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Selberg Supertrace Formula for Super Riemann Surfaces, Analytic Properties of Selberg Super Zeta-Functions and Multiloop Contributions for the Fermionic String

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Abstract. In this paper a complete derivation of the Selberg supertrace formula for super Riemann surfaces and a discussion of the analytic properties of the Selberg super zeta-functions is presented. The Selberg supertrace formula is based on Laplace-Dirac operators $\square_m$ of weight m on super Riemann surfaces. The trace formula for all $m \in \mathbf{Z}$ is derived and it is shown that one must discriminate between even and odd m. Particularly the term in the trace formula proportional to the identity transformation is sensitive to this discrimination. The analytic properties of the two Selberg super zeta-functions are discussed in detail, first with, and the second without consideration of the spin structure. We find for the Selberg super zeta-functions similarities as well as differences in comparison to the ordinary Selberg zeta-function. Also functional equations for the two Selberg super zeta-functions are derived. The results are applied to discuss the spectrum of the Laplace-Dirac operators and to calculate their determinants. For the spectrum it is found that the nontrivial Eigenvalues are the same for $\square_m$ and $\square_0$ up to a constant depending on m, which is analogous to the bosonic case. The analytic properties of the determinants can be deduced from the analytic properties of the Selberg super zeta-functions, and it is shown that they are well-defined. Special cases ($m = 0, 2$) for the determinants are important in the Polyakov approach for the fermionic string. With these results it is deduced that the fermionic string integrand of the Polyakov functional integral is well-defined.

I. Introduction

The Selberg trace formula has turned out to be a powerful tool to analyse the spectra of Laplacians on Riemann surfaces and to calculate their determinants. In

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addition, the Selberg zeta-function serves as a function with which many calculations and considerations can be simplified. In this paper I want to present a discussion generalizing these features to the "super"-case.

The main interest in this field emerges from the Polyakov approach to string theory [49]. In this prescription the perturbation expansion is given by a summation of all Riemann surfaces with increasing genus $g = 0, 1, 2, \ldots$ and an additional integration over all variations of a Riemann surface with a given genus g, i.e. the integration over the Teichmüller space. This partition function in the genus g, i.e. the multiloop expansion, has been in detail discussed by e.g. D'Hoker and Phong [16], Gilbert [21] and Namazie and Rajjev [45] for the bosonic string and by D'Hoker and Phong [18] for the fermionic string. In the bosonic string theory as well as in the fermionic string theory it turns out that the Selberg (super-)trace formula serves as a tool to express the (super-)determinants of the relevant Laplace(-Dirac) operators as ratios of Selberg (super) zeta-functions. There is the alternative to express these terms by means of Theta-functions as was pointed out by Alvarez-Gaumé et al. [1] and Manin [40]. The final task is, of course, the study of the superstring [24, 25] and the heterotic string as developed by Gross et al. [33]. The question of the superstring is relatively easy. In the fermonic string, the spinors on the Riemann surface are defined with some spinor structure, which can be independently chosen for left- and right movers. In type II superstring theory (type I theories contain open strings, whereas type II theories only closed strings), these spinor structures must be summed over to project (GSO-projection [22]) onto the correct sector of the Neveu–Schwarz–Rammond theory [18]. It turns out that at the one-loop level, for type II superstrings the resulting sum for the partition function vanishes by the use of a famous Jacobi identity on Theta-functions ("aequatio identica satis abstrusa"), indicating the presence of 10-dimensional space-time supersymmetry, i.e. the equality at each mass level of the numbers of bosonic and fermionic states (for details see e.g. [18]).

Throughout this paper I work with type II theories in flat space-time having critical dimension $d = 10$. The question of heterotic strings is not discussed in this paper.

However, the Selberg trace formula and its super generalization has much more applications than only for string theory. There is also the question of the spectrum of Laplace(-Dirac) operators on (super-) Riemann surfaces. The Selberg (super-)trace formula with appropriately chosen test functions gives answers for those considerations.

Furthermore there is the study of quantum chaos. Classical motion on bounded domains on the Poincaré upper half-plane (or the Poincaré disc, respectively) turns out to be chaotic (Bernoulli-like) and there are an infinite set of closed geodesics in these domains. An remarkable achievement in this field has been presented in papers of Aurich and Steiner [4], where up to 200 million geodesics were determined, and by Aurich, Sieber and Steiner [5], where these geodesics have been used to determine the first low lying Eigenvalues of the Laplace operator on the simplest symmetric compact domain for $g = 2$, corresponding to the regular octagon. Also McKean [38] and Steiner [57] have used the Selberg trace formula to obtain information about determinants of Laplace operators and to study some properties of the Selberg zeta-function.

This paper is devoted to the program to generalize as much as possible notions of the bosonic case to the fermionic one. To make this paper selfcontained, I introduce in Chap. II the notion of superconformal transformation on the Poincaré super upper half-plane $\mathscr{S}\mathscr{H}$. This includes a description of the super Möbius transformations and the construction of the metric on the super Poincaré upper half-plane $\mathscr{S}\mathscr{H}$ by the Vielbein approach of $2+2$-dimensional supergravity.

The further contents of the chapters will be as follows: In Chap. III the Selberg supertrace formula for automorphic forms of weight m on compact super Riemann surfaces is discussed, the latter are visualized as bounded domains on the super Poincaré upper half-plane $\mathscr{S}\mathscr{H}$. I will derive the corresponding trace formula in the super generalisation for Laplace-Dirac operators with weight m. This gives the super-generalization of the Selberg trace formula as introduced by Selberg [56] and discussed in great detail by Hejhal [36]. A supergeneralization of the Selberg trace formula was already given by Baranov et al. [7], but with some details not taken into account, i.e. in their discussion the term corresponding to the unit transformation (except $m=0$) was missing. I derive this term explicitly and thus complete their work. However, I do not claim to be mathematically rigorous.

The fourth chapter is devoted to the discussion of the two Selberg super zeta-functions and contains entirely new results. These zeta-functions were originally introduced by Selberg [56] to study spectra of Laplacians on compact Riemann surfaces of genus g. The super Selberg zeta-functions are similarly defined as the usual Selberg zeta-function. I find similarities but also important differences for Z_0 and Z_1 in comparison with the usual Selberg zeta-function. Functional relations for Z_0, Z_1 and a relation linking these two functions are derived.

In the fifth Chap. I apply my results to fermionic string theory. This includes first the discussion of the spectra of the Laplace-Dirac operators, and second the calculation of their determinants. It is shown that the relevant determinants which have to be considered in the Polyakov functional integral exist and are regular. Discussions of the superdeterminants are already due to Baranov et al. [7] and Aoki [2]. In ref. [7] ratios of superdeterminants corresponding to different copies of super Fuchsian groups were considered (due to lack of knowledge of the analytic behaviour of super zeta-functions). In ref. [2] attempts have been made to express the superdeterminants by the super zeta-functions, where the functional equation for the usual Selberg zeta-function has been used. These indirect reasonings will be avoided here. Furthermore the behaviour of the superdeterminants of the operators $\square_m$ in the case of degenerate super Riemann surfaces is discussed.

Chapter VI contains a summary and concluding remarks.

II. The Poincaré Super Upper Half-Plane $\mathscr{S}\mathscr{H}$

In this paper **super Riemann surfaces** are considered as special supermanifolds [20, 34, 46, 50, 54] of the DeWitt-type [15, 51]. These supermanifolds $\mathscr{M}$ have a trivial topology in the direction of the soul coordinates, they are fiber bundles over their body $\mathscr{M}_B$. The reason for this property is the fact that an open set in $\mathscr{M}$ is always the cartesian product of an open set of $\mathbf{R}^m$ with the entire space of the soul coordinates. If this restriction is omitted one gets Rogers-supermanifolds which allow a more complicated structure in the soul coordinates [53]. From the point of view of physics only the DeWitt-supermanifolds seem to be of interest [15]; for further details see e.g. Rabin and Crane [51]. An introduction into superanalysis can be found e.g. in the books of Berezin [9] and DeWitt [15].

In the fermionic string theory one is interested in superconformal symmetry. The notion of superspaces and supermanifolds enables one to represent these symmetry transformations as pure "geometrical" transformations in the coordinates $(z,\theta)\in \mathbf{C}_c\times\mathbf{C}_a$ (the indices "a" and "c" denote the set of anticommuting and commuting (complex) numbers, respectively). Let us consider the transformation $(\varepsilon\in\mathbf{C}_a)$:

$$\tilde{z}=z+\theta\varepsilon,\quad \tilde{\theta}=\theta+\varepsilon. \tag{1}$$

Lagrangians are constructed from fields and their derivatives. Therefore one is led to use supersymmetric differential operators. This is nothing but to choose a Vielbein of complex dimension (1, 1), which is invariant under the transformation (1). Now rewrite Eq. (1) in homogeneous coordinates

$$\begin{pmatrix}\tilde{z}\\ \tilde{\theta}\\ 1\end{pmatrix}=\underbrace{\begin{pmatrix}1 & -\varepsilon & 0\\ 0 & 1 & \varepsilon\\ 0 & 0 & 1\end{pmatrix}}_{=1_3+\varepsilon X=\exp(\varepsilon X)}\begin{pmatrix}z\\ \theta\\ 1\end{pmatrix} \tag{2}$$

and realize the infinitesimal generator

$$X=\begin{pmatrix}0 & -1 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\end{pmatrix} \tag{3}$$

as a differential operator on the set of all superanalytic functions $F(\mathscr{M})$ on the supermanifold $\mathscr{M}$:

$$\begin{aligned}&\mathscr{L}_X:\mathscr{F}(\mathbf{C}_c\times\mathbf{C}_a)\to F(\mathbf{C}_c\times\mathbf{C}_a),\quad f(z,\theta)\mapsto\left[\frac{d}{d\varepsilon}f(\exp(\varepsilon X)(z,\theta))\right]_{\varepsilon=0}\\ &\Rightarrow\mathscr{L}_X=-\theta\partial_z+\partial_\theta.\end{aligned} \tag{4}$$

The operator $\mathscr{L}_X$ obviously has odd parity. Invariance of an operator $D\in T_p\mathscr{M}$ is now equivalent to the restriction $[D,\mathscr{L}_X]=0$, where $[A,B]=AB-(-1)^{AB}BA$ is the supercommutator of two operators A and B. For the operator D one chooses

$$D=\theta\partial_z+\partial_\theta. \tag{5}$$

The operator D is something like the square root of ∂_z since $D^2=\partial_z$.

Let us consider a general superanalytic coordinate transformation

$$\tilde{z}=\tilde{z}(z,\theta),\quad \tilde{\theta}=\tilde{\theta}(z,\theta). \tag{6}$$

Then D transforms as follows

$$D=(D\tilde{\theta})\tilde{D}+(D\tilde{z}-\tilde{\theta}D\tilde{\theta})\tilde{D}^2. \tag{7}$$

Now, a superanalytic coordinate transformation is called **superconformal**, if the (0, 1)-dimensional subspace of the tangential space generated by the action of D is invariant under the coordinate transformation, i.e.

$$D=(D\tilde{\theta})\tilde{D}. \tag{8}$$

Thus we have the

Definition. A **Super Riemann Surface** is a complex (1, 1)-dimensional supermanifold, whose coordinate transformations are superconformal mappings.

One introduces homogeneous coordinates and can represent the (1, 1)-dimensional complex projective space as $(\xi, z_1, z_2) \in \mathbf{C}_a \times \mathbf{C}_c^2 \backslash \{0\} \equiv \mathbf{P}_{(1,1)}(\Lambda_\infty)$. The group $SPL(2, \mathbf{C})$ of **superconformal automorphisms** is the natural super-generalisation of the Möbius transformations. Its generators are the operators $L_0, L_1, L_{-1}, G_{1/2}$ and $G_{-1/2}$ of the Neveu–Schwarz section of the Virasoro super algebra of the fermionic string [in ref. [2] $SPL(2, \mathbf{C})$ is denoted as $OSp(2|1, \mathbf{C})$]. On $\mathbf{P}_{(1,1)}(\Lambda_\infty)$ these transformations can be realized as linear transformations, which are superconformal in the local coordinates. Also we have the constraint that $SPL(2, \mathbf{C})_{\mathrm{Body}} = SL(2, \mathbf{C})$. A resonable Ansatz reads:

$$SPL(1,2; \mathbf{C}_a \times \mathbf{C}_c^2)$$

$$:= \left\{ \gamma = \begin{pmatrix} e & \alpha & \beta \\ \delta & a & b \\ \gamma & c & d \end{pmatrix} : a, b, c, d, e \in \mathbf{C}_c; \alpha, \beta, \gamma, \delta \in \mathbf{C}_a; ad - bc = 1; \operatorname{sdet} \gamma = 1 \right\}. \tag{9}$$

Locally the transformation $x' = \gamma x, x, x' \in \mathbf{P}_{(1,1)}(\Lambda_\infty)$ for $\gamma \in SPL(1,2; \mathbf{C}_a \times \mathbf{C}_c^2)$ reads as

$$z' = \frac{\delta\theta + az + b}{\gamma\theta + cz + d} \equiv \frac{A}{B}, \quad \theta' = \frac{e\theta + \alpha z + \beta}{\gamma\theta + cz + d} \equiv \frac{\Gamma}{B}. \tag{10}$$

Superconformal invariance gives the constraint:

$$Dz' = \theta' D\theta' \Rightarrow (DA)B - A(DB) = \Gamma(D\Gamma). \tag{11}$$

Comparison of the coefficients yields $e = 1 + (3/2)\beta\alpha, \gamma = d\alpha - c\beta, \delta = b\alpha - a\beta$. Inserting into (10) gives finally

$$z' = \frac{az + b}{cz + d} + \theta \frac{\alpha z + \beta}{(cz + d)^2}, \quad \theta' = \frac{\alpha z + \beta}{cz + d} + \frac{\theta}{cz + d}\left(1 + \frac{\beta\alpha}{2}\right). \tag{12}$$

Let us define the quantities N_γ and χ_γ by [7, 41, 43]

$$\chi_\gamma [N_\gamma^{1/2} + N_\gamma^{-1/2}] = (a + d)\left(1 - \frac{\alpha\beta}{2}\right) - \alpha\beta = \operatorname{str} \gamma + 1. \tag{13}$$

N_γ is called the **norm** of an hyperbolic $\gamma \in \Gamma$ and χ_γ describes the corresponding spin structure. χ_γ can take on the values ± 1 and has to be chosen as $\chi_\gamma = \operatorname{sign}(a + d)$ or, respectively [43]

$$\chi_\gamma = \begin{cases} 1 & \operatorname{str} \gamma + 1 > 2 \\ -1 & \operatorname{str} \gamma + 1 < -2. \end{cases} \tag{14}$$

N_{γ_0} will denote the norm of a primitive $\gamma_0 \in \Gamma$, where elements $\gamma \in \Gamma$ which are not powers (greater or equal to 2) of any element in Γ are called primitive elements of Γ in analogy to the usual bosonic case. Γ is called a super Fuchsian group, the subgroup $SPL(2, \mathbf{R})$ of $SPL(2, \mathbf{C})$, thus the group of superconformal automorphisms of $\mathcal{SH}$. Its body N_{γ_B} is the corresponding norm of an element $\gamma_B \in PSL(2, \mathbf{R})$, the group of hyperbolic transformation on the Poincaré upper half-plane. In analogy to the classical bosonic case I denote by $l_\gamma = \ln N_\gamma$ the length of a closed geodesic corresponding to a hyperbolic $\gamma \in \Gamma$. Of course, l_{γ_0} is the length corresponding to a primitive γ_0. In the bosonic case these generators are also called boosts, because they correspond to explicit Lorentz transformation on the pseudosphere Λ^2 which is analytical equivalent to the Poincaré upper half-plane $\mathcal{H}$ [27–31]. They obey the important constraint

$$(\gamma_0 \gamma_1^{-1} \cdots \gamma_{2g-2} \gamma_{2g-1}^{-1})(\gamma_0^{-1} \gamma_1 \cdots \gamma_{2g-2}^{-1} \gamma_{2g-1}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{1}_2. \tag{15}$$

For a $SPL(2, \mathbf{R})$ a hyperbolic transformation is always conjugate to the transformation

$$z' = N_\gamma z, \quad \theta' = \chi_\gamma \sqrt{N_\gamma}\, \theta, \tag{16}$$

or in matrix representation:

$$\text{hyperbolic } \gamma \in \Gamma \text{ conjugate to } \begin{pmatrix} 1 & 0 & 0 \\ 0 & \chi_\gamma N_\gamma^{-1/2} & 0 \\ 0 & 0 & \chi_\gamma N_\gamma^{-1/2} \end{pmatrix}. \tag{17}$$

To normalize γ correctly by $\operatorname{sdet} \gamma = 1$ one has to multiply all matrix-entries of Eq. (9) by $K = 1 - \frac{1}{2}\beta\alpha = 1 + \frac{1}{2}\alpha\beta$. Therefore:

$$\gamma = K \begin{pmatrix} 1 + \frac{3}{2}\beta\alpha & \alpha & \beta \\ b\alpha - a\beta & a & b \\ d\alpha - c\beta & c & d \end{pmatrix} = \begin{pmatrix} 1 + \beta\alpha & \alpha & \beta \\ b\alpha - a\beta & a(1 - \frac{1}{2}\beta\alpha) & b(1 - \frac{1}{2}\beta\alpha) \\ d\alpha - c\beta & c(1 - \frac{1}{2}\beta\alpha) & d(1 - \frac{1}{2}\beta\alpha) \end{pmatrix} \tag{18}$$

and the inverse transformation reads:

$$\gamma^{-1} = \begin{pmatrix} 1 + \beta\alpha & c\beta - d\alpha & b\alpha - a\beta \\ -\beta & d(1 - \frac{1}{2}\beta\alpha) & -b(1 - \frac{1}{2}\beta\alpha) \\ -\alpha & -c(1 - \frac{1}{2}\beta\alpha) & a(1 - \frac{1}{2}\beta\alpha) \end{pmatrix}. \tag{19}$$

To formulate **super uniformisation** let us first remember the uniformisation theorem for Riemann surfaces (e.g. [10]):

Theorem. *Every compact Riemann surface is conformally equivalent to $\mathcal{M}/\Gamma$, where $\mathcal{M} = \hat{C}$ (Riemann sphere), $\mathcal{M} = \mathbf{C}$ (for the torus) or $\mathcal{M} = \mathcal{H}$ (upper half-plane) where Γ is a discrete, fix-point free subgroup of the conformal automorphisms of $\mathcal{M}$.*

Since $\hat{C}$, $\mathbf{C}$ and $\mathcal{H}$ are simply connected, and super Riemann surfaces are fiber bundles over their body, there exist generalisations $S\hat{C}$, $S\mathbf{C}$ and $\mathcal{SH}$. The conformal automorphisms of $\mathbf{C}$ and $\mathcal{H}$ are subgroups of $SL(2, \mathbf{C})$. This is not true in general for the superconformal automorphisms of $S\hat{C}$ and $\mathcal{SH}$. But for application in physics we need in general a metric and we can restrict ourselves to "metrizable" super Riemann surfaces. Superconformal automorphisms of a "metrizable" super Riemann surface, which leave the metric invariant, are always subgroups of $SPL(2, \mathbf{C})$.

With the DeWitt definition of open sets, a subgroup $\Gamma \subset SPL(2,\mathbf{R})$ acts discrete and without fix-points, iff $\Gamma_{\mathrm{Red}} \equiv \Gamma_{\mathrm{Body}} \subset SL(2,\mathbf{R})$.

Theorem [51]. *Every "metrizable" super Riemann surface Σ is superconformally equivalent to $\mathscr{M}/\Gamma$ with $\mathscr{M} = S\hat{C}$, $S\mathbf{C}$ or $\mathscr{S}\mathscr{H}$ and Γ is a discrete fix-point free subgroup of the superconformal automorphisms on $\mathscr{M}$.*

The coefficients in Eq. (9) are specified by $a,b,c,d \in \mathbf{R}_c$ and $\alpha, \beta \in \mathbf{C}_a$, $\bar{\alpha} = i\alpha$, $\bar{\beta} = i\beta$. The fundamental group of a compact Riemann surface of genus g can be defined by $2g$ generators satisfying the relation (15). In the super case we have analogously:

$$(\gamma_0\gamma_1^{-1}\cdots\gamma_{2g-2}\gamma_{2g-1}^{-1})(\gamma_0^{-1}\gamma_1\cdots\gamma_{2g-2}^{-1}\gamma_{2g-1}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{20}$$

The real Teichmüller space $\mathscr{T}$ of a compact Riemann surface with genus g has (real) dimension $d_{\mathscr{T}} = 6g - 6$, whereas super Teichmüller space $S\mathscr{T}$, respectively, super Moduli space [44, 46, 58], has dimension $d_{S\mathscr{T}} = (6g - 6, 4g - 4)$.

