

Asymptotics of the Cut Discontinuity and Large Order Behaviour from the Instanton Singularity: The Case of Lattice Schrödinger Operators with Exponential Disorder

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Abstract. We discuss the relation between the singularity structure of the Borel transform, the asymptotics of the cut discontinuity and the large order behaviour of perturbation theory. In an explicit example – a tight binding model with exponential disorder – we show how to obtain the first instanton singularity from a cluster expansion in the Borel variable, and as an application we determine the exact decay of the density of states as $E \rightarrow \infty$. The method opens some perspectives for similar problems arising from different models of Mathematical Physics.

1. Introduction

Perturbation expansions in Statistical Mechanics and QFT generally do not converge but are only asymptotic to the function under consideration. Even worse, asymptotic expansions do not determine their sum uniquely but there is a particularly convenient method invented by Borel, which under certain circumstances makes it possible to give an integral representation of the uniquely determined sum. The following result is well known.

Theorem 1. (*Watson-Nevanlinna*). *Let f be a function analytic in the half-plane*

$$D(R) = \{z \in \mathbb{C} : \operatorname{Re} z > R\}, \quad (1.1)$$

and have there an asymptotic expansion with remainder estimate (for arbitrary N)

$$\left| f(z) - \sum_{n=0}^{N-1} a_n z^{-n} \right| \leq A \sigma^N N! |z|^{-N}. \quad (1.2)$$

Then the Borel transform

$$B(t) = \sum_{n=0}^{\infty} \frac{a_n}{n!} t^n \quad (1.3)$$

converges (at least) in the circle $\{t \in \mathbb{C} : |t| < \sigma^{-1}\}$ and has an analytic continuation to the region $S(\sigma) = \{t : \operatorname{dist}(t, R_+) < 1/\sigma\}$ satisfying the bound

$$|B(t)| \leq \operatorname{const} \exp(|t| \cdot R) \quad (1.4)$$

uniformly in every $S(\sigma')$ with $\sigma' > \sigma$. Furthermore, f is given by the absolutely convergent integral

$$f(z) = z \int_0^{\infty} \exp(-tz) B(t) dt \quad (1.5)$$

valid for all $z \in D(R)$.

For a proof see [12, 15].

Next suppose that f is analytic in the larger domain

$$D(R, \alpha) = \bigcup_{|\theta| \leq \alpha} \exp(i\theta) D(R), \quad 0 \leq \alpha < \pi,$$

and that (1.2) holds there. If $\alpha > \frac{\pi}{2}$, $D(R, \alpha)$ has to be interpreted as a multisheeted region. From Theorem 1 it follows that the Borel transform can be analytically continued into the larger t -region

$$S(R, \alpha) = \bigcup_{|\theta| \leq \alpha} \exp(i\theta) S(R),$$

and for $z \in \exp(i\theta) D(R)$, $|\theta| \leq \alpha$, the representation (1.5) is generalized to

$$f(z) = z \int_0^{\infty e^{i\theta}} B(t) \exp(-tz) dt = e^{i\theta} z \int_0^{\infty} B(e^{i\theta} t) \exp(-e^{i\theta} tz) dt. \quad (1.6)$$

Certainly the constants R, A, σ can depend on α and some of them may diverge for $\alpha \rightarrow \pi$.

If f is analytic in $D(R, \pi)$ and has an asymptotic expansion (1.2) uniformly in every $D(R, \alpha)$, $\alpha < \pi$, then as a consequence the Borel transform is analytic in the complex t -plane cut along the negative real axis from $-\infty$ to $-1/c$ for some $c \leq \sigma$. [Actually c^{-1} is the radius of convergence of the series (1.3).]

