pi K(0, 0)^2}\int \frac{K(z, 0)u_m^* ds}{z}\int \frac{K(z, 0)^* u_n ds}{z^*}\right]$$

with error $200R^2r^{-3}\gamma^{-1}\varepsilon^2 + 20000R^6r^{-8}\varepsilon^2 + 3.10^8(R/r)^{10}\gamma^{-2}\varepsilon^2 \leqq 4.10^8(R/r)^{10}\gamma^{-2}\varepsilon^2$. Here and subsequently all integrals are taken round $\partial\Omega$.

Finally we apply Proposition 1.5, which says that $1/(2\pi\gamma_1) = \sum S_{mn}u_m(\infty)u_n(\infty)^*$ (since $v_n(\infty) = 0$ for all n). Hence:

$$\begin{aligned} \frac{1}{2\pi\gamma_1} &= \sum |u_n(\infty)|^2 - 2\pi\varepsilon\Big|\sum u_n(0)^* u_n(\infty)\Big|^2 \\ &+ \frac{\varepsilon}{2\pi K(0, 0)^2}\Big|\sum\Big(u_n(\infty)\int \frac{K(z, 0)u_n^* ds}{z}\Big)\Big|^2 \end{aligned}$$

with error $4.10^8(R/r)^{10}\gamma^{-2}\varepsilon^2 \sum |u_m(\infty)|^2 = 4.10^8(R/r)^{10}\gamma^{-2}\varepsilon^2/(2\pi\gamma)$. Multiplying by $2\pi\gamma$ and using the fact that $\sum u_n(z)u_n(\zeta)^* = K(z, \zeta)$, we have:

$$\frac{\gamma}{\gamma_1} = 1 - 4\pi^2\gamma\varepsilon|K(0, \infty)|^2 + \frac{\gamma\varepsilon}{K(0, 0)^2}\left|\int \frac{K(z, 0)K(z, \infty)^* ds}{z}\right|^2 \tag{11}$$

with error $4.10^8(R/r)^{10}\gamma^{-2}\varepsilon^2$.

Now the last term simplifies. On $\partial\Omega$, $f(z)\psi(z)dz \geqq 0$, so that $ds = (\psi(z)/|\psi(z)|)f(z)dz = (K(z, \infty)/K(z, \infty)^*)(f(z)/i)\,dz$. Therefore:

$$\int \frac{K(z, 0)K(z, \infty)^* ds}{z} = \frac{1}{i}\int \frac{K(z, 0)K(z, \infty)f(z)dz}{z}$$
$$= -2\pi K(0, 0)K(0, \infty)f(0)$$

since $K(z, 0)K(z, \infty)f(z)$ is analytic on $\bar{\Omega}$ and vanishes at ∞. Substituting in (11), we have:

$$\frac{\gamma}{\gamma_1} = 1 - 4\pi^2\gamma\varepsilon|K(0, \infty)|^2\{1 - |f(0)|^2\} \text{ with error } 4.10^8(R/r)^{10}\gamma^{-2}\varepsilon^2 .$$

Now $4\pi^2\gamma\varepsilon|K(0, \infty)|^2\{1 - |f(0)|^2\} \leqq 4\pi^2\gamma\varepsilon K(0, 0)K(\infty, \infty) \leqq 8(R/r)^2\gamma^{-1}\varepsilon < 10^{-4}$. Also $4.10^8(R/r)^{10}\gamma^{-2}\varepsilon^2 \leqq 1/25$. So we can invert to obtain:

$$\frac{\gamma_1}{\gamma} = 1 + 4\pi^2\gamma\varepsilon|K(0, \infty)|^2\{1 - |f(0)|^2\} \text{ with error } 10^9(R/r)^{10}\gamma^{-2}\varepsilon^2 ;$$
$$\gamma_1 = \gamma + 4\pi^2\gamma^2\varepsilon|K(0, \infty)|^2\{1 - |f(0)|^2\}$$
$$= \gamma + 2\pi\varepsilon|\psi(0)|\{1 - |f(0)|^2\} \text{ with error } 10^9(R/r)^{10}\gamma^{-1}\varepsilon^2 .$$

It is as well to explain the curious choice of the functions v_n in the above proof. The only essential property of v_n we used is that it vanishes at ∞ and is analytic on $\bar{\Omega}$ except for a pole at 0 near which $v_n(z) = (2\pi)^{-1/2}\varepsilon^{n-1/2}z^{-n} + \cdots$. The simpler choice $v_n(z) = (2\pi)^{-1/2}\varepsilon^{n-1/2}z^{-n}$ shortens the proof but yields an error bound dependent on the length of $\partial\Omega$, which would have been unsuitable for the next section.

3. Extension to arbitrary compact sets.

We shall now show how the above results extend to arbitrary compact sets E. In particular, we show how to define the Garabedian function of E, thus solving a problem considered in [2] and [3].