To construct the metric on $\mathscr{S}\mathscr{H}$ let us consider the Vierbein E^A. The general method for constructing the Vierbein in a curved $2+2$-dimensional super space was given by Howe [37]. Because a $2+2$-dimensional superspace is conformally flat, if there exists a coordinate system in which the metric is proportional to the flat metric, one starts with the Vierbein $\hat{E}_M{}^A$ in flat superspace

$$\hat{E}_M{}^A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -\theta & 0 & 1 & 0 \\ 0 & \bar{\theta} & 0 & 1 \end{pmatrix}, \quad \hat{E}_A{}^M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \theta & 0 & 1 & 0 \\ 0 & -\bar{\theta} & 0 & 1 \end{pmatrix}, \tag{21}$$

where $\hat{E}_A{}^M = (\hat{E}_M{}^A)^{-1}$ is the inverse Vierbein. This gives for the quantities $\hat{E}^A = dz^M \hat{E}_M{}^A$:

$$\begin{aligned} \hat{E}^z &= dz + \theta d\theta, \quad \hat{E}^\theta = d\theta \\ \hat{E}^{\bar{z}} &= d\bar{z} - \bar{\theta} d\bar{\theta}, \quad \hat{E}^{\bar{\theta}} = d\bar{\theta}. \end{aligned} \tag{22}$$

Under a super Weyl transformation the Vierbein $\hat{E}_M{}^A$ changes as

$$\hat{E}_M{}^A \to E_M{}^A = \begin{cases} E_M{}^a = \Lambda(Z)\hat{E}_M{}^a, & (a = z, \bar{z}), \\ E_M{}^\alpha = \Lambda^{1/2}(Z)\hat{E}_M{}^\alpha - i\hat{E}_M{}^a(\gamma_a)^{\alpha\beta} D_\beta \Lambda^{1/2}(Z), & (\alpha = \theta, \bar{\theta}), \end{cases} \tag{23}$$

where $D_\alpha = E_\alpha{}^M \partial_M$, $\Lambda(Z)$ the scaling function and (γ_a) the γ-matrices which in my notation read (raising and lowering of spin-indices are performed by the totally antisymmetric $\varepsilon_{\alpha\beta}$-tensor):

$$(\gamma_z)^{\alpha\beta} = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}, \quad (\gamma_{\bar{z}})^{\alpha\beta} = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}. \tag{24}$$

Since the Vierbein E^A should be (up to phase factors – see eg. [28]) invariant under the action of $SPL(2,\mathbf{R})$ the appropriate scaling function reads $\Lambda(Z) = Y^{-1}$, where Y is given by $Y := \operatorname{Im} z + \frac{\theta\bar{\theta}}{2} = y + i\theta_1\theta_2$, if I further set $\theta = \theta_1 + i\theta_2$ and $\bar{\theta} = \theta_1 - i\theta_2$, where θ_1 and θ_2 are real Grassmannians. Note: $\bar{Y}=Y$. The E^A are now given as (see [51, 59]):

$$E^z=\frac{dz+\theta d\theta}{Y},\quad E^\theta=\frac{d\theta}{Y^{1/2}}+\frac{i\theta-\bar\theta}{2Y^{3/2}}(dz+\theta d\theta),$$
$$E^{\bar z}=\frac{d\bar z+\bar\theta d\bar\theta}{Y},\quad E^{\bar\theta}=\frac{d\bar\theta}{Y^{1/2}}-\frac{i\theta+\bar\theta}{2Y^{3/2}}(d\bar z-\bar\theta d\bar\theta). \tag{25}$$

The $SPL(2,\mathbf{R})$ invariant line element can be constructed by [59]:

$$ds^2=E^zE^{\bar z}-2E^\theta E^{\bar\theta}=\frac{1}{Y^2}[d\bar z dz-i\bar\theta d\bar z d\bar\theta-i\theta d\bar\theta dz-(2Y+\theta\bar\theta)d\theta d\bar\theta]. \tag{26}$$

Rewriting $ds^2=dq^a{}_ag_bdq^b$ we obtain the metric tensor on $\mathscr{S}\mathscr{H}$

$$({}_ag_b)=\frac{1}{2Y^2}\begin{pmatrix}0&1&0&-i\theta\\1&0&-i\bar\theta&0\\0&i\bar\theta&0&-(2Y+\theta\bar\theta)\\i\theta&0&2Y+\theta\bar\theta&0\end{pmatrix}. \tag{27}$$

and its superdeterminant reads: $\mathrm{sdet}({}_ag_b)=-\dfrac{1}{4Y^2}$. Let us contruct the $SPL(2,\mathbf{R})$ invariant volume element on $\mathscr{S}\mathscr{H}$ as

$$dV(Z)=\sqrt{|\mathrm{sdet}({}_ag_b)|}\,dzd\bar z d\theta d\bar\theta=\frac{dzd\bar z d\theta d\bar\theta}{2Y}. \tag{28}$$

Note the difference in the power of Y to the $PSL(2,\mathbf{R})$ invariant volume element on $\mathscr{H}$: $dV(z)=dxdy/y^2$. The **super hyperbolic distance** between two points $q^{(1)}$ and $q^{(2)}$ on $\mathscr{S}\mathscr{H}$ is defined as [41, 60]:

$$d(q^{(1)},q^{(2)})=\int_{q^{(1)}}^{q^{(2)}}ds=\int_{t_1}^{t_2}\sqrt{\left(\frac{ds}{dt}\right)^2}dt=\omega(t_2-t_1), \tag{29}$$

where ω is a phase factor with $|\omega|=1$ [59]. This can be rewritten as

$$\cosh d(q^{(1)},q^{(2)})=1+\tfrac{1}{2}R(q^{(1)},q^{(2)})-2r(q^{(1)},q^{(2)}), \tag{30}$$

where [set $q^{(1)}\equiv Z, q^{(2)}\equiv W=(u+iv,\nu_1+\nu_2)$, $V=v+\nu\bar\nu/2$]:

$$R(Z,W)=\frac{|z-\omega-\theta\nu|^2}{YV}=\frac{|z-w-\theta\nu|^2}{(y+\theta\bar\theta/2)(v+\nu\bar\nu/2)}, \tag{31}$$

$$r(Z,W)=\frac{2\theta\bar\theta+i(\nu-i\bar\nu)(\theta+i\bar\theta)}{4Y}+\frac{2\nu\bar\nu+i(\theta-i\bar\theta)(\nu+i\bar\nu)}{4V}$$
$$+\frac{(\nu+i\bar\nu)(\theta+i\bar\theta)\,\mathrm{Re}\,(z-w-\theta\nu)}{4YV}. \tag{32}$$

All these two-point quantities are $SPL(2,\mathbf{R})$ invariant. Following ref. [59] the super Laplae–Beltrami operator can be constructed as

$$\Delta_{\mathrm{SLB}}=(-1)^a g^{-1/4}p_a g^{1/2\,a}g^b{}_b p g^{-1/4}. \tag{33}$$

This super Laplacian is a straightforward generalisation of the classical bosonic one and is in general the simplest one which is invariant under general point canonical transformations (see [30, 47] for a discussion of the classical bosonic case). The quantum Hamiltonian on a super Riemann manifold is then given by

$$H = -\frac{1}{2m}\Delta_{\mathrm{SLB}} = -\frac{1}{2m}[{}^a g^b \partial_b \partial_a + (-1)^a({\Gamma_a}^a g^b + ({\partial_a}^a g^b))\partial_b]. \tag{34}$$

Thus for the Hamiltonian on the Poincaré super upper half-plane:

$$H_{\mathscr{S}\mathscr{H}} = -\frac{1}{2m}{}^a g^b \partial_b \partial_a = -\frac{Y}{m}\left[(2Y - \theta\bar{\theta})\partial_z \partial_{\bar{z}} + i\bar{\theta}\partial_z \partial_{\bar{\theta}} - i\theta\partial_z \partial - \theta\bar{\theta}\partial_\theta \partial_{\bar{\theta}}\right]. \tag{35}$$

The Hamiltonian $H_{\mathscr{S}\mathscr{H}}$ can be factorized. Let us define

$$\begin{aligned} \square &= 2YD\bar{D} = 2Y(\partial_{\bar{\theta}}\partial_\theta + \theta\bar{\theta}\partial_z\partial_{\bar{z}} - \theta\partial_{\bar{\theta}}\partial_z - \bar{\theta}\partial_\theta\partial_z), \\ D &= \partial_\theta + \theta\partial_z, \quad \bar{D} = -\partial_{\bar{\theta}} + \bar{\theta}\partial_{\bar{z}} \end{aligned} \tag{36}$$

[i.e. $D = D_\theta$, $\bar{D} = D_{\bar{\theta}}$, in the Vierbein notation of Eq. (21)]. It can be easily shown that the important relation holds:

$$\Delta_{\mathscr{S}\mathscr{H}} = \square^2. \tag{37}$$

Generally I will refer to the operator $\square$ as the **Laplace-Dirac operator** on $\mathscr{S}\mathscr{H}$. With the invariant volume element on $\mathscr{S}\mathscr{H}$, $\Delta_{\mathscr{S}\mathscr{H}}$ and $\square$ are hermitian with respect to the scalar product

$$(\Phi_1, \Phi_2) = \int dV(Z)\Phi_1\bar{\Phi}_2. \tag{38}$$

The operator $\square$ is the zero-case of the more general operator $\square_m$ which is defined by (I use a slightly different notation as in Baranov et al. [7, 8] and Aoki [2]; in refs. [2, 7] a description is given, how such operators can be constructed in a systematic approach):

$$\square_m = 2YD\bar{D} + m(i\theta - \bar{\theta})\bar{D}. \tag{39}$$

This is the important operator for the fermionic string. In ref. [7] also the operator $\hat{\square}_m$ is introduced which is constructed by a linear isomorphism

$$\square_m = Y^{m/2}\left(\hat{\square}_m + \frac{m}{2}\right)Y^{-m/2}. \tag{40}$$

Hence we have an unitary equivalence of $\square_m$ and $\hat{\square}_m + \frac{m}{2}$. Explicitly $\hat{\square}_m$ reads:

$$\hat{\square}_m = 2YD\bar{D} + \frac{m}{2}(i\theta - \bar{\theta})(\bar{D} + iD). \tag{41}$$

I denote this unitary equivalence by $\square_m \cong \hat{\square} + \frac{m}{2}$. Let us consider a differentiable superfunction on $\mathscr{S}\mathscr{H}$

$$\Phi(z, \bar{z}, \theta, \bar{\theta}) = A(z, \bar{z}) + \frac{1}{\sqrt{y}}[\theta\chi(z, \bar{z}) + \bar{\theta}\tilde{\chi}(z, \bar{z})] + \frac{1}{y}\theta\bar{\theta}B(z, \bar{z}). \tag{42}$$

With the notation $-\Delta_m = -4y^2\partial_z\partial_{\bar z} + imy\partial_x = -y^2(\partial_x^2+\partial_y^2) + imy\partial_x$ we obtain the following equivalence relation [7]:

$$\hat{\Box}_m \Phi = s\Phi \Leftrightarrow \begin{cases} -\Delta_{-m}A = s(1-s)A, \quad B = \frac{s}{2}A \\ -\Delta_{-m-1}\chi = (\frac{1}{4}-s^2)\chi, \quad -\Delta_{-m+1}\tilde\chi = (\frac{1}{4}-s^2)\tilde\chi, \\ \tilde\chi\left(s-\frac{m}{2}\right) = -2y\partial_{\bar z}\chi - \frac{i}{2}(m-1)\chi, \end{cases} \tag{43}$$

where s is an even supernumber. Thus, the solution of the Eigenvalue problem is formally the same as the classical bosonic one. However, the periodic boundary conditions [for e.g. $m=0$: $\Psi(\gamma Z) = \Psi(z)$, $\gamma \in SPL(2,\mathbf{R})$] must be interpreted in the super language. By taking the body in all quantities, one recovers, of course, the bosonic problem. The equivalence relation legitimates to set $s = \frac{1}{2} + ip$ ($p \in \mathbf{R}$, so-called "small" Eigenvalues neglected). This reproduces the positivity of the operator $-\Delta_m$.

Path integral treatments for the free motion on the entire upper half-plane can be found in ref. [27, 29] for $m=0$ and in ref. [26] for $m \neq 0$. As is easily checked, $\Phi_1 = Y^s$ and $\Phi_2 = (\theta_1+\theta_2)y^{-s}$ satisfy (43), i.e. Φ_1 and Φ_2 are an even and an odd solution of the Laplace-Dirac operator $\hat{\Box}_m$, respectively, with Eigenvalue s:

$$\hat{\Box}_m \Phi_i = s\Phi_i, \quad (i=1,2). \tag{44}$$

Starting from Eq. (43) it is straightforward to calculate the even and odd Eigenfunctions of $\hat{\Box}_m$. In ref. [43] this has been done for the Laplacians $\Box_0$ with result

$$\Phi_{p,k}^{\text{even}}(x,y,\theta,\bar\theta) = \sqrt{\frac{2ip\sinh\pi p}{\pi^3}}\left(1+\frac{1+2ip}{4y}\theta\bar\theta\right)e^{ikx}\sqrt{y}K_{ip}(|k|), \tag{45}$$

$$\left.\begin{aligned} \Phi_{p,k}^{\text{odd}}(x,y,\theta,\bar\theta) &= \frac{1}{\sqrt{y}}[\theta\phi_{p,k}(x,y) + \bar\theta\tilde\phi_{p,k}(x,y)], \\ \phi_{p,k}(x,y) &= \sqrt{\frac{\cos[\pi(c+ip)]}{2\pi^2 k(c+ip)^{\sigma_k-1}}}\, e^{ikx}W_{\sigma_k/2,s}(2|k|y), \\ \tilde\phi_{p,k}(x,y) &= i(c+ip)^{\sigma_k}\sqrt{\frac{\cos[\pi(c+ip)]}{2\pi^2 k(c+ip)^{\sigma_k-1}}}\, e^{ikx}W_{-\sigma_k/2,s}(2|k|y) \end{aligned}\right\} \tag{46}$$

with $k \in \mathbf{R}$, $p > 0$, $-\frac{1}{2} \leqq c \leqq \frac{1}{2}$, $\sigma_k = \operatorname{sign}(k)$ and a point $(p,k) = (0,0)$ is understood as excluded. K_ν is a modified Bessel function and $W_{\nu,\mu}$ a Whittaker function. The functions Φ are orthonormal in the sense of the scalar product (38) [43]:

$$(\Phi_{p,k}, \Phi_{q,l}) = \delta(k-l)\delta(p+q). \tag{47}$$

The heat kernel of the Laplacians $\Box_m$ and $\Box_m^2$, respectively, has been calculated by Aoki [2] and can be constructed with the help of the heat kernel of the operator

Δ_m on the Poincaré upper half-plane. However, as has been pointed out by Oshima [48], in ref. [2] the term corresponding to the discrete spectrum is missing. An investigation of the heat-kernel of $\square_m$ and $\square_m^2$, respectively, in fact shows that in addition to the continuous spectrum, there is also a discrete spectrum with Eigenvalues $s=\frac{m}{2}-l$ and $s=l+1-\frac{m}{2}$ $\left(l=0,1,\ldots,N_m<\frac{m-1}{2}\right)$ for the corresponding even and odd Eigenfunction, respectively. This is, of course, due to the spectrum of Δ_m on $\mathscr{H}$ [13,26].

The scalar-product (38) does not form in general a Hilbert space inner product in the sense that it is not positive definite – quite puzzling in view of DeWitt's book [15]. The asymptotic behavior of the heat-kernel (see Chap. V and [2]) suggests that one has to give up either positivity or diagonalizability of self-adjoint operators (or both), to have a super reparametrization invariant notion of super Hilbert space [3].

Let us consider the partition function for the fermionic string. The relevant action to be used in the Polyakov approach [12,14,37] reads:

$$S(g,X,\chi,\psi)=\frac{1}{4}\int_M d^2\sigma\sqrt{g}\left[\tfrac{1}{2}g^{mn}\partial_m X^\mu\partial_n X^\mu+i\bar{\psi}^\mu\gamma^m\partial_m\psi_\mu\right.$$
$$\left.-F^\mu F_\mu-\bar{\chi}_a\gamma^m\gamma^a\psi^\mu\partial_m X_\mu+\tfrac{1}{8}\bar{\psi}^\mu\psi_\mu\bar{\chi}_a\gamma^a\gamma^b\chi_b\right]. \tag{48}$$

Here denote:
1. M: the two-dimensional world sheet,
2. $g_{mn}=e_m{}^a e_n{}^b\delta_{ab}$: metric on the world sheet,
3. ψ^μ: real (Majorana-)spinor,
4. χ_a: spin $\frac{3}{2}$-gravitino field,
5. F^μ: nondynamical field which is needed to close the supersymmetric algebra off shell [37,55]:

$$\delta X=i\bar{\varepsilon}\psi,\quad \delta\Psi=\partial_a X\gamma^a\varepsilon+F\varepsilon,\quad \delta F=i\bar{\varepsilon}\gamma^a\partial_a\psi, \tag{49}$$

where ε is a two-dimensional spinor. One sets $F=0$, since the equation of motion just reads $F=0$.
6. γ^a $(a=0,1,5)$ denote the γ-matrices

$$\gamma^0=\begin{pmatrix}0&1\\-1&0\end{pmatrix},\quad \gamma^1=\begin{pmatrix}0&1\\1&0\end{pmatrix},\quad \gamma^5=\begin{pmatrix}1&0\\0&-1\end{pmatrix}, \tag{50}$$

and a bar over quantities denotes complex conjugation.

The action (48) is invariant under five fundamental symmetries [18]
i) Reparametrization invariance,

$$\begin{aligned}\delta e_m{}^a&=\delta V^n\partial_n e_m{}^a+e_n{}^a\partial_m\delta V^n,\\ \delta\chi_m&=\delta V^n\partial_n\chi_m+\chi_n\partial_m\delta V^n,\\ \delta X^\mu&=\delta V^n\partial_n X^\mu,\\ \delta\Psi^\mu&=\delta V^n\partial_n\Psi^\mu.\end{aligned} \tag{51}$$

ii) Supersymmetry transformations,

$$\delta e_m{}^a = i\xi\gamma^a\chi_m, \quad \delta\chi_m = 2D_m\xi, \quad \delta X^\mu = \xi\Psi^\mu,$$
$$\delta\Psi^\mu_\alpha = -\tfrac{1}{2}i(\gamma^n)_\alpha\beta\xi_\beta(\chi_n\Psi^\mu) + i(\gamma^m)_\alpha{}^\beta\xi_\beta\partial_m X^\mu. \tag{52}$$

Here $D_m = \partial_m - \frac{1}{2}\omega_m\gamma^5$ are covariant derivatives taken with the connection $\omega_m = e_m{}^a\varepsilon^{pq}\partial_p e_q{}^b\delta_{ab} - \frac{i}{2}\chi_m\gamma_5\gamma^n\chi_n$, where ε_{ab} is the totally antisymmetric tensor for the raising and lowering of spinor indices.

iii) Weyl transformations,

$$\delta e_m{}^a = \Lambda e_m{}^a, \quad \delta\Psi^\mu = -\tfrac{1}{2}\Lambda\Psi^\mu,$$
$$\delta\chi_m n = \tfrac{1}{2}\Lambda\chi_m, \quad \delta X^\mu = 0. \tag{53}$$

iv) super-Weyl transformations

$$\delta\chi_m = \gamma_m\lambda, \quad \delta(\text{anything else}) = 0. \tag{54}$$

v) Local Lorentz transformations,

$$\delta e_m{}^a = l\varepsilon^{ab}e_{mb}, \quad \delta\Psi^\mu = \tfrac{1}{2}l\gamma_5\Psi^\mu,$$
$$\delta\chi_m = \tfrac{1}{2}l\gamma_5\chi_m, \quad \delta X^\mu = 0. \tag{55}$$

(With δV^n an infinitesimal vector field, ξ an infinitesimal spinor, Λ and λ an infinitesimal scaling-function and l an infinitesimal Lorentz transformation, respectively.)

It is very important that the action (48) can be cast into a compact form if a superspace notion is used [37]. Define

$$\Phi(Z) = X(x) + i\theta^\dagger\gamma^0\psi(x) + \frac{i}{2}\theta^\dagger\gamma^0\theta F(x). \tag{56}$$

Then the action (48) can be rewritten as

$$S = \tfrac{1}{4}\int dx^1 dx^2 d\theta_1 d\theta_2 eL, \quad L = E_\alpha{}^M\partial_m\Phi(Z)E^{aN}\partial_N\Phi(Z). \tag{57}$$

The equations of motion read

$$D_\alpha\bar{D}^\alpha\Phi(Z) = 0, \tag{58}$$

where $D_\alpha = E_\alpha{}^M\partial_m$. Mapping (in the sense that we want to study partition functions) a closed compact world sheet into a fundamental domain (of a super Fuchsian group) on the Poincaré super upper half-plane $\mathscr{SH}$ we get the Laplace-Dirac operator $\square_0 = 2YD\bar{D}$ which we have to study. The partition function is then calculated as follows:

$$Z = \sum_{g=0}^{\infty} Z_g, \quad Z_g = \int\mathscr{D}g_{ab}\int\mathscr{D}\chi_a\int\mathscr{D}X^\mu\int\mathscr{D}\psi^\mu e^{-S(g,X,\chi,\psi)}. \tag{59}$$

$\mathscr{D}X^\mu$ denotes the imbeddings in space-time ($d = 10$). Analogous considerations as in the bosonic case yield [6–8]:

$$Z_g = \int_{S\mathscr{M}_g} d\mu_{\mathrm{SWP}}[\mathrm{sdet}'(-\square_0^2)]^{-5/2}[\mathrm{sdet}(-\square_2^2)]^{1/2}, \tag{60}$$

where (in the notation I am using) $\mathscr{S}\mathscr{M}_g$ is the super moduli space [44, 58] and $d\mu_{\mathrm{SWP}}$ the super Weil-Peterson measure. The factor $[\mathrm{sdet}(-\square_2^2)]^{1/2}$ is the contribution from the Faddev–Poppov ghosts determinant.

III. The Selberg Supertrace Formula

Let us consider the $SPL(2,\mathbf{R})$ transformation as given in Chap. II,

$$z' = \frac{\delta\theta + az + b}{\gamma\theta + cz + d} \equiv \frac{A}{B}, \quad \theta' = \frac{e\theta + \alpha z + \beta}{\gamma\theta + cz + d} \equiv \frac{\Gamma}{B}, \tag{1}$$

where $A(Z) = az + b - \theta\delta$, $B(Z) = cz + d - \theta\gamma$ and $\Gamma(Z) = \alpha z + \beta + e\theta$ with $e = 1 + \frac{3}{2}\beta\alpha$, $\gamma = d\alpha - c\beta$ and $\delta = b\alpha - a\beta$. The numbers a, b, c, d satisfy the relation $ad - bc = 1$ and are real even supernumbers. The numbers α and β are odd supernumbers with the property $\bar{\alpha} = i\alpha$, $\bar{\beta} = i\beta$. I also use the notation $Z' = (z', \theta') = \gamma(z, \theta) = \gamma Z$. A, B and Γ must be multiplied by $K = 1 + \frac{1}{2}\alpha\beta$ to give the correct normalization $\mathrm{sdet}\, T = 1$. I denote these quantities by $\hat{A}, \hat{B}$ and $\hat{\Gamma}$, respectively.

Let us introduce some important notions:

Definition 1. Let $\Gamma \subset SPL(2,\mathbf{R})$ be a **discrete subgroup** and $U \subset \mathscr{S}\mathscr{H}$ a **fundamental domain** of Γ which tesselates $\mathscr{S}\mathscr{H}$.

Definition 2. Let $\gamma \in \Gamma$. I call a function $f(Z)(Z \in \mathscr{S}\mathscr{H})$ a **super automorphic function** of weight m iff it is satisfying the relation $f(\gamma Z) = j_\gamma^m(Z) f(Z)$, where j_γ^m is given by

$$j_\gamma^m := F_\gamma^m |F_\gamma|^{-m} = \left(\frac{F_\gamma}{\bar{F}_\gamma}\right)^{m/2}, \quad F_\gamma := D\theta' = \frac{1}{\hat{B}(Z)}, \quad (z', \theta') = \gamma(z, \theta). \tag{2}$$

The task is to construct the relevant operator for the super trace formula which maps super automorphic functions into super automorphic functions.