An example of the above set-up is furnished by the series $\sum a_n z^{-n}$ with coefficients

$$a_n = (-1)^n a c^n \Gamma(1 + b + n), \quad b > -1 \quad (1.7)$$

which alternate in sign. Note that $(-1)^n a_n$ are the Stieltjes moments of a uniquely determined positive measure $d\mu(x)$ which is easily obtained by writing

$$a_n = (-1)^n c^n a \int_0^{\infty} x^{b+n} e^{-x} dx = (-1)^n \frac{a}{c} \int_0^{\infty} x^n \left(\frac{x}{c}\right)^b e^{-x/c} dx, \quad (1.8)$$

i.e. $d\mu(x)$ is absolutely continuous with density

$$q(x) = \frac{a}{c} (x/c)^b \exp(-x/c). \quad (1.9)$$

The series $\sum a_n z^{-n}$ is asymptotic to the function $f(z)$ given by the Stieltjes formula

$$f(z) = z \int_0^{\infty} (x+z)^{-1} d\mu(x). \quad (1.10)$$

The Borel transform $B(t)$ can be computed from (1.5) or by using the fact that in this case $B(t)$ is the characteristic function of the measure $d\mu$:

$$B(t) = \int_0^{\infty} \exp(-tz) d\mu(x) = a \Gamma(1+b)/(1+ct)^{b+1}. \quad (1.11)$$

For the particular case that b is an integer or zero $B(t)$ has a pole at $t = -c^{-1}$ and no other singularities in the complex t -plane. The function f is analytic in the complex z -plane cut along the negative real axis. The cut discontinuity is given by

$$\operatorname{Im} f(re^{i\pi}) = -r\pi q(r), \quad (1.12)$$

and using (1.9) we get

$$\operatorname{Im} f(re^{i\pi}) = a\pi(r/c)^{b+1} \exp(-r/c). \quad (1.13)$$

Note that in the case

$$a_n = (-1)^n a n^b c^n n!, \quad b \text{ integer}, \quad (1.14)$$

which is the leading large n asymptotics of (1.7), the Borel transform can also be computed exactly by summing up the generalized geometric series [18]

$$B(t) = a \sum_{n=0}^{\infty} (-1)^n n^b c^n = a \sum_{m=0}^{b+1} \alpha_m (1+ct)^{-m} \quad (1.15)$$

with $\alpha_{b+1} = b!$. As in (1.7) for b integer the dominant pole is of order $b+1$ with coefficient $b!$.

In the physics literature there are attempts to find the asymptotic behaviour of the cut discontinuity from the large order behaviour of the a_n (see [2] for an application), but generally $\operatorname{Im} f$ cannot be determined from the asymptotics of the a_n as $n \rightarrow \infty$ [1]. However, we want to emphasize that both the decay of the cut discontinuity $\operatorname{Im} f(-r)$ for $r \rightarrow \infty$ as well as the large order asymptotics of a_n are exactly determined by the singularity of the Borel transform nearest to the origin in the complex t -plane.

In QFT and Statistical Mechanics the singularities of $B(t)$ on the negative real axis are related to particles called instantons and there is much work being done on extracting a behaviour of type (1.9) for $\operatorname{Im} f(-r)$, $r \rightarrow \infty$, from the instanton structure of $B(t)$ [10].

In this paper we propose a method of finding the singularity of $B(t)$ closest to the origin and from this the behaviour of $\operatorname{Im} f(-r)$, $r \rightarrow \infty$. We find a series representation of $B(t)$ which converges in the cut t -plane and thus offers a tool to work *outside* the circle of convergence of the series (1.3). The method is fairly general and can be applied at least to $\lambda P(\varphi)$ models on a lattice as well as to disordered systems. In order not to obscure the strategy and to avoid more involved computations, we restrict ourselves to the case of a tight-binding model with exponentially distributed disorder although other models can be treated as well.

2. The Exponential Model

Anderson's tight-binding model describes the motion of an electron on a regular lattice under the influence of a random potential. The space of wave-functions is $l^2(\mathbb{Z}^v)$ and the Hamiltonian is given by

$$H = H_0 + V, \quad (2.1)$$

where H_0 is the lattice Laplacian

$$H_0(i, j) = (-\Delta)(i, j) = \begin{cases} 2\nu & i=j \\ -1 & |i-j|=1 \\ 0 & \text{otherwise,} \end{cases} \quad (i, j \in \mathbb{Z}^\nu) \quad (2.2)$$

and the potential V consists of independent and identically distributed random variables $V(i)$, $i \in \mathbb{Z}^\nu$, with common distribution $d\lambda(V)$. In order to make clear the ideas of Sect. 1 we discuss the particularly simple case of an exponential distribution

$$d\lambda(V) = \begin{cases} 0 & \text{if } V < 0 \\ e^{-V} dV & \text{if } V \geq 0. \end{cases} \quad (2.3)$$

It is well known [11] that the spectrum of H is the positive real axis a.s. with respect to $\prod_{i \in \mathbb{Z}^\nu} d\lambda(V_i)$.