Let E be compact. We shall suppose meantime that $\gamma(E) > 0$. E can be expressed as the intersection of a decreasing sequence $\{E_n\}$ in $\mathscr{S}$. Hence ψ_{E_n} and a_{E_n} are defined. Fix $\zeta \in \Omega(E)$, and choose n_0 so that $\zeta \in \Omega(E_n)$ whenever $n > n_0$. By Theorem 2.2 there exist $\varepsilon_0 > 0$, $k > 0$, such that $\forall n > n_0$, $\forall \varepsilon < \varepsilon_0$:

$$|\gamma(E_n \cup D(\zeta; \varepsilon)) - \gamma(E_n) - \varepsilon a_{E_n}(\zeta)| \leqq k\varepsilon^2 . \tag{12}$$

That is, for all $\varepsilon < \varepsilon_0$, the sequence $\{\varepsilon a_{E_n}(\zeta)\}_{n>n_0}$, considered as an element of the Banach space of bounded sequences with the supremum norm, is within a distance $k\varepsilon^2$ of the sequence $\{\gamma(E_n \cup D(\zeta; \varepsilon)) - \gamma(E_n)\}_{n>n_0}$, which converges to $\gamma(E \cup D(\zeta; \varepsilon)) - \gamma(E)$ by 1.1. Thus $\{a_{E_n}(\zeta)\}$ is within a distance $k\varepsilon$ of the closed subspace c of convergent sequences, for all ε, and is therefore itself in c. Call its limit $a_E(\zeta)$. a_E is the *slope function* of E. Letting $n \to \infty$ in (12) now gives, for all $\varepsilon < \varepsilon_0$:

$$|\gamma(E \cup D(\zeta; \varepsilon)) - \gamma(E) - \varepsilon a_E(\zeta)| \leqq k\varepsilon^2 .$$

This shows also that the limit $a_E(\zeta)$ is independent of the choice of the sequence $\{E_n\}$.

Now, for each n, $|\psi_{E_n}(\zeta)| = a_{E_n}(\zeta)/(2\pi\{1 - |f_{E_n}(\zeta)|^2\})$, and this converges pointwise in $\Omega(E)$. Moreover, $\{\psi_{E_n}\}$ is a normal sequence, since, if F is a compact subset of $\Omega(E)$, $\{\psi_{E_n}\}$ is uniformly bounded on F by the remark following Lemma 2.1. It follows that for some sequence λ_n of points on the unit circle, $\{\lambda_n\psi_{E_n}\}$ converges uniformly on compact subsets of $\Omega(E)$. In fact we may take $\lambda_n = 1$, since $\psi_{E_n}(\infty) = 1/(2\pi i)$. So $\{\psi_{E_n}\}$ converges uniformly on compact sets. Call its limit, $\{\psi_E\}$, the *Garabedian function* of E. Hence also $a_{E_n}(\zeta) = 2\pi|\psi_{E_n}(\zeta)|\{1 - |f_{E_n}(\zeta)|^2\}$ converges uniformly on compact sets (and not merely pointwise, as ascertained already).

Now suppose that $\gamma(E) = 0$. We define $\psi_E(\zeta) = 1/(2\pi i)$, $a_E(\zeta) = 1$ for $\zeta \in \Omega(E)$. This is consistent with the relation $a_E(\zeta) = 2\pi|\psi_E(\zeta)|\{1 - |f_E(\zeta)|^2\}$ since $f_E(\zeta) = 0$. $\gamma(E \cup D(\zeta; \varepsilon)) = \varepsilon$ for all $\varepsilon > 0$ by 1.3, and so the relation $\gamma(E \cup D(\zeta; \varepsilon)) = \gamma(E) + \varepsilon a_E(\zeta) + O(\varepsilon^2)$ holds trivially. If $\{E_n\}$ is a sequence in $\mathscr{S}$ decreasing to E, then $\psi_{E_n}(\zeta) \to 1/(2\pi i) = \psi_E(\zeta)$ uniformly on compact sets by the remark following Lemma 2.1.

Finally, if E is compact, and $\{E_n\}$ is *any* sequence of compact sets decreasing to E, the same working as above shows that $\psi_{E_n} \to \psi_E$ and $a_{E_n} \to a_E$ uniformly on compact sets.

We have therefore proved:

THEOREM 3.1. *The Garabedian function $\psi_E(\zeta)$ and the slope function $a_E(\zeta)$ can be defined for all compact sets E, in such a way that:*

(1) *The definitions coincide with the existing meanings if $E \in \mathscr{S}$;*

(2) *If $\{E_n\}$ is a sequence of compact sets decreasing to E, then $\psi_{E_n} \to \psi_E$ and $a_{E_n} \to a_E$ uniformly on compact subsets of $\Omega(E)$;*

(3) *$\gamma(E \cup D(\zeta; \varepsilon)) = \gamma(E) + \varepsilon a_E(\zeta) + O(\varepsilon^2)$ for all $\zeta \in \Omega(E)$, and the bound in the error term depends only on $\gamma(E)$ and on the ratio of the greatest and least distances of points of E from ζ; and*

(4) *$a_E(\zeta) = 2\pi|\psi(\zeta)|\{1 - |f(\zeta)|^2\}$ for all $\zeta \in \Omega(E)$.*

The slope function is related to the problem of subadditivity of γ. If E is connected, then $a_E(\zeta) \leqq 1$: this is a re-statement of Bieberbach's distortion theorem. Subadditivity of γ would obviously imply $a_E(\zeta) \leqq 1$ for all compact E.

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Added in proof. N. Suita recently has independently proved the uniqueness of the Garabedian function much more simply ("On a metric induced by Analytic Capacity," Kōdai Math. Sem. Rep. 25 (1973), 215-218).

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