Definition 3. Let us consider the integral operator L

$$L\phi(Z) = \int_{\mathscr{S}\mathscr{H}} dV(W) k_m(Z, W) \phi(W). \tag{3}$$

We call L the **Selberg super integral operator** on $\mathscr{S}\mathscr{H}$, where $k_m(Z, W)$ is the integral-kernel of an operator valued function of the operator $\square_m$.

We introduce the functions $\Phi(x)$ and $\Psi(x)$ sufficiently decreasing at ∞. Since the Laplace-Dirac operator $\square$ is a $SPL(2,\mathbf{R})$ invariant operator its integral kernel (and the integral kernel of functions of $\square$) must depend on $SPL(2,\mathbf{R})$ invariant quantities. Therefore one makes the Ansatz

$$k(Z, W) = k_0(Z, W) = \Phi[R(Z, W)] - r(Z, W)\Psi[R(Z, W)]. \tag{4}$$

The invariants $r(Z, W)$ and $R(Z, W)$ have been defined in Eqs. (II.31, 32). For the heat-kernel of the operator $\square_m^2$ the integral kernel k_{heat} can be explicitly calculated [2, 43] and has the form of Eq. (4). It consists, of course, of two contributions, coming from the discrete and continuous part of the spectrum of $\square_m$, respectively. Let $m \in \mathbf{N}_0$. k_m is now defined by

$$k_m(Z, W) = J^m(Z, W) k(Z, W), \tag{5}$$

where

$$J^m(Z,W)=\left(\frac{z-\bar{w}+i\theta\bar{v}}{\bar{z}-w+i\bar{\theta}v}\right)^{m/2},\quad (Z,W\in\mathscr{SH}). \tag{6}$$

J^m has for $J^m(Z,W)\to J^m(\gamma Z,\gamma W)$ the transformation property:

$$J^m(\gamma Z,\gamma W)=\left(\frac{z'-\bar{w}'+i\theta'\bar{v}'}{\bar{z}'-w'+i\bar{\theta}'v'}\right)^{m/2}=j_\gamma^m(Z)J^m(Z,W)j_\gamma^{-m}(W). \tag{7}$$

The Selberg super integral operator is then given by

$$Lf(z)=\int_{\mathscr{SH}} dV(Z)J^m(Z,W)\{\Phi[R(Z,W)]-r(Z,W)\Psi[R(Z,W)]\}f(W) \tag{8}$$

and maps super automorphic functions into super automorphic functions.

Let f be a super automorphic function and $g=Lf$. Then:

$$\begin{aligned}g(Z)&=\int_{\mathscr{SH}} dV(W)k_m(Z,W)f(W)=\sum_{\{\gamma\}_p}\int_{\gamma U} dV(W)k_m(Z,W)f(W)\\&=\sum_{\{\gamma\}_p}\int_U dV(W)k_m(Z,\gamma W)f(\gamma W)=\sum_{\{\gamma\}_p}\int_U dV(W)k_m(Z,\gamma W)j_\gamma^m(W)f(W)\\&=\int_U dV(W)K(Z,W)f(W),\end{aligned} \tag{9}$$

where U denotes a fundamental domain of the super Fuchsian group and

Definition 4.

$$K(Z,W)\equiv\sum_{\{\gamma\}_p}k_m(Z,\gamma W)j_\gamma^m(W) \tag{10}$$

is the **super automorphic kernel**.

Let us consider the supertrace of L. L represents an integral operator of an operator valued function h of the Dirac operator $\square_m$, i.e. $L\cong h(\square_m)$. On the one hand we have

$$\operatorname{str}(L)=\operatorname{str}[h(\square_m)]=\sum_{n=0}^{\infty}[h(\lambda_{n,m}^B)-h(\lambda_{n,m}^F)]. \tag{11}$$

$\lambda_{n,m}^{B(F)}$ denote Bose- and Fermi Eigenvalues of $\square_m$, respectively.

On the other, we have for the transformation $W=\gamma Z=(N_\gamma z,\chi_\gamma\sqrt{N_\gamma}\theta)$:

$$j_\gamma^m=\frac{F_\gamma^m}{|F_\gamma|^m}=\left(\frac{D\chi_\gamma\sqrt{N_\gamma}\theta}{|D\chi_\gamma\sqrt{N_\gamma}\theta|}\right)^m=\chi_\gamma{}^m, \tag{12}$$

and therefore we obtain for $\operatorname{str}(L)$:

$$\operatorname{str}(L)=\int_U dV(Z)K(Z,Z)=\sum_{\{\gamma\}_p}\int_U dV(Z)k_m(Z,\gamma Z)j_\gamma^m(Z)\equiv\sum_{\{\gamma\}_p}\chi_\gamma^m A(\gamma), \tag{13}$$

where $A(\gamma)$ is given by

$$\begin{aligned}A(\gamma)&=\int_U dV(Z)k_m(Z,\gamma Z)=\int_U\frac{dx\,dy\,d\theta\,d\bar{\theta}}{2Y}J^m(Z,\gamma Z)[\Phi(R)-r\Psi(R)]\\&=\frac{1}{2}\int_1^{N_0}dy\int_{-\infty}^{\infty}dx\int\frac{d\theta\,d\bar{\theta}}{y+\frac{\theta\bar{\theta}}{2}}J^m(Z,\gamma Z)[\Phi(R)-r\Psi(R)].\end{aligned} \tag{14}$$

Immediately one can state the term corresponding to the identity transformation $[J^m(Z,Z)=(-1)^{m/2}=i^m]$

$$A_0^{(m)}\equiv A(I)=\frac{i^m}{2}\int_1^{N_{y0}}dy\int_{-\infty}^{\infty}dx\int\frac{d\theta d\bar\theta}{Y}\Phi(0)=i^m\pi(g-1)\Phi(0), \tag{15}$$

since $\operatorname{vol}(RS_g)=4\pi(g-1)$, where RS_g denotes a Riemann surface of genus g. For the explicit evaluation of (14) we need the following

Theorem. *Let L be the super Selberg operator and ϕ any Eigenfunction of $\hat\Box_m$ in $\mathcal{SH}$ with $\hat\Box_m\phi=s\phi$. Then*

$$\int_{\mathcal{SH}}dV(Z)k_m(W,Z)\phi(Z)=h(s)\phi(W), \tag{16}$$

where the superfunction h depends only on s and the kernel k. The value of $h(s)$ is thus independent of the function ϕ.

Proof. In [7] the proof of this theorem is given for all functions ϕ (even and odd), all $m\in\mathbf{Z}$ and all $W\in\mathcal{SH}$. Since we need the theorem only at a specific value, i.e. $W=Z_0=(i,0)$, I restrict myself to that relatively easy case, which has also the advantage that the case of odd functions drops out. Thus for $W=Z_0=(i,0)$:

$$J^m(i,Z)=\left(\frac{x-i(y+1)}{x+i(y+1)}\right)^{m/2},\qquad Y^{s-1}=y^{s-1}+\frac{s-1}{2}y^{s-2}\theta\bar\theta,$$
$$R(i,Z)=\frac{x^2+(y-1)^2}{y}\left(1-\frac{\theta\bar\theta}{2y}\right),\quad r(i,Z)=\frac{\theta\bar\theta}{2y}. \tag{17}$$

Let ϕ an even superfunction as in Eq. (II.42) without linear in $\theta,\bar\theta$-terms, i.e. ϕ is of the form $\phi=A(x,y)\left(1+\frac{s}{2y}\theta\bar\theta\right)$. Insertion yields:

$$\begin{aligned}
L\phi(Z_0)&=\int dV(Z)k_m(Z_0,Z)\phi(Z)=\frac{1}{2}\int_0^\infty\frac{dy}{y}\int_{-\infty}^{\infty}dx\left(\frac{x-i(y+1)}{x+i(y+1)}\right)^{m/2}A(x,y)\\
&\quad\cdot\int d\theta d\bar\theta\left[\Phi(R_i)+\frac{s-1}{2y}\theta\bar\theta\Phi(R_i)-\frac{\theta\bar\theta}{2y}R_i\Phi'(R_i)-\frac{\theta\bar\theta}{2y}\Psi(R_i)\right]\\
&=\frac{1}{4}\int_0^\infty\frac{dy}{y^2}\int_{-\infty}^{\infty}dx\left(\frac{x-i(y+1)}{x+i(y+1)}\right)^{m/2}A(x,y)\underbrace{[-(s-1)\Phi(R_i)+R_i\Phi'(R_i)+\Psi(R_i)]}_{\equiv\tilde\Phi(R_i)}\\
&=\frac{1}{4}\int_0^\infty\frac{dy}{y^2}\int_{-\infty}^{\infty}dx\left(\frac{x-i(y+1)}{x+i(y+1)}\right)^{m/2}A(x,y)\tilde\Phi(R_i).
\end{aligned}\tag{18}$$

Since $\tilde\Phi$ depends only on $R_0(z,i)$, where $R_0(z,w)=\frac{|z-w|^2}{yv}$ is an $SL(2,\mathbf{R})$ invariant quantity, the last equation can be interpreted in terms of the Selberg trace formula for automorphic forms of weight m [36] with integral kernel $\tilde\Phi$. Now, an operator $\tilde L$ on the Poincaré upper half-plane whose kernel depends only on R_0 is in fact a function of the Laplace operator Δ_m. It follows that $\tilde L$ multiplies ϕ by

$$h(s):=\int_0^\infty y^{s-3/2}\tilde Q(y+y^{-1}-2)dy,\quad\text{where}\quad\tilde Q(y)=\int_0^\infty\tilde\Phi(x^2+y)dx. \tag{19}$$

This completes the proof. ■

Let us turn to the calculation of $A(\gamma)$. The invariants r and R are given with the hyperbolic transformation $W=\gamma Z=(N_\gamma z, \chi_\gamma\sqrt{N_\gamma}\theta)$ as:

$$R(Z,\gamma Z)=\frac{(N_\gamma-1)^2(x^2+y^2)}{N_\gamma y^2}\left(1-\frac{\theta\bar\theta}{2}\right)\equiv R_0\left(1-\frac{\theta\bar\theta}{2}\right),$$

$$r(Z,\gamma Z)=(2-\chi_\gamma N_\gamma^{1/2}-\chi_\gamma N_\gamma^{-1/2})\frac{\theta\bar\theta}{2y}. \tag{20}$$

For the J^m term:

$$J^m(Z,\gamma Z)=\left(\frac{z-N_\gamma\bar z+i\chi_\gamma\sqrt{N_\gamma}\theta\bar\theta}{\bar z-N_\gamma z+i\chi_\gamma\sqrt{N_\gamma}\theta\bar\theta}\right)^{m/2}$$

$$=\left(\frac{\zeta-2i\cosh\frac{u}{2}}{\zeta+2i\cosh\frac{u}{2}}\right)^{m/2}\left(1-\frac{i\theta\bar\theta}{y}\frac{m\zeta\chi_\gamma}{\zeta^2+4\cosh^2\frac{u}{2}}\right), \tag{21}$$

where use has been made of the substitutions: $\zeta=\sqrt{\frac{N_\gamma-1}{N_\gamma}}\frac{y}{x}, u=\ln N_\gamma$. Y^s is an Eigenfunction of $\hat\Box_m$ with Eigenvalue $\lambda=s$. Setting $W=Z_0(i,0)$ the theorem gives therefore the multiplication by the function $h(s)$:

$$h(s)=\frac{1}{2}\int_0^\infty dy\int_{-\infty}^\infty dx\int\frac{d\theta d\bar\theta}{Y}J^m(i,Z)\{\Phi[R(i,Z)]-r(i,Z)\Psi[R(i,Z)]\}Y^s$$

$$=\frac{1}{4}\int_0^\infty dy\int_{-\infty}^\infty dx\left(\frac{x-i(y+1)}{x+i(y+1)}\right)^{m/2}y^{s-2}\left[-(s-1)\Phi\left(\frac{x^2+(y-1)^2}{y}\right)\right.$$

$$\left.+\frac{x^2+(y-1)^2}{y}\Phi'\left(\frac{x^2+(y-1)^2}{y}\right)+\Psi\left(\frac{x^2+(y-1)^2}{y}\right)\right], \tag{22}$$

where the $(m/2)^{th}$ power is to be a principle value (see [36], p. 454). Now performing in the y-integral a partial integration for $\mathrm{Re}(s)>1$:

$$-(s-1)\int_0^\infty dy\,y^{s-2}\left(\frac{x-i(y+1)}{x+i(y+1)}\right)^{m/2}\Phi\left[\frac{x^2+(y-1)^2}{y}\right]$$

$$=\int_0^\infty dy\,y^{s-1}\left(\frac{x-i(y+1)}{x+i(y+1)}\right)^{m/2}$$

$$\cdot\left[\Phi'\left(\frac{x^2+(y-1)^2}{y}\right)\frac{x^2+(y-1)^2}{y}-\frac{imx}{x^2+(y+1)^2}\Phi\left(\frac{x^2+(y-1)^2}{y}\right)\right]. \tag{23}$$

Therefore for $h(s)$:

$$h(s)=\frac{1}{8}\int_{-\infty}^\infty du\,e^{u(s-1/2)}\int_{4\sinh^2 u/2}^\infty\frac{dx}{(x+4)^{m/2}}$$

$$\cdot\left[\frac{\Psi(x)+2(e^u-1)\Phi'(x)}{\sqrt{x-4\sinh^2\frac{u}{2}}}[\alpha^m(\xi(x),u)+\alpha^m(-\xi(x),u)]\right.$$
$$\left.-ime^{u/2}\Phi(x)\frac{\alpha^m(\xi(x),u)-\alpha^m(-\xi(x),u)}{x+4}\right]. \qquad (24)$$

In the calculations the substitutions $x=\sqrt{y}\xi$ and $y=e^u$ have been made, followed by $x=\xi^2+4\sinh^2\frac{u}{2}$. Further the abbreviation $\alpha^m(\xi,u)=\left(\xi-2i\cosh\frac{u}{2}\right)^m$ has been used. Thus we see that for appropriate h the operator $h(\hat{\Box}_m)$ equals to an integral operator L of the form (3) whose integral kernel $k_m(Z,W)$ is related to h by the equations

$$h(s)=\int_{-\infty}^{\infty} du e^{u(s-1/2)}g(u), \quad (s=\tfrac{1}{2}+ip),$$

$$g(u)=\frac{1}{2\pi}\int_{-\infty}^{\infty} dp e^{-iup}h(\tfrac{1}{2}+ip), \qquad (25)$$

$$g(u)=\frac{1}{8}\int_{4\sinh^2 u/2}^{\infty}\frac{dx}{(x+4)^{m/2}}\left[\frac{\Psi(x)+2(e^u-1)\Phi'(x)}{\sqrt{x-4\sinh^2\frac{u}{2}}}(\alpha_+^m+\alpha_-^m)-ime^{u/2}\Phi(x)\frac{\alpha_+^m-\alpha_-^m}{x+4}\right],$$

with the abbreviation $\alpha_\pm^m=\alpha^m[\pm\xi(x),u]$. Since $\Box_m$ and $\hat{\Box}_m+\frac{m}{2}$ are unitary equivalent, we can study traces of $\hat{\Box}_m$ instead of $\Box_m$. But some care is needed. Going back to $\Box_m$, which is the relevant operator in the fermionic string, then $Y^{s-m/2}$ is an Eigenfunction of $\Box_m$ with Eigenvalue $\lambda=s$; thus

$$\hat{\Box}_m Y^s=sY\Rightarrow\left(\hat{\Box}_m+\frac{m}{2}\right)Y^s=\left(s+\frac{m}{2}\right)Y^s$$
$$\Rightarrow\Box_m Y^{s-m/2}=\left(\hat{\Box}_m+\frac{m}{2}\right)Y^{s-m/2}=sY^s. \qquad (26)$$

Consider now h as an operator valued function of $\Box_m$ we have $h(\Box_m)\cong h\left(\hat{\Box}_m+\frac{m}{2}\right)$. Therefore one hase to replace in the calculation of $h(s)$ as a multiplier of the kernel of $h(\Box_m)Y^s$ by $Y^{s-m/2}$. Considering s as an Eigenvalue of $\hat{\Box}_m$, this yields for the multiplier of the kernel of $h(\Box_m)$

$$h\left(s+\frac{m}{2}\right)=\int_{-\infty}^{\infty} du e^{u(s-1/2)}g(u)$$
$$g(u)=\frac{1}{2\pi}\int_{-\infty}^{\infty} dp e^{-iup}h\left(ip+\frac{m+1}{2}\right), \qquad (27)$$

where $g(u)$ is explicitly given in terms of Φ and Ψ as in Eqs. (25). Note that the contributions $\frac{m}{2}$ coming from $h(s+m/2)$ and $Y^{s-m/2}$ cancel. To distinguish between the functions h in Eqs. (25) and (27) I often denote h in Eq. (27) by $h_m(s) \equiv h\left(s+\frac{m}{2}\right)$. Let us consider several combinations of $g(u)$ and $g(-u)$ for later use

$$g(u)+g(-u)$$
$$=\frac{1}{4}\int_{4\sinh^2 u/2}^{\infty}\frac{dx}{(x+4)^{m/2}}\left[\frac{\Psi(x)+4\sinh^2\frac{u}{2}\Phi'(x)}{\sqrt{x-4\sinh^2\frac{u}{2}}}(\alpha_+^m+\alpha_-^m)-im\cosh\frac{u}{2}\Phi(x)\frac{\alpha_+^m-\alpha_-^m}{x+4}\right], \tag{28}$$

$$g(u)e^{-u/2}+g(-u)e^{u/2}\Big|$$
$$=\frac{1}{4}\int_{4\sinh^2 u/2}^{\infty}\frac{dx}{(x+4)^{m/2}}\left[\frac{\cosh\frac{u}{2}\Psi(x)}{\sqrt{x-4\sin^2\frac{u}{2}}}(\alpha_+^m+\alpha_-^m)-im\Phi(x)\frac{\alpha_+^m-\alpha_-^m}{x+4}\right], \tag{29}$$

$$g(u)-g(-u)$$
$$=\frac{1}{4}\sinh\frac{u}{2}\int_{4\sinh^2 u/2}^{\infty}\frac{dx}{(x+4)^{m/2}}\left[\frac{4\cosh\frac{u}{2}\Phi'(x)}{\sqrt{x-4\sinh^2\frac{u}{2}}}(\alpha_+^m+\alpha_-^m)-im\Phi(x)\frac{\alpha_+^m-\alpha_-^m}{x+4}\right]. \tag{30}$$

We have now the relevant terms to calculate $A(\gamma)$:

$$A(\gamma)=\int_1^{N_{\gamma 0}}dy\int_{-\infty}^{\infty}dx\int\frac{d\theta d\bar\theta}{4Y}J^m(Z,\gamma Z)\{\Phi[R(Z,\gamma Z)]-r(Z,\gamma Z)\Psi[R(Z,\gamma Z)]\}$$
$$=\frac{1}{2}\int_1^{N_{\gamma 0}}dy\int_{-\infty}^{\infty}dx\int d\theta d\bar\theta\left(\frac{1}{y}-\frac{\theta\bar\theta}{y^2}\right)\left(\frac{x(N_\gamma-1)-iy(N_\gamma+1)}{x(N_\gamma-1)+iy(N_\gamma+1)}\right)^{m/2}$$
$$\cdot\left(1-im\frac{x(N_\gamma-1)\chi_\gamma\sqrt{N_\gamma}\theta\bar\theta}{x^2(N_\gamma-1)^2+y^2(N_\gamma+1)^2}\right)$$
$$\cdot\left[\Phi(R_0)-\frac{\theta\bar\theta}{y}R_0\Phi'(R_0)-2\left(1-\chi_\gamma\cosh\frac{u}{2}\right)\frac{\theta\bar\theta}{2y}\Psi(R_0)\right].$$

Performing the $\theta\bar\theta$-integration and the substitution $x=y\xi$ gives,

$$A(\gamma)=\frac{1}{2}\int_1^{N_{\gamma 0}}\frac{dy}{y}\int_{-\infty}^{\infty}d\xi\left(\frac{\xi(N_\gamma-1)-i(N_\gamma+1)}{\xi(N_\gamma-1)+i(N_\gamma+1)}\right)^{m/2}$$
$$\cdot\left[R_0\Phi'(R_0)+\left(1-\chi_\gamma\cosh\frac{u}{2}\right)\Psi(R_0)+\tfrac{1}{2}\Phi(R_0)+\frac{im\xi(N_\gamma-1)\chi_\gamma\sqrt{N_\gamma}\theta\bar\theta}{\xi^2(N_\gamma-1)^2+(N_\gamma+1)^2}\Phi(R_0)\right].$$

Performing another substitution, $\xi=\zeta\sqrt{N_\gamma}/(N_\gamma-1)=\zeta\Big/\left(2\sinh\frac{u}{2}\right)$, where $N_\gamma\equiv e^u$

and $R_0 = \zeta^2 + 4\sinh^2\frac{u}{2}$, the y-integration yields

$$A(\gamma) = \frac{\ln N_{\gamma_0}}{4\sinh^2\frac{u}{2}} \int_{-\infty}^{\infty} d\zeta \left(\frac{\zeta - 2i\cosh\frac{u}{2}}{\zeta + 2i\cosh\frac{u}{2}}\right)^{m/2}$$

$$\cdot\left[\left(\zeta^2 + 4\sinh^2\frac{u}{2}\right)\Phi'\left(\zeta^2 + 4\sinh^2\frac{u}{2}\right) + \left(1 - \chi_\gamma\cosh\frac{u}{2}\right)\Psi\left(\zeta^2 + 4\sinh^2\frac{u}{2}\right)\right.$$

$$\left. + \frac{1}{2}\Phi\left(\zeta^2 + 4\sinh^2\frac{u}{2}\right) + \frac{im\zeta\chi_\gamma}{\zeta^2 + 4\cosh^2\frac{u}{2}}\Phi\left(\zeta^2 + 4\sinh^2\frac{u}{2}\right)\right]. \tag{31}$$

Let us consider the Φ and Φ'-terms in (31) and perform a partial integration in ζ

$$\int_{-\infty}^{\infty} d\zeta \left(\frac{\zeta - 2i\cosh\frac{u}{2}}{\zeta + 2i\cosh\frac{u}{2}}\right)^{m/2} \left[\zeta^2\Phi'\left(\zeta^2 + 4\sinh^2\frac{u}{2}\right) + \frac{1}{2}\Phi\left(\zeta^2 + 4\sinh^2\frac{u}{2}\right)\right]$$

$$= \int_{-\infty}^{\infty} d\zeta \left\{\left(\frac{\zeta - 2i\cosh\frac{u}{2}}{\zeta + 2i\cosh\frac{u}{2}}\right)^{m/2} \zeta^2\Phi'\left(\zeta^2 + 4\sinh^2\frac{u}{2}\right)\right.$$

$$\left. - \frac{1}{2}\zeta\frac{d}{d\zeta}\left[\Phi\left(\zeta^2 + 4\sinh^2\frac{u}{2}\right)\left(\frac{\zeta - 2i\cosh\frac{u}{2}}{\zeta + 2i\cosh\frac{u}{2}}\right)^{m/2}\right]\right\}$$

$$= -im\cosh\frac{u}{2}\int_{-\infty}^{\infty} d\zeta \left(\frac{\zeta - 2i\cosh\frac{u}{2}}{\zeta + 2i\cosh\frac{u}{2}}\right)^{m/2} \frac{\zeta\Phi\left(\zeta^2 + 4\sinh^2\frac{u}{2}\right)}{\zeta^2 + 4\cosh^2\frac{u}{2}}. \tag{32}$$

With the substitution $x = \zeta^2 + 4\sinh^2\frac{u}{2}, dx = 2\zeta d\zeta = 2\sqrt{x - 4\sinh^2\frac{u}{2}}d\zeta$ we get

$$A(\gamma) = \frac{\ln N_{\gamma_0}}{8\sinh\frac{u}{2}} \int_{4\sinh^2 u/2}^{\infty} \frac{dx}{\left(x + 4\sinh^2\frac{u}{2}\right)^{m/2}}$$

$$\cdot\left[\frac{\left(1 - \chi_\gamma\cosh\frac{u}{2}\right)\Psi(x) + 4\sinh^2\frac{u}{2}\Phi'(x)}{\sqrt{x - 4\sinh^2\frac{u}{2}}}(\alpha_+^m + \alpha_-^m)\right.$$

$$\left. + im\left(\chi_\gamma - \cosh\frac{u}{2}\right)\Phi(x)\frac{\alpha_+^m - \alpha_-^m}{x + 4}\right]$$

(and finally by Eqs. (28) and (29))

$$A(\gamma)=\frac{\ln N_{\gamma_0}}{N_\gamma^{1/2}-N_\gamma^{-1/2}}[g(u)+g(-u)-\chi_\gamma(g(u)e^{-u/2}+g(-u)e^{u/2})]. \tag{33}$$

This is the result of ref. [7]. Therefore the supertrace formula reads:

$$\begin{aligned}\operatorname{str} L&=i^m(g-1)\pi\Phi_m(0)\\&+\sum_{\{\gamma\}_p}\frac{\chi_\gamma{}^m\ln N_{\gamma_0}}{N_\gamma^{1/2}-N_\gamma^{-1/2}}[g(u)+g(-u)-\chi_\gamma(g(u)e^{-u/2}+g(-u)e^{u/2})],\end{aligned} \tag{34}$$

where $u=\ln N_\gamma=l_\gamma$ and $g(u)$ is given by

$$g(u)=\frac{1}{2\pi}\int_{-\infty}^{\infty}dp\,e^{-iup}h\left(ip+\frac{m+1}{2}\right). \tag{35}$$

Furthermore an index m has been added in $\Phi_m(x)$ to denote the dependence on m. Our final task is to eliminate $\Phi_m(0)$.