In this paper we shall consider the averaged Green's function

$$\overline{G(E; x, y)} = \int \langle x | (H - E)^{-1} | y \rangle \prod_{i \in \mathbb{Z}^\nu} d\lambda(V_i), \quad (2.4)$$

where the ket $|y\rangle$ is shorthand for the wave-function located in $y \in \mathbb{Z}^\nu$. The starting point of our discussion is the random path expansion

$$\overline{G(E; x, y)} = \sum_{\omega: x \rightarrow y} \prod_{j \in \omega} (2\nu - E + V_j)^{-n_j(\omega)} d\lambda(V_j), \quad (2.5)$$

which is just a Neumann expansion of the resolvent around its diagonal part. Here ω is a lattice path from x to y and $n_j(\omega)$ is the number of times ω visits $j \in \mathbb{Z}^\nu$ (see [3] for details).

It is fairly easy to prove along the lines of [4] that in the variable $z = -E$, \bar{G} fulfills the conditions of the Watson-Nevanlinna theorem, but we shall construct a representation of type (1.5) directly. From now on we put $z = -E$. Using

$$\int_0^\infty (2\nu + V + z)^{-n} e^{-V} dV = \frac{1}{(n-1)!} \int_0^\infty \frac{t^{n-1} e^{-(2\nu+z)t}}{1+t} dt, \quad (2.6)$$

we can rewrite the expansion (2.5) as $\left(n(\omega) = \sum_{j \in \omega} n_j(\omega)\right)$,

$$\overline{G(E; x, y)} = \sum_{\omega: x \rightarrow y} \prod_{j \in \omega} \frac{1}{(n_j-1)!} \int_0^\infty \frac{t^{n_j-1} e^{-(2\nu+z)t}}{1+t} dt. \quad (2.7)$$

Our expansion (2.7) [or (2.5)] is similar in structure to a cluster (high-temperature- or polymer-) expansion of Statistical Mechanics. All expansions of this type have the form

$$f(z) = \sum_{\text{Graphs } g} \prod_{i=1}^{i(g)} f_{i,g}(z). \quad (2.8)$$

Now, if the functions $f_{i,g}(z)$ are Laplace transforms of functions $\widehat{f}_{i,g}(t)$

$$f_{i,g}(z) = \int_0^\infty e^{-tz} \widehat{f}_{i,g}(t) dt, \quad (2.9)$$

then we can hope to express $f(z)$ as a Laplace transform, too

$$f(z) = \int_0^\infty e^{-tz} \sum_{\text{Graphs } g} (\widehat{f}_{1,g} * \dots * \widehat{f}_{i(g),g})(t) dt. \quad (2.10)$$

The convolution $*$ is given by

$$(\widehat{f}_1 * \widehat{f}_2)(t) = \int_0^t \widehat{f}_1(t') \widehat{f}_2(t-t') dt' = \int_0^t \widehat{f}_1(t-t') \widehat{f}_2(t') dt'. \quad (2.11)$$

If the expansion on the right-hand side of (2.10) converges absolutely we almost have an explicit expression for the Borel transform $B(t)$ of $f(z)$. Note, however, that in the Watson-Nevanlinna theorem the Laplace transform is written in z -multiplied form. So the inverse Laplace transform $\hat{f}(t)$ of $f(z)$ is related to $B(t)$ by an integration-by-parts-formula

$$f(z) = z \int_0^\infty e^{-tz} B(t) dt = B(0) + \int_0^\infty e^{-tz} \hat{f}(t) dt \quad (2.12)$$

[indeed, in our case $B(0)=0$].

If $f(z)$ obeys an expansion

$$f(z) \sim \sum_{n=0}^\infty a_n z^{-n} \quad (z \rightarrow \infty), \quad (2.13)$$

then a local analytic germ of $\hat{f}(t)$ is given by

$$\hat{f}(t) = \sum_{n=1}^\infty a_n t^{n-1} / (n-1)! \quad (2.14)$$

However, it is not a priori clear that $\hat{f}(t)$ obeys some exponential bound (1.4), so the right-hand side of (2.12) need not exist.