Let us first consider $m=0$. By Eqs. (30) we have

$$g(u)-g(-u)=2\sinh\frac{u}{2}\cosh\frac{u}{2}\int_{4\sinh^2 u/2}^{\infty}\frac{\Phi_0'(x)dx}{\sqrt{x-4\sinh^2\frac{u}{2}}}. \tag{36}$$

Let us denote $\left(w=4\sinh^2\frac{u}{2}\right)$:

$$Q_0(w)=\frac{1}{\sinh u}[g(u)-g(-u)]=\int_w^\infty\frac{\Phi_0'(x)}{\sqrt{x-w}}dx. \tag{37}$$

Further consider the integral

$$\begin{aligned}-\frac{1}{\pi}\int_x^\infty\frac{dw}{\sqrt{w-x}}Q_0(w)&=\int_x^\infty\frac{dw}{\sqrt{w-x}}\int_w^\infty\frac{\Phi_0'(y)}{\sqrt{y-w}}dy\\&=-\frac{1}{\pi}\int_x^\infty dy\,\Phi_0'(y)\int_w^y dw(w-x)^{-1/2}(x-w)^{-1/2}\\&=-\frac{1}{\pi}B(\tfrac{1}{2},\tfrac{1}{2})\int_x^\infty dy\,\Phi_0'(y)=\Phi_0(x).\end{aligned} \tag{38}$$

Here use has been made in the last step of the integral [23, p. 285]:

$$\int_a^b(x-a)^{\mu-1}(b-x)^{\nu-1}dx=(b-a)^{\nu+\mu-1}B(\nu,\mu) \tag{39}$$

and $B(x,y)=\Gamma(x)\Gamma(y)/\Gamma(x+y)$ is the Beta function. Thus we have the inversion formula

$$\Phi_0(x)=-\frac{1}{\pi}\int_x^\infty\frac{dw}{\sqrt{w-x}}Q_0(w). \tag{40}$$

Therefore

$$\Phi_0(0) = -\frac{1}{\pi}\int_0^\infty \frac{dw}{\sqrt{w}} Q_0(w) = -\frac{1}{\pi}\int_0^\infty \frac{du}{\sinh\frac{u}{2}}[g(u) - g(-u)]$$

$$= -\frac{1}{2\pi^2}\int_0^\infty \frac{du}{\sinh\frac{u}{2}}\int_{-\infty}^\infty dp[e^{-iup}h(ip+\tfrac{1}{2}) - e^{iup}h(ip+\tfrac{1}{2})]$$

$$= \frac{i}{\pi^2}\int_{-\infty}^\infty dp\, h(ip+\tfrac{1}{2})\int_0^\infty \frac{\sin up}{\sinh\frac{u}{2}}du = \frac{i}{\pi}\int_{-\infty}^\infty h(ip+\tfrac{1}{2})\tanh \pi p dp,$$

where the integral [23, p. 503]:

$$\int_0^\infty \frac{\sin(ax)}{\sinh(bx)}dx = \frac{\pi}{2b}\tanh\frac{a\pi}{2b} \tag{41}$$

has been used. Finally (in [7] a factor of two is missing):

$$A_0^{(0)} = i(g-1)\int_{-\infty}^\infty h(ip+\tfrac{1}{2})\tanh \pi p dp. \tag{42}$$

It is possible to construct the inversion formulas for, e.g. the $m=1$ and $m=2$ cases explicitly by starting from Eq. (30). But this is rather tedious and cannot be easily generalized to all $m\in\mathbf{Z}$. Therefore I must develop a symmetric approach to invert Eq. (30), i.e. to express $\Phi_m(x)$ by an integral (or integrals) over $g(u)-g(-u)$. The general inversion formula must then be evaluated for $\Phi_m(0)$.

Let us consider Eq. (30) by reinserting the variable $\xi = \sqrt{x - 4\sinh^2\frac{u}{2}}$:

$$\begin{aligned} g(u) - g(-u) &\\ = \frac{1}{2}\sinh\frac{u}{2}\int_{-\infty}^\infty d\xi &\left(\frac{\xi - 2i\cosh\frac{u}{2}}{\xi + 2i\cos\frac{u}{2}}\right)^{m/2} \\ \cdot &\left[4\cosh\frac{u}{2}\Phi_m'\left(\xi^2 + 4\sin^2\frac{u}{2}\right) - \frac{im\xi}{\xi^2 + 4\cosh^2\frac{u}{2}}\Phi_m\left(\xi^2 + 4\sinh^2\frac{u}{2}\right)\right]. \end{aligned} \tag{43}$$

Let be $m \neq 0$. I perform a partial integration in the second term, where it is assumed that all the relevant terms are sufficiently decreasing at ∞:

$$\int_{-\infty}^\infty d\xi\left[\left(\frac{\xi - 2i\cosh\frac{u}{2}}{\xi + 2i\cosh\frac{u}{2}}\right)^{m/2}\frac{1}{\xi^2 + 4\cosh^2\frac{u}{2}}\right]\cdot\left[\xi\Phi_m\left(\xi^2 + 4\sinh^2\frac{u}{2}\right)\right]$$

$$= -\frac{1}{2im\cosh\frac{u}{2}} \int_{-\infty}^{\infty} d\xi \left(\frac{\xi - 2i\cos\frac{u}{2}}{\xi + 2i\cosh\frac{u}{2}}\right)^{m/2}$$

$$\cdot\left[\Phi_m\left(\xi^2 + 4\sinh^2\frac{u}{2}\right) + 2\xi^2\Phi_m'\left(\xi^2 + 4\sinh^2\frac{u}{2}\right)\right]. \tag{44}$$

With the abbreviation $w = 4\sinh^2\frac{u}{2}$ this gives in Eq. (44):

$$g(u) - g(-u) = \frac{1}{2} i^m \tanh\frac{u}{2} \int_{-\infty}^{\infty} d\xi \left(\frac{\sqrt{w+4} + i\xi}{\sqrt{w+4} - i\xi}\right)^{m/2}$$

$$\cdot[(w + \xi^2 + 4)\Phi_m'(w + \xi^2) + \tfrac{1}{2}\Phi_m(w + \xi^2)]. \tag{45}$$

Let us define [Q must not be confused with Q_0 in Eq. (37)]

$$Q(w) = 2\coth\frac{u}{2}[g(u) - g(-u)], \quad \tilde{\Phi}_m(x) = i^m[(x+4)\Phi_m'(x) + \tfrac{1}{2}\Phi_m(x)] \tag{46}$$

and get the integral relation $\left(w = 4\sinh^2\frac{u}{2} \geqq 0\right)$:

$$Q(w) = \int_{-\infty}^{\infty} d\xi \left(\frac{\sqrt{w+4} + i\xi}{\sqrt{w+4} - i\xi}\right)^{m/2} \tilde{\Phi}_m(w + \xi^2). \tag{47}$$

For this integral relation I can apply an inversion formula given by Hejhal [36, p. 454] which yields for $\tilde{\Phi}$:

$$\tilde{\Phi}_m(x) = -\frac{1}{\pi} \int_{-\infty}^{\infty} Q'(x + t^2) \left(\frac{\sqrt{x + 4 + t^2} - t}{\sqrt{x + 4 + t^2} + t}\right)^{m/2} dt. \tag{48}$$

Reinserting $\tilde{\Phi}$ we get a differential equation for Φ:

$$\Phi_m'(x) + \frac{1}{2(x+4)}\Phi_m(x) = \frac{-1}{i^m\pi(x+4)} \int_{-\infty}^{\infty} Q'(x + t^2) \left(\frac{\sqrt{x + 4 + t^2} - t}{\sqrt{x + 4 + t^2} + t}\right)^{m/2} dt, \tag{49}$$

which can be easily solved to give the **inversion formula** for Φ:

$$i^m\Phi_m(x) = \frac{1}{\pi\sqrt{x+4}} \int_x^{\infty} \frac{dy}{\sqrt{y+4}} \int_{-\infty}^{\infty} Q'(y + t^2) \left(\frac{\sqrt{y + 4 + t^2} - t}{\sqrt{y + 4 + t^2} + t}\right)^{m/2} dt. \tag{50}$$

This is the main result of this section.

Note. 1) The integration constant in Eq. (50) is given by $\Phi_m(\infty) = 0$.
2) The inversion formula is valid for $m \in \mathbf{Z}$ [see below Eq. (51) for $m = 0$].

To get some confidence in the inversion formula let us consider Eq. (50) for some specific values of m.

1) $m=0$:

$$\Phi_0(x)=\frac{1}{\pi\sqrt{x+4}}\int_x^\infty \frac{dy}{\sqrt{y+4}}\int_{-\infty}^\infty Q'(y+t^2)dt$$

$(w=y^2+t)$

$$=\frac{1}{\pi\sqrt{x+4}}\int_x^\infty \frac{dy}{\sqrt{y+4}}\int_y^\infty \frac{Q'(w)}{\sqrt{w-y}}dw.$$

(Rearrangement of integrations)

$$=\frac{1}{\pi\sqrt{x+4}}\int_x^\infty dw\,Q'(w)\int_x^w \frac{dy}{\sqrt{y+4}\sqrt{w-y}}.$$

$$\left[\text{Elementary integral: } \int\frac{dx}{\sqrt{(ax+b)(cx+d)}}=-\frac{2}{\sqrt{-ac}}\arctan\sqrt{-\frac{x+b/a}{x+d/c}}(ac<0)\right]$$

$$=\frac{2}{\pi\sqrt{x+4}}\int_x^\infty Q'(w)\arctan\sqrt{\frac{w-x}{x+4}}\,dw.$$

(Partial integration)

$$=-\frac{1}{\pi}\int_x^\infty \frac{Q(w)}{(w+4)\sqrt{w-x}}dw$$

$\left(w=4\sinh^2\frac{u}{2}\right)$

$$=-\frac{1}{\pi}\int_{2\operatorname{arsinh}\sqrt{x}/2}^\infty \frac{g(u)-g(-u)}{\sqrt{4\sinh^2\frac{u}{2}-x}}d\mu. \tag{51}$$

This is equivalent with Eq. (40) and shows that the inversion formula is also valid for $m=0$, i.e. the inversion formula is valid for all $m\in\mathbf{Z}$.

2) $m=1$: Similarly as for $m=0$.

$$i\Phi_1(x)=\frac{1}{\pi\sqrt{x+4}}\int_x^\infty \frac{dy}{\sqrt{y+4}}\int_{-\infty}^\infty Q'(y+t^2)\left(\frac{\sqrt{y+4+t^2}-t}{\sqrt{y+4+t^2}+t}\right)^{1/2}dt$$

$$=-\frac{4}{\pi\sqrt{x+4}}\int_{2\operatorname{arsinh}\sqrt{x}/2}^\infty \frac{g(u)-g(-u)}{\sqrt{4\sinh^2\frac{u}{2}-x}}\cosh\frac{u}{2}du. \tag{52}$$

In particular for $x=0$:

$$i\Phi_1(0)=\frac{i}{\pi}\int_{-\infty}^\infty h_1(ip+\tfrac{1}{2})\coth\pi p\,dp, \tag{53}$$

where use has been made of the integral [23, p. 504]

$$\int_0^\infty \sin ax\frac{\cosh\beta x}{\sinh\gamma x}dx=\frac{\pi}{2\gamma}\frac{\sinh\frac{\pi a}{\gamma}}{\cosh\frac{\pi a}{\gamma}+\cos\frac{\pi\beta}{\gamma}}. \tag{54}$$

This gives finally for $A_0^{(1)}$ by Eq. (15)

$$A_0^{(1)} = i(g-1) \int_{-\infty}^{\infty} \coth \pi p h(ip+1) dp. \tag{55}$$

3) $m = 2$:

$$\begin{aligned} i^2 \Phi_2(x) &= \frac{1}{\pi\sqrt{x+4}} \int_x^{\infty} \frac{dy}{\sqrt{y+4}} \int_{-\infty}^{\infty} Q'(y+t^2)\left(\frac{\sqrt{y+4+t^2}-t}{\sqrt{y+4+t^2}+t}\right) dt \\ &= \frac{2}{\pi\sqrt{x+4}} \int_0^{\infty} \frac{dy}{\sqrt{y+4}} \int_0^{\infty} Q'(y+t^2) dt \\ &\quad + \frac{4}{\pi\sqrt{x+4}} \int_0^{\infty} \frac{dy}{\sqrt{y+4}^3} \int_0^{\infty} t^2 Q'(y+t^2) dt. \end{aligned} \tag{56}$$

The first integral is up to a factor Eq. (51). For the second I get:

$$\begin{aligned} &\frac{4}{\pi\sqrt{x+4}} \int_0^{\infty} \frac{dy}{(y+4)^{3/2}} \int_0^{\infty} t^2 Q'(y+t^2) dt \\ &= -\frac{4}{\pi(x+4)} \int_{2\operatorname{arsinh}\sqrt{x}/2}^{\infty} [g(u)-g(-u)] \sqrt{4\sinh^2\frac{u}{2} - x}\, du. \end{aligned} \tag{57}$$

This gives for $m = 2$ the inversion formula

$$\begin{aligned} i^2 \Phi_2(x) = &-\frac{1}{\pi} \int_{2\operatorname{arsinh}\sqrt{x}/2}^{\infty} \frac{g(u)-g(-u)}{\sqrt{4\sinh^2\frac{u}{2} - x}} du \\ &- \frac{4}{\pi(x+4)} \int_{2\operatorname{arsinh}\sqrt{x}/2}^{\infty} [g(u)-g(-u)] \sqrt{4\sinh^2\frac{u}{2} - x}\, du. \end{aligned} \tag{58}$$

In particular for $x = 0$:

$$i^2 \Phi_2(0) = \frac{i}{\pi} \int_{-\infty}^{\infty} h_2(ip+\tfrac{1}{2}) \tanh \pi p dp - \frac{2}{\pi} \int_{-\infty}^{\infty} g(u) \sinh\frac{u}{2} du, \tag{59}$$

and therefore finally

$$A_0^{(2)} = i(g-1) \int_{-\infty}^{\infty} h_2(ip+\tfrac{1}{2}) \tanh \pi p dp + (1-g)[h_2(1) - h_2(0)]. \tag{60}$$

4) $m = 3$: Similarly as for $m = 2$ it is straightforward to show that

$$\left.\begin{aligned} i^3 \Phi_3(0) &= \frac{i}{\pi} \int_{-\infty}^{\infty} h_3(ip+\tfrac{1}{2}) \coth \pi p dp - \frac{2}{\pi} \int_{-\infty}^{\infty} g(u) \sinh u du \\ A_0^{(3)} &= i(g-1) \int_{-\infty}^{\infty} h_3(ip+\tfrac{1}{2}) \coth \pi p dp + (1-g)[h_3(\tfrac{3}{2}) - h_3(-\tfrac{1}{2})]. \end{aligned}\right\} \tag{61}$$

Let us for a moment turn to test functions h for the operator $\hat{\Box}_m$, i.e. let us consider the function $h(s) \equiv h_0(s)$. By Eqs. (25) (relating h and g) $\hat{g}$ and $\hat{Q}$ do not depend on

m ("hatted" quantities belonging to $\hat{\square}_m$). The equations for $\hat{\Phi}_m$ $(m = 0, 1, 2, 3)$ suggest the following general structure for $\hat{\Phi}_m$:

$$i^m \hat{\Phi}_m(0) = \frac{i}{\pi} \int_{-\infty}^{\infty} h(ip + \tfrac{1}{2}) \tanh \pi p\, dp - \frac{2}{\pi} \sum_{k=1}^{m/2} \int_{-\infty}^{\infty} \hat{g}(u) \sinh (k - \tfrac{1}{2}) u\, du \quad (m \text{ even}), \tag{62}$$

$$i^m \hat{\Phi}_m(0) = \frac{i}{\pi} \int_{-\infty}^{\infty} h(ip + \tfrac{1}{2}) \coth \pi p\, dp - \frac{2}{\pi} \sum_{k=1}^{(m-1)/2} \int_{-\infty}^{\infty} \hat{g}(u) \sinh ku\, du \quad (m \text{ odd}). \tag{63}$$

In particular, it remains to show that for all m (even and odd):

$$i^{m+2} \hat{\Phi}_{m+2}(0) - i^m \hat{\Phi}_m(0) = -\frac{2}{\pi} \int_{-\infty}^{\infty} \hat{g}(u) \sinh \frac{m+1}{2} du. \tag{64}$$

Having proved Eq. (64) once, one can go back to the operator $\square_m$ and all related quantities. I prove Eq. (64) by induction for $m \to m + 2$.

1) Since each step forward is by two units in the induction I have to distinguish between the even and odd cases. Equations (42) and (59), respectively Eqs. (53, 55) and (61) show that Eq. (64) is correct for $m = 0$ and $m = 1$, respectively.
2) Let us consider Eq. (64) and insert for $i^m \hat{\Phi}_m(0)$ and $i^{m+2} \hat{\Phi}_{m+2}(0)$:

$$\begin{aligned} &i^{m+2} \hat{\Phi}_{m+2}(0) - i^m \hat{\Phi}_m(0) \\ &= \frac{1}{2\pi} \int_x^{\infty} \frac{dy}{\sqrt{y+4}} \hat{Q}'(y+t^2) \left[\left(\frac{\sqrt{y+4+t^2} - t}{\sqrt{y+4+t^2} + t} \right)^{(m+2)/2} - \left(\frac{\sqrt{y+4+t^2} - t}{\sqrt{y+4+t^2} + t} \right)^{m/2} \right] \\ &= -\frac{1}{\pi} \int_0^{\infty} \frac{dy}{(y+4)^{(m+3)/2}} \int_{-\infty}^{\infty} t \hat{Q}'(y+t^2) (\sqrt{y+4+t^2} - t)^{m+1} dt. \end{aligned}$$

$\left(\text{Partial integration and } w = 4 \sinh^2 \frac{u}{2} \right)$

$$\begin{aligned} &= -\frac{m+1}{4\pi} \int_0^{\infty} \frac{dy}{(y+4)^{(m+3)/2}} \\ &\quad \cdot \int_y^{\infty} \frac{Q(w)}{\sqrt{w+4}\sqrt{w-y}} [(\sqrt{w+4} - \sqrt{w-y})^{m+1} + (\sqrt{w+4} + \sqrt{w-y})^{m+1}]. \end{aligned}$$

With a rearrangement of integrations:

$$\begin{aligned} &i^{m+2} \hat{\Phi}_{m+2}(0) - i^m \hat{\Phi}_m(0) \\ &= -\frac{m+1}{4\pi} \int_0^{\infty} dw \frac{\hat{Q}(w)}{\sqrt{w+4}} \int_0^{w} \frac{dy}{\sqrt{w-y}} \left[\frac{(\sqrt{w+4} - \sqrt{w-y})^{m+1}}{(y+4)^{(m+3)/2}} + \frac{(\sqrt{w+4} + \sqrt{w-y})^{m+1}}{(y+4)^{(m+3)/2}} \right]. \end{aligned}$$

[Substitution $t^2 = 1 - (y+4)/(w+4)$]

$$= -\frac{m+1}{2\pi} \int_0^{\infty} dw \frac{\hat{Q}(w)}{w+4} \int_0^{\sqrt{w/(w+4)}} \left[\frac{(1-t)^{(m-1)/2}}{(1+t)^{(m+3)/2}} + \frac{(1+t)^{(m-1)/2}}{(1-t)^{(m+3)/2}} \right] dt. \tag{65}$$

Let us consider the t-integration and replace $m \to m+2$:

$$(m+3) \int_0^{\sqrt{w/(w+4)}} \left[\frac{(1-t)^{(m+1)/2}}{(1+t)^{(m+5)/2}} + \frac{(1+t)^{(m+1)/2}}{(1-t)^{(m+5)/2}} \right] dt.$$

$\left(\text{Partial integration and } w = \sinh^2 \frac{u}{2} \text{ reinserted}\right)$

$$= 4 \cosh \frac{u}{2} \sinh \frac{m+2}{2} u - (m+1) \int_0^{\sqrt{w/(w+4)}} \left[\frac{(1-t)^{(m-1)/2}}{(1+t)^{(m+3)/2}} + \frac{(1+t)^{(m-1)/2}}{(1-t)^{(m+3)/2}} \right] dt. \tag{66}$$

Thus repeating the calculations of Eq. (65) for $m \to m+2$ and taking into account the result of Eq. (66) yield together with $w = 4 \sinh^2 \frac{u}{2}$:

$$\begin{aligned} &i^{m+4} \hat{\Phi}_{m+4}(0) - i^{m+2} \hat{\Phi}_{m+2}(0) \\ &= -\frac{8}{\pi} \int_0^\infty [\hat{g}(u) - \hat{g}(-u)] \cosh \frac{u}{2} \sinh \frac{m+2}{2} du - [i^{m+2} \hat{\Phi}_{m+2}(0) - i^m \hat{\Phi}_m(0)] \\ &= -\frac{2}{\pi} \int_{-\infty}^\infty \hat{g}(u) \sinh \frac{m+3}{2} du. \end{aligned} \tag{67}$$

This proves the induction! ■

Let us make some remarks concerning the property of the kernel $k(Z, W)$ that in general for $m \neq 0$ it can be represented by a sum of a discrete and continuous spectrum contribution

$$k(Z, W) = k_{\text{disc}}(Z, W) + k_{\text{cont}}(Z, W), \quad (Z, W \in \mathscr{S}\mathscr{H}). \tag{68}$$

As mentioned in Chap. II, in the case of the heat-kernel on $\mathscr{S}\mathscr{H}$, $k(Z, W)$ can be explicitly calculated [2]. In the calculations of the supertrace formula, we have not made any reference to k_{disc} and k_{cont}, respectively. Of course, implicitly these two contributions to the complete kernel are always present and contribute to the trace formula. This is similar as for the usual "bosonic" trace formula and, of course, is important in the calculation of determinants [8, 11, 17, 48]. However, let us look on this feature more explicitly. Let h be an operator valued function of the Dirac-operator $\square_m$. Then the kernel of $h(\square_m)$ is given by

$$\begin{aligned} k(Z, W) = &\int_{-\infty}^\infty dk \sum_{l=0}^{N_M} \left[h\left(\frac{m}{2} + l\right) - h\left(\frac{m}{2} - l - 1\right) \right] \Psi_{l,k}(Z) \Psi_{l,k}^*(W) \\ &+ \int_{-\infty}^\infty dk \int_0^\infty dp \left[h\left(\frac{m+1}{2} + ip\right) - h\left(\frac{m+1}{2} - ip\right) \right] \Psi_{p,k}(Z) \Psi_{p,k}^*(W). \end{aligned} \tag{69}$$

Here $\Psi_{l,k}$ and $\Psi_{p,k}$ denote the Eigenfunctions of the Laplacian $\square_m$ on $\mathscr{S}\mathscr{H}$ with $l = 0, 1, \ldots, N_M \leq \frac{m}{2}, \frac{m-1}{2}$ (m even or odd, respectively). Equation (69) generalizes the result of [2] for some function h and the usual bosonic case [36]. Rewriting Eqs. (62) and (63) in terms of $A_0^{(m)}$ we obtain using Eq. (69)

$$A_0^{(m)} = i(g-1) \int_{-\infty}^{\infty} h\left(ip + \frac{m+1}{2}\right)\binom{\tanh \pi p}{\coth \pi p} dp$$
$$+(1-g)\sum_{l=0}^{N_M}\left[h\left(\frac{m}{2}+l\right)-h\left(\frac{m}{2}-l-1\right)\right] \tag{70}$$

for m even or odd, respectively. Thus we see explicitly the contributions in the $A_0^{(m)}$ term coming from the discrete and continuous part of the spectrum of $\square_m$. It is not difficult to show that for $A(\gamma)$ the corresponding terms coming from the discrete spectrum do not contribute; this can be shown by contour integration and is similar as for the bosonic case [8, 17]. The great advantage of discussing the abstract concept of a trace formula in comparison to an explicit expression like heat-kernels lies thus in the fact that one simply has not to take care of substilities of discrete or continuous spectra. Once stated, the trace formula contains all relevant information.

I summarize. I have formulated the Selberg Supertrace formula on super Riemannian surfaces for operator valued functions of the Laplace-Dirac operator $\square_m$. Let h be a testfunction with the properties (following Baranov et al. [7]):

i) $h(\frac{1}{2}+ip)\in C^\infty(\mathbf{R})$,
ii) $h(\frac{1}{2}+ip)$ need not be an even function in p,
iii) $h(\frac{1}{2}+ip)\propto O\left(\frac{1}{p^2}\right)(p\to\pm\infty)$.
iv) $h(\frac{1}{2}+ip)$ is holomorphic in the strip $|\mathrm{Im}(p)|\leqq 1+\frac{m}{2}+\varepsilon, \varepsilon>0$ to guarantee absolute convergence in the sums of Eq. (75) below (see [36, p. 30]).

Its Fourier transform g is given by:

$$g(u)=\frac{1}{2\pi}\int_{-\infty}^{\infty} dp\, e^{-iup} h\left(ip+\frac{m+1}{2}\right). \tag{71}$$

The term $A_0^{(m)}$ corresponding to the identity transformation reads

$$A_0^{(m)} = i(g-1)\int_{-\infty}^{\infty} h\left(ip+\frac{m+1}{2}\right)\tanh \pi p\, dp$$
$$+(1-g)\sum_{k=1}^{m/2}\left[h\left(\frac{m}{2}+k\right)-h\left(\frac{m}{2}-k-1\right)\right] \qquad (m \text{ even}), \tag{72}$$

$$A_0^{(m)} = i(g-1)\int_{-\infty}^{\infty} h\left(ip+\frac{m+1}{2}\right)\coth \pi p\, dp$$
$$+(1-g)\sum_{k=1}^{(m-1)/2}\left[h\left(\frac{m+1}{2}+k\right)-h\left(\frac{m-1}{2}-k\right)\right] \qquad (m \text{ odd}). \tag{73}$$

The last two equations can be combined and stated in a compact form yielding

$$A_0^{(m)} = (1-g)\int_0^\infty \frac{g(u)-g(-u)}{\sinh\frac{u}{2}} T_m\left(\cosh\frac{u}{2}\right)du, \quad (m\in\mathbf{Z}), \tag{74}$$

where $T_m\left(\cosh\frac{u}{2}\right)=\cosh\frac{m}{2}u$ denotes the m^{th} Chebyshev-polynomial in $\cosh\frac{u}{2}$. Thus for the supertrace formula we get (l_γ primitive geodesic, $\lambda_n^{B(F)}=\frac{1}{2}+p_n^{B(F)}$ ($n\in\mathbb{N}$) are denoting the Bose and Fermi Eigenvalues of $\square$, respectively):

$$\sum_{n=0}^{\infty}[h_m(p_n^B)-h_m(p_n^F)]$$
$$=(1-g)\int_0^\infty \frac{g(u)-g(-u)}{\sinh\frac{u}{2}}\cosh\frac{m}{2}du$$
$$+\sum_{\{\gamma\}_P}\sum_{k=1}^{\infty}\frac{l_\gamma \chi_\gamma{}^{km}}{e^{kl_\gamma/2}-e^{-kl_\gamma}}\left[g(kl_\gamma)+g(-kl_\gamma)-\chi_\gamma{}^k\left(g(kl_\gamma)e^{-kl_\gamma/2}+g(-kl_\gamma)e^{kl_\gamma/2}\right)\right]. \tag{75}$$

Equation (75) completes the work of refs. [6–8] by explicit statement of the inversion formula (50) and the $A_0^{(m)}$-term (74), respectively.

IV. Analytic Properties of the Selberg Super Zeta-Functions

1. The Selberg Super Zeta-Function Z_1. The Selberg super zeta-functions are defined by

$$Z_q(s):=\prod_{\{\gamma\}_P}\prod_{k=0}^{\infty}[1-\chi_\gamma{}^q e^{-(s+k)l_\gamma}],\quad (\mathrm{Re}(s)>1), \tag{1}$$

where q can take on the values $q=0,1$, respectively. χ_γ describes the spin structure and l_γ is the length of a primitive geodesic, as already defined. The γ product is taken over all primitive conjugacy classes $\gamma\in\Gamma$. The Selberg super R-functions are defined by

$$R_q(s):=\frac{Z_q(s)}{Z_q(s+1)}=\prod_{\{\gamma\}_P}[1-\chi_\gamma{}^q e^{-sl_\gamma}],\quad (\mathrm{Re}(s)>1). \tag{2}$$

To study the analytic properties of Z_0 and Z_1 let us consider the Selberg supertrace formula for $m=0$, i.e. (throughout this Chap. I denote by $\lambda_n^{B(F)}=\frac{1}{2}+p_n^{B(F)}$ ($n\in\mathbb{N}$) the Bose and Fermi Eigenvalues of $\square$, respectively):

$$\sum_{n=0}^{\infty}[h(p_n^B)-h(p_n^F)]$$
$$=i(g-1)\int_{-\infty}^{\infty}h(ip+\tfrac{1}{2})\tanh\pi p\,dp$$
$$+\sum_{\{\gamma\}_P}\sum_{k=1}^{\infty}\frac{l_\gamma}{e^{kl_\gamma/2}-e^{-kl_\gamma/2}}\left[g(kl_\gamma)+g(-kl_\gamma)-\chi_\gamma{}^k\left(g(kl_\gamma)e^{-kl_\gamma/2}+g(-kl_\gamma)e^{kl_\gamma/2}\right)\right]. \tag{3}$$

To get information for Z_1 or R_1, respectively, one has to choose a test function $h(p)$ so that the first two terms in the square bracket in the supertrace formula cancel, i.e. $g(u) = -g(-u)$. I choose the function ($\mathrm{Re}(s) > 1$, $\mathrm{Re}(\sigma) > 1$):

$$h_s(p) = 2\left[\frac{\lambda - \frac{1}{2}}{s^2 - (\lambda - \frac{1}{2})^2} - \frac{\lambda - \frac{1}{2}}{\sigma^2 - (\lambda - \frac{1}{2})^2}\right]\Bigg|_{\lambda = 1/2 + ip} = 2ip\left(\frac{1}{s^2 + p^2} - \frac{1}{\sigma^2 + p^2}\right). \tag{4}$$

The second term plays the role of a regulator so that all the involved terms in the supertrace formula are convergent. Thus for $g(u)$:

$$g(u) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iup} h_s(p)\,dp = \frac{2}{\pi}\int_0^{\infty} p \sin up\left(\frac{1}{s^2 + p^2} - \frac{1}{\sigma^2 + p^2}\right)dp, \tag{5}$$

and we see that $g(u)$ is an odd function as required. Using [23, p. 406]:

$$\int_0^{\infty} \frac{x \sin ax}{\beta^2 + x^2}\,dx = \frac{\pi}{2} e^{-a\beta}, \tag{6}$$

we get ($u > 0$) $g(u) = (e^{-su} - e^{-\sigma u})$ and for $u \in \mathbf{R}$

$$g(u) = \operatorname{sign}(u)(e^{-s|u|} - e^{-\sigma|u|}), \tag{7}$$

thus finally for $G(u, \chi)$

$$G(u, \chi_\gamma) = 2\chi_\gamma(e^{-s|u|} - e^{-\sigma|u|}) \sinh\frac{u}{2}. \tag{8}$$

Therefore only the χ_γ-term remains in the supertrace formula which allows to study the properties of Z_1 alone. Inserting $G(u, \chi)$ into the length term yields

$$\begin{aligned}
&\sum_{\{\gamma\}_p} \sum_{k=1}^{\infty} \frac{l_\gamma}{2\sinh\frac{kl_\gamma}{2}} G(kl_\gamma, \chi_\gamma) \\
&= \sum_{\{\gamma\}_p} \sum_{k=1}^{\infty} l_\gamma \chi_\gamma^k (e^{-skl_\gamma} - e^{-\sigma kl_\gamma}) = \sum_{\{\gamma\}_p}\left[\frac{l_\gamma \chi_\gamma e^{-sl_\gamma}}{1 - \chi_\gamma e^{-sl_\gamma}} - \frac{l_\gamma \chi_\gamma e^{-\sigma l_\gamma}}{1 - \chi_\gamma e^{-\sigma l_\gamma}}\right] \\
&= \frac{R_1'(s)}{R_1(s)} - \frac{R_1'(\sigma)}{R_1(\sigma)}.
\end{aligned} \tag{9}$$

In the last step the property of the logarithmic derivative of the Selberg super R-functions has been used, i.e. for $\mathrm{Re}(s) > 1$:

$$\frac{d}{ds} \ln R_q(s) = \frac{d}{ds} \ln \prod_{\{\gamma\}_p} [1 - \chi_\gamma^q e^{-sl_\gamma}] = \sum_{\{\gamma\}_p} \frac{l_\gamma \chi_\gamma^q e^{-sl_\gamma}}{1 - \chi_\gamma^q e^{-sl_\gamma}}. \tag{10}$$

The A_0 term gives

$$\begin{aligned}
A_0 &= i(g - 1) \int_{-\infty}^{\infty} h_s(p) \tanh \pi p\,dp \\
&= \frac{4}{\pi}(1 - g) \int_0^{\infty} \frac{du}{\sinh\frac{u}{2}}\left[\int_0^{\infty} \frac{p \sin up}{p^2 + s^2}\,dp - \int_0^{\infty} \frac{p \sin up}{p^2 + \sigma^2}\,dp\right] \\
&= 4(1 - g) \int_0^{\infty} e^{-((s + \sigma)/2)u} \frac{\sinh\frac{\sigma - s}{2}}{\sinh\frac{u}{2}}\,du,
\end{aligned} \tag{11}$$

where the integrals (6) and (III.41) have been used. Using now [23, p. 356]

$$\int_0^\infty e^{-\mu x}\frac{\sinh\beta x}{\sinh bx}dx=\frac{1}{2b}\left[\Psi\left(\frac{1}{2}+\frac{\mu+\beta}{2b}\right)-\Psi\left(\frac{1}{2}+\frac{\mu-\beta}{2b}\right)\right], \tag{12}$$

where $\Psi(z)=\Gamma'(z)/\Gamma(z)$, $z\in\mathbf{C}$, we obtain finally for A_0

$$A_0=4(g-1)[\Psi(s+\tfrac{1}{2})-\Psi(\sigma+\tfrac{1}{2})]. \tag{13}$$

Let us denote by $\Delta n_0^{(0)}=n_0^B-n_0^F$ the difference between the number of even and odd zero modes of the Dirac operator $\Box$. Thus we get the supertrace formula for the function h_s

$$\sum_{n=1}^{\infty}[h_s(p_n^B)-h_s(p_n^F)]-\Delta n_0^{(0)}\left[\frac{1}{(s-\frac{1}{2})(s+\frac{1}{2})}-\frac{1}{(\sigma-\frac{1}{2})(\sigma+\frac{1}{2})}\right]$$
$$=4(g-1)[\Psi(s+\tfrac{1}{2})-\Psi(\sigma+\tfrac{1}{2})]+\frac{R_1'(s)}{R_1(s)}-\frac{R_1'(\sigma)}{R_1(\sigma)}. \tag{14}$$

First let us discuss the trivial structure of zeros and poles of R_1 and Z_1 in the complex s-plane. We can read off the analytic properties of the R_1-function:

- For $s=\frac{1}{2}$: there is a pole, zero or a regular point depending on whether $\Delta n_0^{(0)}>0$, $\Delta n_0^{(0)}<0$ or $\Delta n_0^{(0)}=0$, respectively.
- For $s=-\frac{1}{2}$: there is a zero of multiplicity $4(g-1)+\Delta n_0^{(0)}$ (assuming that $-\Delta n_0^{(0)}>4(g-1)$).
- For $s=-\frac{1}{2}-k$ ($k\in\mathbf{N}$): there are zeros with multiplicity $4(g-1)$.

Here in the discussion has been used that $\operatorname{Res}\Psi(z)|_{z=-k}=-1$ ($k\in\mathbf{N}_0$). Therefore we get the analytic properties of Z_1:

- For $s=\frac{1}{2}$: there is a pole, zero or a regular point depending on whether $\Delta n_0^{(0)}>0$, $\Delta n_0^{(0)}<0$ or $\Delta n_0^{(0)}$, respectively.
- For $s=-\frac{1}{2}-k$ ($k\in\mathbf{N}_0$): there are zeros with multiplicity $4(k+1)(g-1)$.

Second let us turn to the nontrivial zeros and poles of these two functions (first so called "small Eigenvalues" not considered). Since

$$h_s(p)=2ip\left[\frac{1}{(s+ip)(s-ip)}-\frac{1}{(\sigma+ip)(\sigma-ip)}\right] \tag{15}$$

one has

$$\operatorname{Res}[h_s(p_n^B)]|_{s=ip_n^B}=1,\quad \operatorname{Res}[h_s(p_n^B)]|_{s=-ip_n^B}=-1 \tag{16}$$

with signs of the residua reversed for the Fermi Eigenvalues. Thus we see that $R_1(s)$ has

- for $s=ip_n^{B(F)}$: there are zeros (poles) of the same multiplicity as the corresponding Eigenvalue of $\Box$,
- for $s=-ip_n^{B(F)}$: reversed situation for poles and zeros.

Note the crucial dependence on the signs.

Since $R_1(1\pm ip)$ is regular we can conclude by $Z_1(1\pm ip)=R_1(1\pm ip)\cdot Z_1(2\pm ip)$ that $Z_1(s)$ is regular on the line $\operatorname{Re}(s)=1$. Furthermore this gives by $Z_1(ip)=R_1(ip)\cdot$

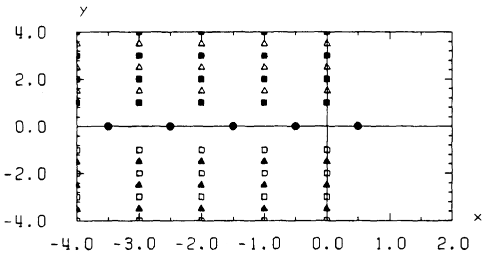

Fig. 1. Zeros and poles of the zeta-function Z_1

$Z_1(1+ip)$ that $Z_1(s)$ has on the line $\mathrm{Re}(s)=0$ the same properties as $R_1(s)$, i.e. zeros (poles) for $s=ip_n^{B(F)}$ and poles (zeros) for $s=-ip_n^{B(F)}$. Repeating this procedure for $Z_1(ip-k)=R_1(ip-k)\cdot Z_1(ip-k+1)$ ($k\in\mathbf{N}$), we see that we get an infinite number of critical lines for Z_1 located at $\mathrm{Re}(s)=-k$ ($k\in\mathbf{N}_0$). Therefore we get the analytic properties of Z_1 for the nontrivial zeros and poles ($k\in\mathbf{N}_0$):

- For $s=ip_n^{B(F)}-k$: there are zeros (p_N^B) and poles (p_n^F) with the same multiplicity as the corresponding Eigenvalue of $\square$.
- For $s=-ip_n^{B(F)}-k$: there are poles (p_N^B) and zeros (p_n^F) with the same multiplicity as the corresponding Eigenvalue of $\square$.

Finally, let us discuss the case of so-called small Eigenvalues ($0\leqq\lambda\leqq\frac{1}{2}$), which are also unknown and likely do not exist for small g [5]. We can see from Eq. (14) that for R_1 they are located in the complex s-plane at $-\frac{1}{2}\leqq s\leqq\frac{1}{2}$.

- for $s=\lambda_n^B-\frac{1}{2}$ there are zeros and
- for $s=-(\lambda_n^F-\frac{1}{2})$ there are poles of the same multiplicity as the corresponding Eigenvalue of $\square$, respectively.

By the same considerations as for the other nontrivial zeros and poles we get the structure for the Z_1-function ($k\in\mathbf{N}_0$):

- for $s=\lambda_n^{B(F)}-\frac{1}{2}-k$ there are zeros (poles) and
- for $s=-(\lambda_n^{B(F)}-\frac{1}{2})-k$ there are poles (zeros) of the same multiplicity as the corresponding Eigenvalue of $\square$, respectively.

All these Eigenvalues are of course, even numbers, i.e. elements of $\mathbf{C}_c$. Therefore we can conclude that the supertrace formula can be extended meromorphically to all $s\in\Lambda_\infty$ and that R_1 and Z_1 are meromorphic functions in Λ_∞.

In Fig. 1 I have displayed the analytic properties of the Z_1-function. The trivial zeros are indicated by filled dots, the position of the bosonic zeros and poles by filled and empty squares, respectively, and the position of the fermionic zeros and poles by filled and empty triangles, respectively. The small Eigenvalues are not considered. The x- and y-axis are taken at the body of Λ_∞, i.e. $(\Lambda_\infty)_{\mathrm{Body}}=\mathbf{C}$. The y-axis is taken in arbitrary units.