In the following we shall show that in the case of the exponential model formula (2.12) is correct. Since no confusion will arise, we shall adopt the same Borel transform for both the Watson-Nevanlinna Borel transform and the inverse Laplace transform.

Theorem 2. *In the exponential model the random path expansion of the Borel transform*

$$\hat{f}(t) = \sum_{\omega: x \rightarrow y} (\widehat{f}_1 * \dots * \widehat{f}_{|\omega|})(t) \quad \text{with} \quad \widehat{f}_i(t) = \frac{1}{(n_i-1)!} t^{n_i-1} e^{-2\gamma t} / (1+t), \quad (2.15)$$

and $|\omega| = \# \{\text{points visited by } \omega\}$ converges absolutely and uniformly in any sector $D_{\varepsilon, R} = \{t: 0 \leq |t| \leq R, |\arg t| \leq \pi - \varepsilon\}$ and in any circle $C_\varepsilon = \{t: |t| \leq 1 - \varepsilon\}$, and therefore defines a function analytic in $\mathbb{C} \setminus (-\infty, -1]$. Furthermore $f(t)$ is of exponential type uniformly in every $D_{\varepsilon, \infty}$, so the Laplace transform

$$\overline{G(-z; x, y)} = \int_0^\infty e^{-tz} \hat{f}(t) dt \quad (2.16)$$

exists and is analytic in $\left\{z: |\arg z| < \frac{3\pi}{2} - \varepsilon, |z| > z_0(\varepsilon)\right\}$.

Proof. We restrict ourselves to the case $x=y=0$. Let $\arg t = \varphi$, and take the integrations along the straight line joining 0 and t . The $e^{-2\nu t}$ -term is preserved under convolutions and can be extracted. Putting $r=|\omega|-1$ we write the remaining convolutions as

$$\int_0^t d\tau_r \frac{(t-\tau_r)^{n_{r+1}-1}}{1+t-\tau_r} \int_0^{\tau_r} d\tau_{r-1} \frac{(\tau_r-\tau_{r-1})^{n_r-1}}{1+\tau_r-\tau_{r-1}} \dots \int_0^{\tau_2} d\tau_1 \frac{(\tau_2-\tau_1)^{n_2-1}}{1+\tau_2-\tau_1} \frac{\tau_1^{n_1-1}}{1+\tau_1}. \quad (2.17)$$

The denominators are estimated by

$$|1+\tau_i-\tau_{i-1}|^{-1} \leq \begin{cases} 1/\sin|\varphi| & |\varphi| > \frac{\pi}{2} \\ 1 & \text{otherwise,} \end{cases} \quad (2.18)$$

and we are left with the integrals

$$\begin{aligned} & \int_0^{|t|} d|\tau_r| (|t|-|\tau_r|)^{n_{r+1}-1} \\ & \cdot \int_0^{|\tau_r|} d|\tau_{r-1}| (|\tau_r|-|\tau_{r-1}|)^{n_r-1} \dots \int_0^{|\tau_2|} d|\tau_1| (|\tau_2|-|\tau_1|)^{n_2-1} |\tau_1|^{n_1-1} \\ & = \frac{\prod_{i=1}^{r+1} (n_i-1)!}{\left[\left(\sum_{i=1}^{r+1} n_i \right) - 1 \right]!} |t|^{(\sum n_i) - 1}. \end{aligned} \quad (2.19)$$

Inserting (2.18) and (2.19) into (2.15), we obtain

$$|\hat{f}(t)| \leq \sum_{\omega: 0 \rightarrow 0} \frac{t^{n(\omega)-1}}{(n(\omega)-1)!} \left(\frac{1}{|\sin \varphi|} \right)^{|\omega|+1} \leq e^{2\nu c|t|} \quad (2.20)$$

for some $c > 0$. (If $|\varphi| < \frac{\pi}{2}$, then $c=1$ and if $t \in C_\varepsilon$, then $\sin \varphi$ has to be replaced by ε .) Combining (2.20) and Vitali's theorem completes the proof. \square