Let us consider Eq. (14) in the limit $\sigma\to\frac{1}{2}$ and get

$$\lim_{\sigma\to 1/2}\left[\frac{R_1'(\sigma)}{R_1(\sigma)}+\frac{\Delta n_0^{(0)}}{\sigma^2-\frac{1}{4}}-4(g-1)\Psi(\sigma+\tfrac{1}{2})\right]=A_1+4(g-1)\gamma_E, \tag{17}$$

where $\Psi(1) = -\gamma_E$ is the Euler's constant $\gamma_E = 0.57721\ldots$ and A_1 is given by

$$\left.\begin{aligned} A_1 &= \frac{R_1'(\frac{1}{2})}{R_1(\frac{1}{2})}, && (\Delta n_0^{(0)} = 0),\\ &= \frac{R_1^{(1-\Delta n_0^{(0)})}(\frac{1}{2})}{(1-\Delta n_0^{(0)})R_1^{(-\Delta n_0^{(0)})}(\frac{1}{2})}, && (\Delta n_0^{(0)} < 0),\\ &= \frac{\oint(\sigma-\frac{1}{2})^{\Delta n_0^{(0)}-2}R_1(\sigma)d\sigma}{\oint(\sigma-\frac{1}{2})^{\Delta n_0^{(0)}-1}R_1(\sigma)d\sigma}, && (\Delta n_0^{(0)} > 0). \end{aligned}\right\} \tag{18}$$

Therefore

$$\sum_{n=1}^{\infty}\left[\frac{2ip_n^B}{s^2+(p_n^B)^2} - \frac{2ip_n^B}{\frac{1}{4}+(p_n^B)^2} - \frac{2ip_n^F}{s^2+(p_n^F)^2} + \frac{2ip_n^F}{\frac{1}{2}+(p_n^F)^2}\right] - 4(g-1)\gamma_E - A_1$$
$$= \frac{\Delta n_0^{(0)}}{(s-\frac{1}{2})(s+\frac{1}{2})} + 4(g-1)\Psi(s+\tfrac{1}{2}) + \frac{R_1'(s)}{R_1(s)}. \tag{19}$$

h_s has the symmetry $h_s = h_{-s}$. Writing down Eq. (19) for $s \to -s$ and subtracting it from Eq. (19) gives with $\Psi(\frac{1}{2}+s) = \Psi(\frac{1}{2}-s) + \pi\tan\pi s$ [39, p. 14] the functional equation in differential form for the R_1-function,

$$\frac{d}{ds}\ln R_1(s)R_1(-s) = -4(g-1)\pi\tan\pi s. \tag{20}$$

Of course, every information about the nontrivial zeros is lost. This equation can be integrated yielding

$$R_1(s)R_1(-s) = \tilde{A}_1(\cos\pi s)^{4(g-1)}, \tag{21}$$

where $\tilde{A}_1$ is a constant given e.g. by $\tilde{A}_1 = R_1(s_0)R_1(-s_0)(\cos\pi s_0)^{4(1-g)}$ with some $s_0 \in \mathbf{C}$, which is however, independent of s_0. We have, e.g. (no small Eigenvalue $\lambda = \frac{1}{2}$ assumed) for $s_0 = 0$: $\tilde{A}_1 = R_1^2(0)$.

2. The Selberg Super Zeta-Function Z_0. In this section I derive the analytic properties of the Selberg super zeta-function Z_0 and present a functional equation connecting the two Selberg super zeta-functions Z_0 and Z_1.[1] Let us consider the test function ($\mathrm{Re}(s) > \frac{3}{2}$):

$$h_s(p) = \left.\frac{1}{\lambda(1-\lambda) - s(1-s)}\right|_{\lambda = ip+(1/2)} = \frac{1}{p^2+(s-\frac{1}{2})^2}. \tag{22}$$

---

[1] It is also possible to derive the analytic properties of Z_0 similarly as the reasoning for Z_1 as in the previous section. The choice of the test function ($\mathrm{Re}(\sigma) > 1$, $\mathrm{Re}(\sigma) > 1$):

$$h_s(p) = \left.\frac{2\lambda}{s^2-\lambda^2} - \frac{2\lambda}{\sigma^2-\lambda^2}\right|_{\lambda=(1/2)+ip} = \frac{1+2ip}{s^2-(\frac{1}{2}+ip)^2} - \frac{1+2ip}{\sigma^2-(\frac{1}{2}+ip)^2}$$

turns out to be the correct one

This gives at once $A_0 = 0$ because h_s is an even function in p. Furthermore for $g(u)$:

$$g(u) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-iup}}{(s-\frac{1}{2})^2 + p^2} dp = \frac{1}{2s-1} e^{-(s-(1/2))|u|}, \tag{23}$$

([39, p. 431]). Thus for $G(u, \chi)$:

$$G(kl_\gamma, \chi_\gamma) = \frac{e^{-(s-(1/2))kl_\gamma}}{s-\frac{1}{2}} \left(1 - \chi_\gamma{}^k \cosh \frac{kl_\gamma}{2}\right). \tag{24}$$

Therefore we get for the right-hand side of the supertrace formula

$$\begin{aligned}
&\sum_{\{\gamma\}_p} \sum_{k=1}^{\infty} \frac{l_\gamma}{2 \sinh \frac{kl_\gamma}{2}} \frac{e^{-(s-(1/2))kl_\gamma}}{s-\frac{1}{2}} \left(1 - \chi_\gamma{}^k \cosh \frac{kl_\gamma}{2}\right) \\
&= \frac{1}{2s-1} \sum_{\{\gamma\}_p} \sum_{k=1}^{\infty} \frac{l_\gamma}{1 - e^{-kl_\gamma}} [2e^{-skl_\gamma} - \chi_\gamma{}^k e^{-(s-(1/2))kl_\gamma} - \chi_\gamma{}^k e^{-(s+(1/2))kl_\gamma}] \\
&= \frac{1}{2s-1} \frac{d}{ds} \ln \left[\frac{Z_0^2(s)}{Z_1(s-\frac{1}{2}) Z_1(s+\frac{1}{2})}\right].
\end{aligned} \tag{25}$$

Here use has been made of the properties of the logarithmic derivative of the super zetz-functions:

$$\frac{d}{ds} \ln Z_q(s) = \frac{d}{ds} \ln \sum_{\{\gamma\}_p} \prod_{k=0}^{\infty} [1 - \chi_\gamma{}^q e^{-(s+k)l_\gamma}] = \sum_{\{\gamma\}_p} \sum_{n=1}^{\infty} \frac{l_\gamma \chi_\gamma{}^{qn} e^{-snl_\gamma}}{1 - e^{-nl_\gamma}}. \tag{26}$$

Thus we find the supertrace formula for the test function h_s

$$\sum_{n=1}^{\infty} [h_s(p_n^B) - h_s(p_n^F)] - \frac{\Delta n_0^{(0)}}{s(1-s)} = \frac{1}{2s-1} \frac{d}{ds} \ln \left[\frac{Z_0^2(s)}{Z_1(s-\frac{1}{2}) Z_1(s+\frac{1}{2})}\right]. \tag{27}$$

Due to our knowledge of the analytic properties of the Z_1-function we can deduce the analytic properties of the Z_0-function. Therefore:

- $s = -k$ ($k \in \mathbf{N}_0$): There are trivial zeros with multiplicity $(g-1)(4k+2)$.

Since both sides of Eq. (27) must be regular for $s = \frac{1}{2} \pm ip_n - k$ ($k \in \mathbf{N}_0$) I get further

- $s = \frac{1}{2} + ip_n^{B(F)} - k$: There are zeros ($p_n^B$) and poles ($p_n^F$),
- $s = -\frac{1}{2} - ip_n^{B(F)} - k$: there are poles ($p_n^B$) and zeros ($p_n^F$),

with the same multiplicity as the corresponding Eigenvalue, respectively. Similarly, as for Z_1, we get an infinite number of critical lines. Note that there is no zero for $s = 1$ as for the ordinary Selberg zeta-function [36]. By the same considerations as for Z_1 we get the structure of the Z_0-function for the "small Eigenvalues" ($k \in \mathbf{N}_0$):

- for $s = \lambda_n^{B(F)} - k$ there are zeros (poles) and
- for $s = 1 - \lambda_n^{B(F)} - k$ there are poles (zeros) of the same multiplicity as the corresponding Eigenvalue of $\Box$, respectively.

A functional equation for R_0 can be derived, as can be seen in the next section.

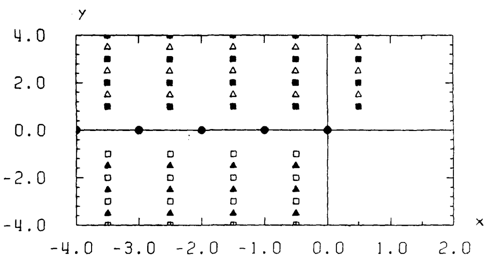

Fig. 2. Zeros and poles of the zeta-function Z_0

Of course Eq. (27) and Z_0 can be extended meromorphically to all $s \in \Lambda_\infty$. In Fig. 2 I have sketched the analytic structure of Z_0. The trivial zeros are indicated by field dots, the position of the bosonic zeros and poles by filled and empty squares, respectively, and the position of the fermionic zeros and poles by filled and empty triangles, respectively. The small Eigenvalues are not considered. The x- and y-axis are again taken at the body of Λ_∞, i.e. $(\Lambda_\infty)_{\text{Body}} = \mathbf{C}$. The y-axis is taken in arbitrary units.

The test function h_s is invariant under the change $s \to 1 - s$. Performing this substitution in Eq. (27) and subtracting it from (27) yields the functional equation

$$\frac{d}{ds}\ln\left[\frac{Z_0^2(s)}{Z_1(s-\frac{1}{2})Z_1(s+\frac{1}{2})}\right] = \frac{d}{ds}\ln\left[\frac{Z_0^2(1-s)}{Z_1(\frac{1}{2}-s)Z_1(\frac{3}{2}-s)}\right]. \tag{28}$$

Let us consider the functional equation (20) for the R_1-function and perform the substitution $s \to \frac{1}{2} - s$. By expressing the R_1-function by the quotient of the Z_1-functions, this yields

$$\frac{d}{ds}\ln\left[\frac{Z_1(\frac{1}{2}-s)Z_1(s-\frac{1}{2})}{Z_1(\frac{3}{2}-s)Z_1(s+\frac{1}{2})}\right] = 4\pi(g-1)\cot\pi s. \tag{29}$$

Thus we find by combining Eqs. (28) and (29) the functional equation in differential form connecting Z_0 and Z_1:

$$\frac{d}{ds}\ln\left[\frac{Z_1(\frac{1}{2}-s)Z_0(s)}{Z_1(\frac{1}{2}+s)Z_0(1-s)}\right] = 2\pi(g-1)\cot\pi s. \tag{30}$$

The functional equation can be integrated yielding (in ref. [8] the $(\sin\pi s)^{2(g-1)}$-dependence is missing):

$$\frac{Z_1(\frac{1}{2}-s)Z_0(s)}{Z_1(\frac{1}{2}+s)Z_0(1-s)} = C_0(\sin\pi s)^{2(g-1)}, \tag{31}$$

where C_0 is, e.g. given by $Z_1(\frac{1}{2}-s_0)Z_0(s_0)/[Z_1(\frac{1}{2}+s_0)Z_0(1-s_0)(\sin\pi s_0)^{2(1-g)}]$ with some $s_0 \in \mathbf{C}$ which is, however, independent of s_0, e.g. for $s_0 = \frac{1}{2}$: $C_0 = Z_1(0)/Z_1(1) = R_1(0) = \sqrt{\tilde{A}_1}$.

3.. The Super Zeta-Function Z_S. To get around the difficulties of the combination of the Z_0 and Z_1 functions for general test functions h in the Selberg supertrace formula let us (following Matsumoto, Uehara and Yasui [42]) define the super zeta-function Z_S:

$$\begin{aligned} Z_S(s) &:= \prod_{\{\gamma\}_p}\prod_{n=0}^{\infty} \operatorname{sdet}\left[1-\operatorname{diag}(1, e^{-l_\gamma}, \chi_\gamma e^{-(l_\gamma/2)}, \chi_\gamma e^{-(l_\gamma/2)})e^{-(s+n)l_\gamma}\right] \\ &= \prod_{\{\gamma\}_p}\prod_{n=0}^{\infty} \frac{[1-e^{-(s+n)l_\gamma}][1-e^{-(s+n+1)l_\gamma}]}{[1-\chi_\gamma e^{-(s+n+1/2)l_\gamma}]^2} = \frac{Z_0(s)Z(s+1)}{Z_1^2(s+\frac{1}{2})}. \end{aligned} \tag{32}$$

Let us consider the resolvent of $\square_0^2$: $R_s(\square_0^2) = (s^2 - \square_0^2)^{-1}(\operatorname{Re}(s) > 1)$. Therefore

$$h(p) = \frac{1}{s^2-\lambda^2}\bigg|_{\lambda=(1/2)+ip} = \frac{1}{(s^2-\frac{1}{4}) - ip + p^2}. \tag{33}$$

We first calculate the Fourier transform of $h(p)$:

$$g(u) = \frac{1}{2\pi}\int_{-\infty}^{\infty} h(p)e^{-iup}\,dp = g_1(u) + g_2(u), \tag{34}$$

where

$$\begin{aligned} g_1(u) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\cos up}{(s^2-\frac{1}{2}) - ip + p^2}\,dp = g_1(-u), \\ g_2(u) &= \frac{-i}{2\pi}\int_{-\infty}^{\infty} \frac{\sin up}{(s^2-\frac{1}{2}) - ip + p^2}\,dp = -g_2(-u). \end{aligned} \tag{35}$$

Using the integrals [23, p. 407]:

$$\left.\begin{aligned} \int_{-\infty}^{\infty} \frac{(b+cx)\sin ax}{p+qx+x^2}\,dx &= \left[\frac{cq-b}{\sqrt{p-q^2}}\sin aq + c\cos aq\right]\pi e^{-a\sqrt{p-q^2}} \\ \int_{-\infty}^{\infty} \frac{(b+cx)\cos ax}{p+qx+x^2}\,dx &= \left[\frac{b-cq}{\sqrt{p-q^2}}\cos aq + c\sin aq\right]\pi e^{-a\sqrt{p-q^2}} \end{aligned}\right\}. \tag{36}$$

We get for $u > 0$:

$$\begin{aligned} g_1(u) &= \frac{1}{2s}\cosh\frac{u}{2}e^{-us}, \\ g_2(u) &= \frac{1}{2s}\sinh\frac{u}{2}e^{-us}. \end{aligned} \tag{37}$$

Therefore ($u \in \mathbf{R}$):

$$g(u) = \frac{1}{2s}e^{(u/2)-s|u|}, \tag{38}$$

which gives for $G(u,\chi)$

$$G(u,\chi_\gamma) = \frac{1}{s}e^{-us}\left(\cosh\frac{u}{2} - \chi_\gamma\right), \tag{39}$$

and the right-hand side of the supertrace formula reads:

$$\frac{1}{2s}\sum_{\{\gamma\}_p}\sum_{k=1}^{\infty}\frac{l_\gamma}{e^{kl_\gamma/2}-e^{-kl_\gamma/2}}e^{-skl_\gamma}(e^{kl_\gamma/2}+e^{-kl_\gamma/2}-2\chi_\gamma{}^k)=\frac{1}{2s}\frac{Z_S'(s)}{Z_S(s)}. \tag{40}$$

For the A_0-term:

$$\begin{aligned}A_0&=i(g-1)\int_{-\infty}^{\infty}h(p)\tanh\pi p\,dp\\&=i\frac{g-1}{\pi}\int_0^{\infty}\frac{du}{\sinh\frac{u}{2}}\int_{-\infty}^{\infty}\frac{\sin up}{(s^2-\frac{1}{4})-ip+p^2}dp=\frac{1-g}{s^2},\end{aligned} \tag{41}$$

where the integral (36a) has been used. Therefore we have for the resolvent kernel the supertrace formula

$$\sum_{n=1}^{\infty}\left[\frac{1}{s^2-(\lambda_n^B)^2}-\frac{1}{s^2-(\lambda_n^F)^2}\right]+\frac{\Delta n_n^{(0)}+g-1}{s^2}=\frac{1}{2s}\frac{Z_S'(s)}{Z_S(z)}. \tag{42}$$

Equation (42) and Z_S can be extended meromorphically to all $s\in\Lambda_\infty$. We can read off the simple analytic structure of Z_S:

- $s=0$ there is a zero with multiplicity $2(g-1+\Delta n_0^{(0)})$,
- $s=\pm(\frac{1}{2}+ip_n^B)$ there are zeros (poles) and
- $s=\pm(\frac{1}{2}+ip_n^F)$ there are poles (zeros),

with the same multiplicity as the corresponding Eigenvalue of $\Box$, respectively. A very simple functional relation can be deduced from Eq. (42), reading

$$\frac{d\ln Z_S(s)}{ds}=\frac{d\ln Z_S(-s)}{ds}. \tag{43}$$

In terms of Z_0 and Z_1 Eq. (43) gives (in comparison to ref. [42] one has to take the limit $\alpha=1$ in the formulas):

$$\frac{d}{ds}\ln\frac{Z_0(s)Z_0(s+1)}{Z_1^2(s+\frac{1}{2})}=\frac{d}{ds}\ln\frac{Z_0(-s)Z_0(1-s)}{Z_1^2(\frac{1}{2}-s)}. \tag{44}$$

Equation (43) or (44), respectively, integrated gives $Z_S(s)=Z_S(-s)$, thus $Z_S(s)$ is an even function in s. Combining Eqs. (20), (30) and (44) I deduce the functional equation for the R_0 function, which reads:

$$\frac{d}{ds}\ln R_0(s)R_0(-s)=4\pi(g-1)\cot\pi s. \tag{45}$$

Equation (45) can be integrated to give

$$R_0(s)R_0(-s)=B_0(\sin\pi s)^{4(g-1)}, \tag{46}$$

where the constant B_0 is e.g. given by $B_0=R_0(s_0)R_0(-s_0)(\sin\pi s_0)^{4(1-g)}$ with some $s_0\in\mathbf{C}$, where B_0 is independent of s_0. We have, e.g. for $s=\pm\frac{1}{2}$. $B_0=Z_0(-\frac{1}{2})/Z_0(\frac{3}{2})=R_0(-\frac{1}{2})$. Of course, any information about the nontrivial zeros and poles is lost.

A similar relation holds also for the ordinary Selberg zeta-function:

$$R(s)R(-s)=\frac{Z(s)Z(-s)}{Z(1+s)Z(1-s)} \tag{47}$$

however, in this case the integration constant is given by $B=2^{4(g-1)}$.

From Eqs. (21), (31) and (46) many relations linking Z_0 and Z_1 for particular arguments can be deduced, e.g.

$$B_0=\frac{Z_0(-\frac{1}{2})}{Z_0(\frac{3}{2})}=\frac{Z_1(-1)Z_1(1)}{Z_1(2)Z_1(0)}=\frac{Z_1^2(0)}{Z_1^2(1)}=C_0^2=\tilde{A}_1. \tag{48}$$

However, I do not see any valuable consequence as, e.g. determining from these relations the constants $\tilde{A}_1, B_0$ and C_0 like for the Selberg zeta-function. It is also not obvious to me to derive from these relations a functional relation for Z_0 or Z_1, respectively, like for the ordinary Selberg zeta-function:

$$Z(s)=Z(1-s)\left[(2\pi)^{1-2s}\frac{G(s)G(s+1)}{G(1-s)G(2-s)}\right]^{2(g-1)} \tag{49}$$

(here $G(z)$ denotes the Barnes G-function, e.g. [23, p. 937; 57]). On the contrary: I believe that such relations do not exist for Z_0 and Z_1, because we have an infinite number of critical lines for these two functions. The immediate consequence of such relations, if they would exist, would be that we could solve the Eigenvalue problem for the operator $\square_0$ by just looking at the poles (for λ_n^B) and zeros (for λ_n^F) at, e.g. the critical line $\mathrm{Re}(s)=-\frac{1}{2}$ for $Z_0(s)$. The values at the critical line $\mathrm{Re}(s)=-\frac{1}{2}$ for $Z_0(s)$ would be related to the line $\mathrm{Re}(s)=\frac{3}{2}$, where Z_0 could be easily calculated by Eq. (1) once a sufficiently large enough set of geodesics $\{l_\gamma\}$ would be known. This is, however, very unlikely (but not a proof).

V. Spectra and Determinants

1. Resolvent and Heat-Kernel. Since $\square_m^2$ is not a positive definite operator I calculate the superdeterminant of $c^2-\square_m^2$ for $\mathrm{Re}(c)>m$ and analytically continue in c. Similar considerations have been done by Aoki [2] by means of the supertrace of the heat kernel of $\square_m^2$. Fitted with the knowledge of the analytical properties of the Selberg super zeta-functions I can avoid the indirect reasoning of Aoki to get the superdeterminants in compact form. For this purpose I use the functional relations for Z_0 and Z_1 of the previous chapter. These functional relations have not been available in [2]; without proof Aoki has used the functional relation of the Selberg zeta-function, assuming that it is also valid in the super case. As discussed at the end of the previous section this seems to be very unlikely that such functional relations exist. Furthermore, statements of the spectrum of the operators $\hat{\square}_m$ and its relation to the spectrum of $\square$ can be made (and similarly for $\square_m$ which I do not consider explicitly).

Let be $m\in\mathbb{N}_0$. Let us calculate the superdeterminants by the ζ-function regularization. This method of regularization was introduced by Ray and Singer [52] in differential geometry and Hawking [35] in field theory. We get:

$$\mathrm{sdet}\,(c^2 - \Box_m^2) = \exp\left[-\frac{\partial}{\partial s}\zeta_m(s;c)\Big|_{s=0}\right]$$

$$\zeta_m(s;c) = \mathrm{str}\,[(c^2 - \Box_m^2)^{-s}] = \frac{1}{\Gamma(s)}\int_0^\infty dt\,t^{s-1}\,\mathrm{str}\,\{\exp[-t^2 - \Box_m^2)]\}, \tag{1}$$

where use has been made of the integral [23, p. 317]:

$$\int_0^\infty x^{\nu-1}e^{-\mu x}dx = \mu^{-\nu}\Gamma(\nu). \tag{2}$$

The function h corresponding to the heat-kernel of $(c^2 - \Box_m^2)$ reads

$$h_{hk}(s) = e^{t[(s+(m/2))^2 - c^2]}. \tag{3}$$

Therefore for $g(u)$

$$g(u) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iup}\,h_{hk}(ip + \tfrac{1}{2})dp = \frac{1}{\sqrt{4\pi t}}\exp\left[-\frac{u^2}{4t} - c^2 t + (m+1)\frac{u}{2}\right]. \tag{4}$$

This gives

$$G(u,\chi) = \frac{1}{\sqrt{\pi t}}e^{-u^2/4t - c^2 t}\left[\cosh(m+1)\frac{u}{2} - \chi\cosh m\frac{u}{2}\right],$$

$$g(u) - g(-u) = \frac{1}{\sqrt{\pi t}}e^{-u^2/4t - c^2 t}\sinh(m+1)\frac{u}{2}. \tag{5}$$

Splitting the calculation of $\zeta_m(s;c)$ into two terms corresponding to the identity transformation and the length term, respectively, gives:

$$\zeta_m(s;c) = \zeta_m^I(s;c) + \zeta_m^\Gamma(s;c). \tag{6}$$

Let us first calculate ζ_m^I:

$$\zeta_m^I(s;c) = \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}A_0^{(m)}(t)dt,$$

$$A_0^{(m)}(t) = \frac{1-g}{\sqrt{\pi t}}e^{-c^2 t}\int_0^\infty e^{-(u^2/4t)}\frac{\sinh(m+1)\frac{u}{2}}{\sinh\frac{u}{2}}\cosh m\frac{u}{2}\,du = (1-g)e^{-c^2 t}\sum_{k=0}^{m} e^{k^2 t}. \tag{7}$$

Equation (7) can be e.g. proved by induction [2, 28].

Similarly:

$$A_0^{(-1)} = 0$$

$$A_0^{(-m)}(t) = (g-1)e^{-c^2 t}\sum_{k=0}^{m-2} e^{k^2 t} \quad (m = 2, 3, \ldots). \tag{8}$$

This gives for ζ_m^I:

$$\zeta_m^I(s;c) = \frac{1-g}{\Gamma(s)}\sum_{k=0}^{m}\int_0^\infty t^{s-1}e^{-(c^2-k^2)t}dt = (1-g)\sum_{k=0}^{m}(c^2 - k^2)^{-s}. \tag{9}$$

For later use I easily calculate

$$\frac{\partial}{\partial s}\zeta_m^I(s;c)\bigg|_{s=0}=(g-1)\sum_{k=0}^{m}\ln(c^2-k^2). \tag{10}$$

Let us calculate ζ_m^Γ in two alternative ways. The first is appropriate to the analysis of the spectrum, the second to the calculation of the superdeterminants.
1) The supertrace formula for the heat-kernel now reads:

$$\sum_{n=1}^{\infty}\{e^{t[(\lambda_{n,m}^B)^2-c^2]}-e^{[t(\lambda_{n,m}^F)^2-c^2]}\}=(1-g)e^{-c^2t}\sum_{k=1}^{m}e^{k^2t}$$
$$+\frac{e^{-c^2t}}{\sqrt{4\pi t}}\sum_{\gamma\in\Gamma}\sum_{k=1}^{\infty}\frac{l_\gamma\chi_\gamma{}^{km}}{\sinh\frac{kl_\gamma}{2}}e^{-(k^2l_\gamma{}^2/4t)}\left[\cosh(m+1)\frac{kl_\gamma}{2}-\chi_\gamma{}^k\cosh m\frac{kl_\gamma}{2}\right], \tag{11}$$

and the A_0 term appropriately replaced for negative integers. With the help of Eqs. (2), (7) and the integral [23, p. 340]:

$$\int_0^\infty x^{\nu-1}e^{-(\beta/x)-\gamma x}dx=2\left(\frac{\beta}{\gamma}\right)^{\nu/2}K_\nu(2\sqrt{\beta\gamma}), \tag{12}$$

we get for the supertrace formula of the generalized resolvent kernel:

$$\sum_{n=0}^{\infty}\left[\frac{1}{[c^2-(\lambda_{n,m}^B)^2]^s}-\frac{1}{[c^2-(\lambda_{n,m}^F)^2]^s}\right]$$
$$=(1-g)\sum_{k=0}^{m}\frac{1}{(c^2-k^2)^s}+\frac{1}{\Gamma(s)}\sum_{\gamma\in\Gamma}\sum_{k=1}^{\infty}\frac{l_\gamma\chi_\gamma{}^{km}}{e^{kl_\gamma/2}-e^{-(kl_\gamma/2)}}\left(\frac{kl_\gamma}{2c}\right)^{s-(1/2)}$$
$$\cdot K_{s-(1/2)}(ckl_\gamma)\left[\cosh\left(\frac{m+1}{2}kl_\gamma\right)-\chi_\gamma{}^k\cosh\left(\frac{m}{2}kl_\gamma\right)\right]. \tag{13}$$

This gives explicitly for $s=1$ (m even):

$$\sum_{n=0}^{\infty}\left[\frac{1}{c^2-(\lambda_{n,m}^B)^2}-\frac{1}{c^2-(\lambda_{n,m}^F)^2}\right]$$
$$=(1-g)\sum_{k=0}^{m}\frac{1}{c^2-k^2}+\frac{1}{2c}\frac{d}{dc}\ln\left[\frac{Z_0\left(\frac{m}{2}+c+1\right)Z_0\left(c-\frac{m}{2}\right)}{Z_1\left(c+\frac{m+1}{2}\right)Z_1\left(c+\frac{1-m}{2}\right)}\right], \tag{14}$$

where the logarithmic derivative of the super zeta-functions has been used. For $s=1$ and m odd:

$$\sum_{n=0}^{\infty}\left[\frac{1}{[c^2-(\lambda_n^B)^2]}-\frac{1}{[c^2-(\lambda_n^F)^2]}\right]$$
$$=(1-g)\sum_{k=0}^{m}\frac{1}{c^2-k^2}+\frac{1}{2c}\frac{d}{dc}\ln\left[\frac{Z_1\left(\frac{m}{2}+c+1\right)Z_1\left(c-\frac{m}{2}\right)}{Z_0\left(\frac{m+1}{2}+c\right)Z_0\left(c+\frac{1-m}{2}\right)}\right]. \tag{15}$$

2) Let first m be an even number. Let us consider the representation ($\operatorname{Re}(s)<1$):

$$t^{s-1}=\frac{2}{\Gamma(1-s)}\int_0^\infty \frac{\lambda+c}{[\lambda(\lambda+2c)]^s}e^{-\lambda(\lambda+2c)t}d\lambda, \tag{16}$$

where this integral representation follows with the help of [23, p. 318]. Therefore we get for $\zeta_m\Gamma(c;s)$ with the help of Eq. (11) and the representation $K_{\pm 1/2}(z)=\sqrt{\pi/2z}e^{-z}$:

$$\begin{aligned}\zeta_m^\Gamma(s;c)&=\frac{\sin\pi s}{\pi}\int_0^\infty \frac{d\lambda}{[\lambda(\lambda+2c)]^s}\\ &\quad\cdot 2\sum_{\gamma\in\Gamma}\sum_{k=1}^\infty \frac{l_\gamma}{1-e^{-kl_\gamma}}e^{-kl_\gamma(\lambda+c+1/2)}\left[\cosh\left(\frac{m+1}{2}kl_\gamma\right)-\chi_\gamma{}^k\cosh\left(\frac{m}{2}kl_\gamma\right)\right]\\ &=\frac{\sin\pi s}{\pi}\int_0^\infty \frac{d\lambda}{[\lambda(\lambda+2c)]^s}\frac{d}{d\lambda}\ln\left[\frac{Z_0\left(\frac{m}{2}+\lambda+c+1\right)Z_0\left(\lambda+c-\frac{m}{2}\right)}{Z_1\left(\frac{m+1}{2}+\lambda+c\right)Z_1\left(\frac{m-1}{2}+\lambda+c\right)}\right].\end{aligned} \tag{17}$$

Let be $f(s)=\sin(\pi s)[\lambda(\lambda+2c)]^{-s}]$. Then $f'(s)|_{s=0}=\pi$ and we get for $\zeta'(0;c)$ ($\operatorname{Re}(s)>m$):

$$\begin{aligned}\zeta'(0;c)&=(g-1)\sum_{k=0}^m\ln(c^2-k^2)\\ &\quad+\int_0^\infty d\lambda\frac{d}{d\lambda}\ln\left[\frac{Z_0\left(\lambda+\frac{m}{2}+c+1\right)Z_0\left(\lambda-\frac{m}{2}+c\right)}{Z_1\left(\lambda+\frac{m+1}{2}+c\right)Z_1\left(\lambda+\frac{1-m}{2}+c\right)}\right]\\ &=(g-1)\sum_{k=0}^m\ln(c^2-k^2)-\ln\left[\frac{Z_0\left(c+1+\frac{m}{2}\right)Z_0\left(c-\frac{m}{2}\right)}{Z_1\left(c+\frac{m+1}{2}\right)Z_1\left(c+\frac{1-m}{2}\right)}\right].\end{aligned} \tag{18}$$

Here it was used that $\lim_{s\to\infty}Z_q(s)=1$, which follows at once from the Euler product representation of the Selberg super zeta-functions. Therefore ($m=0,2,\ldots$):

$$\operatorname{sdet}(c^2-\Box_m^2)=\frac{Z_0\left(c+\frac{m}{2}+1\right)Z_0\left(c-\frac{m}{2}\right)}{Z_1\left(c+\frac{m+1}{2}\right)Z_1\left(c+\frac{1-m}{2}\right)c^{2(g-1)}}\prod_{k=1}^m(c^2-k^2)^{1-g}. \tag{19}$$

Similarly ($m=2,4,\ldots$):

$$\operatorname{sdet}(c^2-\Box_{-m}^2)=\frac{Z_0\left(c-\frac{m}{2}+1\right)Z_0\left(c+\frac{m}{2}\right)}{Z_1\left(c+\frac{m+1}{2}\right)Z_1\left(c+\frac{1-m}{2}\right)}\prod_{k=0}^{m-2}(c^2-k^2)^{g-1}. \tag{20}$$

For m an odd number the roles of Z_0 and Z_1 are just reversed and it follows immediately ($m = 1, 3, \ldots$):

$$\operatorname{sdet}(c^2 - \Box_m^2) = \frac{Z_1\left(c + 1 + \frac{m}{2}\right) Z_1\left(c - \frac{m}{2}\right)}{Z_0\left(c + \frac{m+1}{2}\right) Z_0\left(c + \frac{1-m}{2}\right) c^{2g-2}} \prod_{k=0}^{m-2} (c^2 - k^2)^{1-g}. \quad (21)$$

Similarly ($m = 1, 3, \ldots$):

$$\operatorname{sdet}(c^2 - \Box_{-m}^2) = \frac{Z_1\left(c + 1 - \frac{m}{2}\right) Z_1\left(c + \frac{m}{2}\right)}{Z_0\left(c + \frac{m+1}{2}\right) Z_0\left(c + \frac{1-m}{2}\right)} \prod_{k=1}^{m-2} (c^2 - k^2)^{g-1}. \quad (22)$$

Equations (19–22) are the starting points for the calculation of determinants. Because the super zeta-functions are meromorphic functions in Λ_∞, the same holds for the superdeterminants.

Let us denote by $\hat{\Theta}(t) := \operatorname{str}[\exp(t\Box_0^2)]$. Then we have:

$$\operatorname{sdet}(c^2 - \Box_0^2) = \exp\left\{ -\frac{\partial}{\partial s}\left[\frac{1}{\Gamma(s)} \int_0^\infty dt\, t^{s-1} e^{-tc^2} \hat{\Theta}(t) \Big|_{s=0} \right] \right\}. \quad (23)$$

Even if we are in the position to make statements about the t-integral this would be of no use because we have no information about the implicit moduli-dependence and signs which occur in the Grassmann-part, or respectively in the moduli-part in the super Weil–Petersen integration measure in Eq. (II.60). Thus no statement about the growing properties for $\operatorname{sdet}(c^2 - \Box_0^2)$ for increasing genus can be made, similarly to the argueing of Gross and Periwal [32, 34] for the bosonic string. However, we can make some statement about $\hat{\Theta}$ and can derive an equation expressing $\hat{\Theta}$ by the zeta-function Z_S. Let us consider the supertrace formula for the resolvent kernel:

$$\operatorname{str}(c^2 - \Box_0^2)^{-1} = \int_0^\infty e^{-tc^2} \hat{\Theta}(t) dt = \frac{1}{2c} \frac{Z_S'(c)}{Z_S(c)} - \frac{g - 1 + \Delta n_0^{(0)}}{c^2}. \quad (24)$$

This equation can be inverted by the theory of Laplace transformations yielding (see e.g. [19, pp. 129]):

$$\begin{aligned} \hat{\Theta}(t) &= \frac{1}{2\pi} \int_{b-i\infty}^{b+i\infty} \left[\frac{1}{2c} \frac{Z_S'(c)}{Z_S(c)} - \frac{g - 1 + \Delta n_0^{(0)}}{c^2} \right] dc^2 \\ &= -\frac{1}{\sqrt{4\pi t}} \int_0^\infty u e^{-u^2/4t} (\mathscr{L}^{-1} \ln Z_S)(u) du - (g - 1 + \Delta n_0^{(0)}), \end{aligned} \quad (25)$$

where $\mathscr{L}^{-1}$ denotes the inverse Laplace transformation. In particular this gives

$$\hat{\Theta}(0) = -(g - 1 + \Delta n_0^{(0)}); \quad (26)$$

this result is consistent with Eq. (14). Equation (14) gives also that for $t \to \infty$ the supertrace for the heat-kernel for $\Box_m^2$ diverges according to ($m \in \mathbf{N}$)

$$\hat{\Theta}_m(t) = \operatorname{str}[\exp(t\Box_m^2)] \cong (1 - g - \Delta n_0^{(0)}) e^{m^2 t}, \quad (t \to \infty) \quad (27)$$

(and similarly for negative integers), a result found by Aoki [2].

2. *Discussion of the Spectrum.* The operator $\hat{\Box}$ is simpler to study than the operator $\Box_m$, because the trivial contribution $\frac{m}{2}$ to the Eigenvalues has been subtracted $\left(\text{see Eq. (II.40) for the unitary equivalence between } \Box_m \text{ and } \hat{\Box}_m + \frac{m}{2}\right)$. For applying the supertrace formula for the operator $\hat{\Box}_m$ we must change the formulas appropriately – see Eqs. (III.25). We have

$$A_0^{(m)} = (1-g)\int_0^\infty \frac{g(u)-g(-u)}{\sinh\frac{u}{2}} T_m\left(\cosh\frac{u}{2}\right)du,$$

$$G(u,\chi) = g(u) + g(-u) - \chi[g(u)e^{-u/2} + g(-u)e^{u/2}]. \tag{28}$$

The function h has taken on the argument $ip + \frac{1}{2}$. This gives immediately:

$$g(u) = \frac{1}{\sqrt{4\pi t}} e^{-u^2/4t - c^2t + u/2}, \quad G(u,\chi) = \frac{1}{\sqrt{\pi t}} e^{-u^2/4t - c^2 t}\left(\cosh\frac{u}{2} - \chi\right),$$

$$A_0^{(m)} = \frac{1-g}{\sqrt{\pi t}} e^{-c2t}\int_0^\infty e^{-u^2/4t}\cosh m\frac{u}{2} du = (1-g)e^{-(c^2 - m^2/4)t}. \tag{29}$$

Therefore we get for the supertrace formula for the resolvent of $\hat{\Box}_m^2$ for m even ("hatted" quantities belonging to $\hat{\Box}$):

$$\sum_{n=1}^\infty \left[\frac{1}{c^2 - (\hat{\lambda}_{n,m}^B)^2} - \frac{1}{c^2 - (\hat{\lambda}_{n,m}^F)^2}\right] = \frac{1-g}{\left(c - \frac{m}{2}\right)\left(c + \frac{m}{2}\right)} + \frac{1}{2c}\frac{d}{dc}\ln\left[\frac{Z_0(c+1)Z_0(c)}{Z_1^2(c+\frac{1}{2})}\right]; \tag{30}$$

and similarly for m odd:

$$\sum_{n=1}^\infty \left[\frac{1}{c^2 - (\hat{\lambda}_{n,m}^B)^2} - \frac{1}{c^2 - (\hat{\lambda}_{n,m}^F)^2}\right] = \frac{1-g}{\left(c - \frac{m}{2}\right)\left(c + \frac{m}{2}\right)} + \frac{1}{2c}\frac{d}{dc}\ln\left[\frac{Z_1(c+1)Z_1(c)}{Z_0^2(c+\frac{1}{2})}\right]. \tag{31}$$

Analysing for the particular values $c = \varepsilon$ and $c = \pm\frac{m}{2} + \varepsilon$ we get for m even ($|\varepsilon| \ll 1$):

$$\frac{1}{2c}\frac{d}{dc}\ln\left[\frac{Z_0(c+1)Z_0(c)}{Z_1^2(c+\frac{1}{2})}\right] - \frac{g-1}{\left(c - \frac{m}{2}\right)\left(c + \frac{m}{2}\right)}$$

$$\propto \begin{cases} \dfrac{\Delta\hat{n}_0^{(0)}}{\varepsilon^2} & (c = \varepsilon, m = 0), \\[2ex] \pm\dfrac{g - 1 + \Delta\hat{n}_0^{(0)}}{\varepsilon^2} & (c = \varepsilon, m \neq 0), \\[2ex] \pm\dfrac{1-g}{m\varepsilon} & \left(c = \pm\dfrac{m}{2} + \varepsilon, m \neq 0\right) \end{cases} \tag{32}$$

and regularly otherwise up to the nontrivial zeros and poles of Z_0 and Z_1. For m odd we get for $c = \pm\frac{1}{2} + \varepsilon$ and $c = \pm\frac{m}{2} + \varepsilon$ ($|\varepsilon| \ll 1$):

$$\frac{1}{2c}\frac{d}{dc}\ln\left[\frac{Z_1(c+1)Z_1(c)}{Z_0^2(c+\frac{1}{2})}\right] - \frac{g-1}{\left(c-\frac{m}{2}\right)\left(c+\frac{m}{2}\right)}$$

$$\propto \begin{cases} \mp\dfrac{g-1+\Delta\hat{n}_0^{(0)}}{\varepsilon} & (c = \pm\frac{1}{2}+\varepsilon, m = 1), \\ \mp\dfrac{\Delta\hat{n}_0^{(0)}}{\varepsilon} & (c = \pm\frac{1}{2}+\varepsilon, m \neq 1), \\ \pm\dfrac{1-g}{m\varepsilon} & \left(c = \pm\dfrac{m}{2}+\varepsilon, m \neq 1\right) \end{cases} \tag{33}$$

and regularly otherwise up to the nontrivial zeros and poles of Z_0 and Z_1. Let us discuss two scenarios for $\Delta\hat{n}_0^{(0)}$.

1. $\Delta\hat{n}_0^{(0)} = 0$: This yields for the various trivial modes of $\hat{\Box}_m$ for m even:

m even:		*m odd*:	
$\Delta\hat{n}_0^{(0)} = 0$		$\Delta\hat{n}_0^{(1)} = g-1$	
$\Delta\hat{n}_0^{(m)} = g-1$	$(m \neq 0)$,	$\Delta\hat{n}_{\pm 1/2}^{(m)} = 0$	$(m \neq 1)$,
$\Delta\hat{n}_{\pm m/2}^{(m)} = 1-g$	$(m \neq 0)$;	$\Delta\hat{n}_{\pm m/2}^{(m)} = 1-g$	$(m \neq 1)$.

2. $\Delta\hat{n}_0^{(0)} = 1-g$: In a similar way:

m even:		*m odd*:	
$\Delta\hat{n}_0^{(0)} = 1-g$		$\Delta\hat{n}_0^{(1)} = 0$	
$\Delta\hat{n}_0^{(m)} = 0$	$(m \neq 0)$,	$\Delta\hat{n}_{\pm 1/2}^{(m)} = g-1$	$(m \neq 1)$,
$\Delta\hat{n}_{\pm m/2}^{(m)} = 1-g$;	$(m \neq 0)$;	$\Delta\hat{n}_{\pm m/2}^{(m)} = 1-g$	$(m \neq 1)$.

That trivial-modes or trivial Eigenvalues (as the trivial-modes of $\Box_m$) appear can be understood in the view of the corresponding results for the classical Laplacian $-\Delta_m$ as discussed, e.g. by Hejhal [36, p. 408]. Let $\{\lambda_n^{(m)}\}$ be the set of all Eigenvalues of the Laplace-operator $-\Delta_m = -y^2(\partial_x 2 + \partial_y 2) + imy\partial_x$ and $m \geqq 2$. Then ($n \in \mathbf{N}$):

$$\{\lambda_n^{(m)}\} = \left\{\frac{m}{2}\left(1-\frac{m}{2}\right)\right\}_{k=1}^{d} \cup \left\{\lambda_n^{(m-2)} \,|\, \lambda_n^{(m-2)} \neq \frac{m}{2}\left(1-\frac{m}{2}\right)\right\}, \tag{34}$$

where $d = \delta + (g-1)(m-1)$ and δ takes on the values 0 and 1, depending on m. There are several methods of obtaining this result. E.g. one can first consider the trace formula for the (regularized) resolvent-kernel function and deduce this statement from the analytical properties of the Selberg zeta-function (nontrivial Eigenvalues) and the poles occurring in the A_0 term (trivial-modes); second, one can consider commutation relations of the differential operators ∇_z^k acting on tensorfields which give simple recursion formulas for the Laplacian Δ_m depending on the curvature R of the space in question [11].

Equations (32) and (33) give also the relation of the Eigenvalues $\hat{\lambda}_{n,m}$ of $\hat{\Box}_m$ and $\hat{\lambda}_n$ of $\hat{\Box}_0$. I find due to the analytical structure of the super zeta-functions for the nontrivial Eigenvalues:

$$\hat{\lambda}_{n,m} = \hat{\lambda}_n, \quad (n \in \mathbf{N}, m \in \mathbf{N}). \tag{35}$$

This simple result corresponds to the classical one noted in Eq. (34).