3. Instanton Structure for the Exponential Model

From now on we shall consider the expansion (2.15) for the case $x=y=0$. $\hat{f}(t)$ is a series of convolutions of the $\hat{f}_{i,\omega}(t)$ given by (2.15). Note that $\hat{f}_{i,\omega}(t)$ has a simple pole at $t=-1$ which is the first instanton pole. Higher instanton poles are created by convolution which can be made plausible by the simple integral

$$\int_0^t \frac{1}{1+t-t'} \frac{1}{1+t'} dt' = \frac{2 \ln(1+t)}{2+t}. \quad (3.1)$$

Here the second instanton pole occurs at $t=-2$. This reminds us of the theory of resurgent functions which is the contents of the recent impressive work by Ecalle [6]. A function is called resurgent if its Borel transform f has only isolated poles or logarithmic singularities, i.e. has locally the form

$$\hat{f}(t) = \hat{f}_0(t-t_0) + \ln(t-t_0) \hat{f}_1(t-t_0) + \frac{1}{t-t_0} \hat{f}_2(t-t_0) + \text{higher poles}, \quad (3.2)$$

where \hat{f}_i , $i=0, 1, 2, \dots$ are analytic in a neighbourhood of the origin. Resurgent functions have been recently encountered in several areas of mathematical physics including the renormalisation group approach to dynamical systems and QFT [7, 17].

The singularities of Borel transforms often exhibit some periodic structure, e.g. they may be located at $-t_0, -2t_0, -3t_0, \dots$ for some $t_0 \in \mathbb{C}$. In our case the singularities are located at the negative integers, but since we are only concerned with the first instanton singularity we shall not go beyond the plausibility argument (3.1) (see [5] for some ideas).

Comparing (2.15) and (3.1) suggests that the leading singularity of $f(t)$ at $t = -1$ comes from those paths where no convolution takes place. There is only one such path, namely $\omega = \{0\}$. The contribution of this path is $\hat{f}_{(0)}(t) = (1+t)^{-1}$.

We shall now determine the behaviour of the remainder term at $t = -1$. Only the results important for Sect. 4 will be given.

Lemma 1. *Let $\hat{f}_{(1)}(t) = \hat{f}(t) - \hat{f}_{(0)}(t)$, let t be a point on the half-circle $\{t: \text{Re } t \geq -1, |t+1| = \varepsilon\}$. Then*

$$|\hat{f}_{(1)}(t)| \leq C_1 \ln(1/\varepsilon). \quad (3.3)$$

Furthermore if $t = -1 + i\varepsilon$, ε real, then

$$\left| \frac{d}{dt} \hat{f}_{(1)}(t) \right| \leq C_2 / |\varepsilon|. \quad (3.4)$$

Proof. Again, we perform the integrals (2.17), the path of integration taken along the straight line joining 0 and t . Then, by using

$$|\tau_i - \tau_{i-1}| \leq |t|, \quad (3.5)$$

the numerators are bounded by $|t|^{n(\omega) - |\omega|}$, and we are left with

$$\int_0^{|t|} d|\tau_r| \frac{1}{|1+t-\tau_r|} \int_0^{|\tau_r|} d|\tau_{r-1}| \frac{1}{|1+\tau_r-\tau_{r-1}|} \dots \int_0^{|\tau_2|} d|\tau_1| \frac{1}{|1+\tau_2-\tau_1|} \frac{1}{|1+\tau_1|}. \quad (3.6)$$

Now note that $\text{Re } t \geq -1$, and estimate

$$\frac{1}{|1+\tau_2-\tau_1|} \frac{1}{|1+\tau_1|} = \frac{1}{|2+\tau_2|} \left(\frac{1}{|1+\tau_2-\tau_1|} + \frac{1}{|1+\tau_1|} \right) \leq \left(\frac{1}{|1+\tau_2-\tau_1|} + \frac{1}{|1+\tau_1|} \right) \quad (3.7)$$

(see Fig. 1).