3. Determinants and the Fermionic String Integrand. The starting points for the calculation of determinants of the operator $\Box_m^2$ are Eqs. (19–22) which all can be analytically continued to $c = 0$ (including omission of zero-modes if necessary). Let us first consider Eq. (19) for $m = 0$. Performing the limit $c \to \varepsilon$ for $|\varepsilon| \ll 1$ one gets

$$\operatorname{sdet}(-\Box_0^2) = \frac{1}{(2g-2)!} \cdot \frac{Z_0(1) Z_0^{(2g-2)}(0)}{[\tilde{Z}_1(\frac{1}{2})]^2} \varepsilon^{2\Delta n_0^{(0)}}. \tag{36}$$

Here I have denoted by $\tilde{Z}_1(\frac{1}{2})$ the appropriate derivative or residuum of Z_1 at $s = \frac{1}{2}$, depending whether $\Delta n_0^{(0)} \leqq 0$ or $\Delta n_0^{(0)} > 0$, respectively. To make this quantity well-defined we subtract from $\operatorname{sdet}(-\Box_0^2)$ the zero-mode which is denoted by priming the sdet. Using further the functional relation (IV.31) for Z_0 and Z_1 we get finally:

$$\operatorname{sdet}'(-\Box_0^2) = (-1)^{\Delta n_0^{(0)}} \left[\pi^{g-1} \frac{Z_0(1)}{\tilde{Z}_1(\frac{1}{2})} \right]^2 \frac{Z_1(0)}{Z_1(1)}. \tag{37}$$

For calculating the superdeterminant for m even and $m \geqq 2$ a subtraction of zero- or trivial-modes is not necessary. Proceeding similarly as for $m = 0$ we get for $m = 2, 4, \ldots$:

$$\operatorname{sdet}(-\Box_m^2) = \left[\left(\frac{\pi}{m!} \right)^{g-1} \frac{Z_0\left(1 + \frac{m}{2}\right)}{Z_1\left(\frac{m+1}{2}\right)} \right]^2 \frac{Z_1(0)}{Z_1(1)}. \tag{38}$$

Similarly ($m = 2, 4, \ldots$):

$$\operatorname{sdet}(-\Box_{-m}^2) = \left[\left(\frac{(m-2)!}{\pi} \right)^{g-1} \frac{Z_0\left(\frac{m}{2}\right)}{Z_1\left(\frac{m+1}{2}\right)} \right]^2 \frac{Z_1(1)}{Z_1(0)}. \tag{39}$$

For $m = 1, 3, \ldots$:

$$\operatorname{sdet}(-\Box_m^2) = \left[\left(\frac{\pi}{im!} \right)^{g-1} \frac{Z_1\left(1 + \frac{m}{2}\right)}{Z_0\left(\frac{m+1}{2}\right)} \right]^2 \frac{Z_1(0)}{Z_1(1)}; \tag{40}$$

and $m = 3, 5, \ldots$:

$$\operatorname{sdet}(-\Box^2_{-m}) = \left[\left(\frac{(m-2)!\,i}{\pi}\right)^{g-1} \frac{Z_1\left(\frac{m}{2}\right)}{Z_0\left(\frac{m+1}{2}\right)}\right]^2 \frac{Z_1(1)}{Z_1(0)}. \tag{41}$$

Note the differences to ref. [2] which are due to the additional super zeta-functions. The case of $\Box^2_{-1}$ must be treated separately because of the appearance of zero-modes which must be subtracted. Therefore denoting the omission of zero-modes by priming the super determinant we get

$$\operatorname{sdet}'(-\Box^2_{-1}) = (-1)^{\Delta n_0^{(0)}} \left[\pi^{1-g} \frac{\tilde{Z}_1(\frac{1}{2})}{Z_0(1)}\right]^2 \frac{Z_1(1)}{Z_1(0)}. \tag{42}$$

From Chap. II we know that the relevant string integrand is given by $\operatorname{sdet}'(-\Box_0^2)$ and $\operatorname{sdet}(-\Box_2^2)$. Equations (37) and (38) yield:

$$\begin{aligned}
&[\operatorname{sdet}'(-\Box_0^2)]^{-5/2}[\operatorname{sdet}(-\Box_2^2)]^{1/2} \\
&= (-1)^{5/2\Delta n_0^{(0)}} \left(\pi^{g-1} \frac{Z_0(1)}{\tilde{Z}_1(\frac{1}{2})}\right)^{-5} \left(\frac{\pi}{2}\right)^{g-1} \frac{Z_0(2)}{Z_1(\frac{3}{2})} \left(\frac{Z_1(1)}{Z_1(0)}\right)^2,
\end{aligned}$$

or alternatively

$$= (-1)^{5/2\Delta n_0^{(0)}} \left(\pi^{g-1} \frac{Z_0(1)}{\tilde{Z}_1(\frac{1}{2})}\right)^{-5} \left(\frac{\pi}{2}\right)^{g-1} \frac{Z_0(2)}{Z_1(\frac{3}{2})} \frac{Z_0(\frac{3}{2})}{Z_0(-\frac{1}{2})}, \tag{43}$$

and I conclude that this expression is well defined. Furthermore for Z_g of Eq. (II.60):

$$\begin{aligned}
Z_g &= \int_{sM_g} d\mu_{\mathrm{SWP}} [\operatorname{sdet}'(-\Box_0^2)]^{-5/2}[\operatorname{sdet}(-\Box_2^2)]^{1/2} \\
&= \left(\frac{1}{2\pi^4}\right)^{g-1} \int_{sM_g} d\mu_{\mathrm{SWP}} (-1)^{5/2\Delta n_0^{(0)}} \left(\frac{Z_0(1)}{\tilde{Z}_1(\frac{1}{2})}\right)^{-5} \frac{Z_0(2)}{Z_1(\frac{3}{2})} \left(\frac{Z_1(1)}{Z_1(0)}\right)^2.
\end{aligned} \tag{44}$$

Note the appearance of the various ratios of the Selbrg super zeta-functions. The main difference to Aoki [2] who first calculated super determinants of Laplace-Dirac operators lies in the additional factor $[Z_1(1)/Z_1(0)]^2$ in the superdeterminants. This factor is unambiguously given by the functional equations which have been used to derive Eq. (43) and it changes the super-moduli dependence of the integrand.

Finally we can discuss the behaviour of the fermionic string integrand for the case of degenerate super Riemann surfaces. For this purpose let us consider such a surface, i.e. a pinching process takes place and at least the length of one geodesic vanishes. Let l_0 be the geodesics of $\gamma_0 \in \Gamma$ with $(l_0)_{\mathrm{Body}} < (l_\gamma)_{\mathrm{Body}}$ for all $\gamma \in \Gamma$ with $\gamma \neq \gamma_0$. Let us introduce the partial zeta-functions $\mathscr{Z}_q(s) = \mathscr{Z}_q(s, l_0)$ with

$$\mathscr{Z}_q(s) := \prod_{n=0}^{\infty} [1 - \chi_{\gamma_0}^q e^{-(s+n)l_0}], \quad (q = 0, 1, \operatorname{Re}(s) > 1). \tag{45}$$

For the entire zeta-functions one has

$$Z_q(s)=[\mathscr{Z}_q(s)]^{g(l_0)}\prod_{\substack{\gamma\in\Gamma\\ \gamma\neq\gamma_0}}\prod_{k=0}^{\infty}[1-\chi_{\gamma_0}^q e^{-(s+k)l_\gamma}],\quad (\mathrm{Re}(s)>1), \tag{46}$$

where $g(l_0)$ denotes the multiplicity of l_0. A discussion for the bosonic string is due to Wolpert [61] who showed that for $l_0\to 0$ one has (set $q=0$ in Eq. (45) and interpret all quantities in terms of the bosonic case)

$$\mathscr{Z}(s)\propto l_0^{(1/2)-s}\exp\left(-\frac{\pi^2}{6l_0}+O(l_0)\right),\quad (l_0\to 0). \tag{47}$$

This asymptotic behaviour has the immediate consequence that the bosonic string has a divergence due to geodesics of zero-length. We can generalize this result to the fermionic string. To see this let us start by taking the logarithm of partial zero-function:

$$\begin{aligned}
-\ln\mathscr{Z}_q(s)&=-\sum_{n=0}^{\infty}\ln(1-\chi_{\gamma_0}^q e^{-(s+n)l_0})\\
&=\sum_{n=0}^{\infty}\sum_{k=1}^{\infty}\frac{1}{k}\chi_{\gamma_0}^k e^{-k(s+n)l_0}=\sum_{k=1}^{\infty}\frac{\chi_{\gamma_0}^k e^{-ksl_0}}{k(1-e^{-kl_0})}\\
&=\frac{1}{l_0}\sum_{k=1}^{\infty}\frac{\chi_{\gamma_0}^k e^{-ksl_0}}{k^2}+\frac{1}{2}\sum_{k=1}^{\infty}\frac{\chi_{\gamma_0}^k e^{-ksl_0}}{k}+\frac{l_0}{12}\sum_{k=1}^{\infty}\chi_{\gamma_0}^k e^{-ksl_0}+\cdots,
\end{aligned} \tag{48}$$

where the denominator was expanded as

$$\frac{1}{1-e^{-kl_0}}=\frac{1}{kl_0}+\frac{1}{2}+\frac{kl_0}{12}+O(k^2l_0^2). \tag{49}$$

The logarithm in Eq. (48) was expanded by considering the integrand from a geometric power series. For the various sums we get

$$\begin{aligned}
\frac{1}{l_0}\sum_{k=1}^{\infty}\frac{\chi_{\gamma_0}^k e^{-ksl_0}}{k^2}&=\frac{1}{l_0}z\Phi(z,2,1),\quad (z=\chi_{\gamma_0}e^{-sl_0})\\
&=\frac{1}{l_0}\left(\frac{\pi^2}{6}+\ln\chi_{\gamma_0}-sl_0+(sl_0-\ln\chi_{\gamma_0})\ln(sl_0-\ln\chi_{\gamma_0})\right)+O(l_0),
\end{aligned} \tag{50}$$

where $\Phi(z,s,a):=\sum_{n=0}^{\infty}\frac{z^n}{(a+n)^s}$ is Lerch's transcendent [39, p. 32] with the expansion:

$$\Phi(z,m,a)=z^{-a}\left\{\sum_{\substack{n=0\\ n\neq m-1}}^{\infty}\zeta(m-n,a)\frac{(\ln z)^n}{n!}+\frac{(\ln z)^{m-1}}{(m-1)!}\left[\Psi(m)-\Psi(a)-\ln\left(\ln\frac{1}{z}\right)\right]\right\}. \tag{51}$$

(Note $\zeta(s,1)=\zeta(s),\zeta(2)=\pi^2/6$ and $\Psi(2)-\Psi(1)=1$.) Further

$$\begin{aligned}
\frac{1}{2}\sum_{k=1}^{\infty}\frac{\chi_{\gamma_0}^k e^{-ksl_0}}{k}&=-\tfrac{1}{2}\ln(1-\chi_{\gamma_0}e^{-sl_0})\\
&=-\tfrac{1}{2}\ln(sl_0)+O(l_0),\quad (\chi_{\gamma_0}=1),
\end{aligned} \tag{52}$$

$$\frac{l_0}{12}\sum_{k=1}^{\infty}\chi_{\gamma_0}^k e^{-skl_0}=\frac{l_0}{12}\frac{\chi_{\gamma_0}e^{-sl_0}}{1-\chi_{\gamma_0}e^{-sl_0}}$$
$$=\frac{1}{sl_0}-\frac{1}{2}+O(l_0),\qquad (\chi_{\gamma_0}=1). \tag{53}$$

The last two expansions are valid for $(sl_0)_{\mathrm{Body}}>0$. Furthermore no singularities appear for $\chi_{\gamma_0}=-1$. Therefore we have in the case of $\chi_{\gamma_0}=1$ the expansion

$$-\ln\mathscr{Z}_0(s)=\frac{\pi^2}{6l_0}+(s-\tfrac{1}{2})\ln l_0+\text{const.}+O(l_0), \tag{54}$$

which is equivalent with Eq. (51). For $\chi_{\gamma_0}=-1$ things are changed and we get:

$$-\ln\mathscr{Z}_1(s)|_{\chi_{\gamma_0}=-1}=\frac{1}{l_0}\left(\frac{\pi^2}{6}+i\pi\right)-\frac{i\pi}{l_0}\ln\left(l_0-\frac{i\pi}{s}\right)+s\ln\left(l_0-\frac{i\pi}{s}\right)+\text{const.}+O(l_0). \tag{55}$$

Therefore we have to discriminate between $\chi_{\gamma_0}=1$ and $\chi_{\gamma_0}=-1$. Let us first assume that the character χ_{γ_0} corresponding to the smallest geodesic is positive or that this can be achieved by an appropriate redefinition of the $4g$ generators γ_i,γ_i^{-1} $(i=1,\dots,2g)$. In the relevant combinations for $\operatorname{sdet}(-\square_m^2)$ we get

$$\begin{aligned}
&\frac{\mathscr{Z}_0\left(s-\frac{m}{2}\right)\mathscr{Z}_0\left(s+1+\frac{m}{2}\right)}{\mathscr{Z}_1\left(s+\frac{m+1}{2}\right)\mathscr{Z}_1\left(s+\frac{1-m}{2}\right)}\propto\text{const.},\quad (m\in\mathbf{Z},\text{ even},\ l_0\to 0),\\
&\frac{\mathscr{Z}_1\left(s-\frac{m}{2}\right)\mathscr{Z}_1\left(s+1+\frac{m}{2}\right)}{\mathscr{Z}_0\left(s+\frac{m+1}{2}\right)\mathscr{Z}_0\left(s+\frac{1-m}{2}\right)}\propto\text{const.},\quad (m\in\mathbf{Z},\text{ odd},\ l_0\to 0),
\end{aligned} \tag{56}$$

where the const. may depend on s. Therefore in this case the determinants are proportional to a constant ($m\in\mathbf{Z}$) and thus the fermionic string integrand is finite.

In the case of $\chi_{\gamma_0}=-1$ things are changed and we get in the limit $l_0\to 0$, e.g. for $m=0$,

$$\begin{aligned}
&\frac{\mathscr{Z}_0(s)\mathscr{Z}_0(s+1)}{\mathscr{Z}_1^2(s+\frac{1}{2})}\propto e^{2i\pi/l_0}l_0^{-2s}\left(l_0-\frac{i\pi}{s+\frac{1}{2}}\right)^{2s-1-(2i\pi/l_0)},\\
&\frac{\mathscr{Z}_1(s)\mathscr{Z}_1(s+1)}{\mathscr{Z}_0^2(s+\frac{1}{2})}\propto e^{-2i\pi/l_0}l_0^{2s}\left(l_0-\frac{i\pi}{s}\right)^{(i\pi/l_0)-s}\left(l_0-\frac{i\pi}{s+1}\right)^{(i\pi/l_0)-s-1},
\end{aligned} \tag{57}$$

where again the const. may depend on s. In this case the fermionic string integrand diverges for $l_0\to 0$ as in the bosonic case.

Finally let us consider the product of the superdeterminants of $-\square_0^2$ and $-\square_{-1}^2$:

$$\operatorname{sdet}'(-\square_0^2)\cdot\operatorname{sdet}'(-\square_{-1}^2)=1, \tag{58}$$

which follows directly from Eqs. (37) and (42). Generalizing this interesting result we get (omitting zero-modes if necessary):

$$\mathrm{sdet}'(-\square^2_{-m})\cdot\mathrm{sdet}'(-\square^2_{m-1})=(-1)^{g-1}(m-1)^{2-2g}, \quad (m\in\mathbb{N}). \tag{59}$$

Let be $f_m=\prod_{k=0}^{m}(-k2)^{g-1}$. Redefining the superdeterminants according to $\mathrm{sdet}'(-\square^2_m):=\mathrm{sdet}'(-\square^2_m)/f_m$ $(m\geqq 0)$ and $\mathrm{sdet}'(-\square^2_{-m}):=\mathrm{sdet}'(-\square^2_{-m})\cdot f_{m-2}$ $(m\geqq 1)$ I obtain the relation

$$\mathrm{sdet}'(-\square^2_{-m})\cdot\mathrm{sdet}'(-\square^2_{m-1})=1, \quad (m\in\mathbb{N}). \tag{60}$$

An equation like this was already stated by Baranov and Schwarz [8] by more general considerations. I close with this result, which nicely confirms my own considerations.

VI. Summary

In this paper the Selberg supertrace formula on super Riemann surfaces has been discussed and some of its most important consequences. The Selberg super operator L on $\mathscr{S}\mathscr{H}$ was defined and it was found that the operator L multiplies an arbitrary Eigenfunction of $\square_m$ by the function h, where h is only defined by the Eigenvalue s of this Eigenfunction with respect to $\square_m$ and the integral kernel of L. It was found that the Selberg supertrace formula reads

$$\begin{aligned}
&\sum_{n=0}^{\infty}[h_m(p_n^B)-h_m(p_n^F)]\\
&\quad=(1-g)\int_0^{\infty}\frac{g(u)-g(-u)}{\sinh\frac{u}{2}}\cosh\frac{m}{2}du+\sum_{\{\gamma\}_p}\sum_{k=1}^{\infty}\frac{l_\gamma\chi_\gamma{}^{km}}{e^{kl_\gamma/2}-e^{-kl_\gamma/2}}\\
&\quad\cdot\left[g(kl_\gamma)+g(-kl_\gamma)-\chi_\gamma{}^k\left(g(kl_\gamma)e^{-kl_\gamma/2}+g(-kl_\gamma)e^{kl_\gamma/2}\right)\right].
\end{aligned} \tag{1}$$

The inversion formula which is needed in the supertrace formula to calculate the term $A_0^{(m)}=i^m\pi(g-1)\Phi_m(0)$ which corresponds to the identity transformation was calculated to be given by

$$i^m\Phi_m(x)=\frac{1}{\pi\sqrt{x+4}}\int_x^{\infty}\frac{dy}{\sqrt{y+4}}\int_{-\infty}^{\infty}Q'(y+t^2)\left(\frac{\sqrt{y+4+t^2}-t}{\sqrt{y+4+t^2}+t}\right)^{m/2}dt, \tag{2}$$

where $Q(u)=2\coth\frac{u}{2}[g(u)-g(-u)]$. $A_0^{(m)}$ and the inversion formula for $\Phi_m(x)$ completed the work of Baranov et al. by explicitly stating the $A_0^{(m)}$-term and the inversion formula, respectively.

Chapter IV was devoted to the discussion of the analytic properties of the two Selberg super zeta-functions Z_0 and Z_1. By considering specific test functions the analytic properties of Z_0 and Z_1, respectively, could be discussed. There is no zero of $Z_0(s)$ at $s=1$ which is quite different in comparison to the usual Selberg zeta-function. The crucial importance of $\Delta n_0^{(0)}=\#even(zero-modes)-$

$\#odd(zero - modes)$ of the operator $\square$ has become clear. I could derive a functional equation for Z_0:

$$R_0(s)R_0(-s) = \frac{Z_0(s)Z_0(-s)}{Z_0(s+1)Z_0(1-s)} = \frac{Z_0(-\frac{1}{2})}{Z_0(\frac{3}{2})}(\sin \pi s)^{4(g-1)}. \tag{3}$$

The corresponding functional relation for Z_1 turned out to be:

$$R_1(s)R_1(-s) = \frac{Z_1(s)Z_1(-s)}{Z_1(s+1)Z_1(1-s)} = \frac{Z_1^2(0)}{Z_1(1)}(\cos \pi s)^{4(g-1)}. \tag{4}$$

For both functions we obtain an infinite set of critical lines located for Z_0 at $\mathrm{Re}(s) = \frac{1}{2} - k\,(k \in \mathbf{N}_0)$ and for Z_1 at $\mathrm{Re}(s) = -k\,(k \in \mathbf{N}_0)$. Unfortunately no functional equation for Z_0 or Z_1 as for the ordinary Selberg zeta-function could be found. However, I have argued the unlikelihood that such a relation exists, based on the existence of the infinite number of critical lines. This appearance of an infinite number of critical lines for the two functions Z_0 and Z_1 is surprising, because there is not any classical analogy for this feature. However, in view of the functional relations for $R_0(Z_0)$ and $R_1(Z_1)$ this is a consistent result. The functional relations are of no use for the explicit determination of the spectrum of the Laplace-Dirac operator $\square$. This in turn is the same situation as in the classical case. There is up to now no way into the critical domain of the (super) zeta-functions in the complex plane, where the nontrivial zeros (and/or poles) are located.

By an appropriate test function h I could deduce a functional relation connecting Z_0 and Z_1:

$$\frac{Z_1(\frac{1}{2}-s)Z_0(s)}{Z_1(\frac{1}{2}+s)Z_0(1-s)} = \frac{Z_1(0)}{Z_1(1)}(\sin \pi s)^{2(g-1)}. \tag{5}$$

Having discussed the properties of Z_0 and Z_1 I treated in the final chapter the spectrum and superdeterminants of the Laplacian-Dirac operators $\hat{\square}_m$ and $\square_m$, respectively. Denoting by $\Delta\hat{n}_\lambda^{(m)}$ the difference of the even and odd trivial-modes λ of the operator $\hat{\square}_m$ I discussed two scenarios for $\Delta\hat{n}_0^{(0)}$, i.e. $\Delta\hat{n}_0^{(0)} = 0$ and $\Delta\hat{n}_0^{(0)} = 1 - g$, respectively. For the nontrivial Eigenvalues of $\hat{\square}_m$ I found that they are determined by the nontrivial Eigenvalues of $\square_0$ as $\hat{\lambda}_{n,m} = \hat{\lambda}_{n,0}$ $(n \in \mathbf{N}, m \in \mathbf{N})$. The calculation of the determinants was performed with the well-known zeta-regularization method. The representations showed clearly that the superdeterminants are well-defined quantities. Since the superdeterminants were regular, it could be shown that the fermionic string integrand in the Polyakov approach is well-defined. The remaining integral over the super moduli space reads

$$Z_g = \left(\frac{1}{2\pi^4}\right)^{g-1} \int_{sM_g} d\mu_{\mathrm{SWP}}(-1)^{5/2\Delta n_0^{(0)}} \left(\frac{Z_0(1)}{\tilde{Z}_1(\frac{1}{2})}\right)^{-5} \frac{Z_0(2)}{Z_1(\frac{3}{2})}\left(\frac{Z_1(0)}{Z_1(1)}\right)^2. \tag{6}$$

Unfortunately no statement about the growing properties of this expression for increasing genus g like the analysis of Gross and Periwal could be made.

However, I could discuss what happens for the fermionic string integrand if a pinching takes place. Here I found that divergence as well as convergence can happen, depending on the spin structure.

An interesting feature of the determinants is that there is a typical factor of $Z_1(0)/Z_1(1)$. This factor does not appear in the work of Aoki, who started from quite analogous expressions but used the functional relation of the ordinary Selberg zeta-function instead of the functional relations for the Selberg super zeta-functions. But this factor is an unambiguous consequence of the functional relations which I derived in Chap. IV. I do not see any way to simplify this characteristic factor any further by exploiting all these functional relations. Therefore this factor gives an additional contribution in the super moduli dependence of the superdeterminants and thus also for the fermionic string integrand.

An interesting relation for the determinants was deduced reading

$$\operatorname{sdet}'(-\square^2_{-m})\cdot\operatorname{sdet}'(-\square^2_{m-1})=(-1)^{g-1}(m-1)^{2-2g},\quad (m\in\mathbb{N}). \tag{7}$$

These results which are all direct consequences of the Selberg super trace formula demonstrate in an impressive way the power of the trace formula.

The fact that the fermionic string theory is, formulated in the super analysis formulation, well-defined, is a step forward in the understanding of the whole string theory. However, one must keep in mind that the fermionic string is as well as the bosonic string nothing but a toy-model. To incorporate supersymmetry or to get the standard-model gauge symmetries, the superstring or the heterotic string theory is needed. (The higher-loop partition function for the latter has been constructed by Moore, Nelson and Polchinski [44].) Whereas the incorporation of the superstring can be done by the GSO-projection, it is not obvious to formulate a Selberg trace formula for the heterotic string case and to study its consequences. Again new surprising features may occur. I think that we must face the possibility that we do not know up to now enough mathematics to understand this new physics, and once again physics may be too hard for the physicists.

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