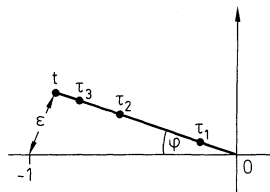


Fig. 1

Multiplying the next term gives

$$\frac{1}{|1 + \tau_3 - \tau_2|} \frac{1}{|1 + \tau_2 - \tau_1|} \leq \frac{1}{|1 + \tau_3 - \tau_2|} + \frac{1}{|1 + \tau_2 - \tau_1|},$$

and

$$\begin{aligned} \frac{1}{|1 + \tau_3 - \tau_2|} \frac{1}{|1 + \tau_1|} &= \frac{1}{|2 + \tau_1 + \tau_3 - \tau_2|} \left(\frac{1}{|1 + \tau_3 - \tau_2|} + \frac{1}{|1 + \tau_1|} \right) \\ &\leq \frac{1}{|1 + \tau_3 - \tau_2|} + \frac{1}{|1 + \tau_1|}. \end{aligned}$$

Proceeding in the same spirit, we obtain 2^r terms of the type

$$\begin{aligned} &\int_0^{|t|} d|\tau_r| \dots \int_0^{|\tau_i|} d|\tau_{i-1}| \frac{1}{|1 + \tau_i - \tau_{i-1}|} \int_0^{|\tau_{i-1}|} \dots \int_0^{|\tau_2|} d|\tau_1| \\ &= \int_0^{|t|} d|\tau_r| \dots \int_0^{|\tau_i|} d\tau_{i-1} \frac{1}{|1 + \tau_i - \tau_{i-1}|} \frac{|\tau_{i-1}|^{i-2}}{(i-2)!} \\ &\leq \int_0^{|t|} d|\tau_r| \dots \frac{|\tau_i|^{i-2}}{(i-2)!} \int_0^{|\tau_i|} d|\tau_{i-1}| \frac{1}{|1 + \tau_i - \tau_{i-1}|}. \end{aligned} \quad (3.8)$$

Next we are going to estimate

$$\int_0^{|\tau_i|} d|\tau_{i-1}| \frac{1}{|1 + \tau_i - \tau_{i-1}|}. \quad (3.9)$$

To do this let $\varphi = \pi - \arg t$, $a = |\tau_i - \tau_{i-1}| = |\tau_i| - |\tau_{i-1}|$. Then $|1 + \tau_i - \tau_{i-1}| = (1 + a^2 - 2a \cos \varphi)^{1/2}$ and,

$$\begin{aligned} \int_0^{|\tau_i|} d|\tau_{i-1}| \frac{1}{|1 + \tau_i - \tau_{i-1}|} &= \int_0^{|\tau_i|} da (1 + a^2 - 2a \cos \varphi)^{-1/2} \\ &\leq \ln \left(\frac{|t| - \cos \varphi + ((|t| - \cos \varphi)^2 + \sin^2 \varphi)^{1/2}}{1 - \cos \varphi} \right) \end{aligned} \quad (3.10)$$

(compare Fig. 2). But

$$\left. \begin{aligned} &((|t| - \cos \varphi)^2 + \sin^2 \varphi)^{1/2} = |1 + t| = \varepsilon, \\ &|t| - \cos \varphi = -(\varepsilon^2 - 1 + \cos^2 \varphi)^{1/2}, \quad \varphi \text{ small.} \end{aligned} \right\} \quad (3.11)$$

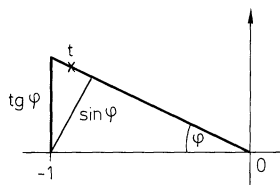


Fig. 2

So for $\varphi \rightarrow 0$ (the delicate case) the argument in (3.10) remains bounded by $2/\varepsilon$, and so it is not hard to see that there exists a $C_1 > 0$ such that

$$\int_0^{|\tau_i|} d|\tau_{i-1}| |1 + \tau_i - \tau_{i-1}|^{-1} \leq C_1 \ln(1/\varepsilon). \quad (3.12)$$

Obtaining a bound for the derivative runs similarly: taking derivatives in (2.17) and making all estimates as above gives several $\ln(1/\varepsilon)$ -terms besides one of the form

$$\int_0^{|t|} d|\tau_r| \frac{1}{|1 + t - \tau_r|^2}$$

which is evaluated according to (3.10) and gives a contribution $C \frac{|t|^{r-1}}{(r-1)!} 1/|\varepsilon|$.

It remains to show that the expansion (2.15) with the appropriate bound for (3.6) still converges. For this we have to estimate (recall that $r = |\omega| - 1$)

$$\begin{aligned} \sum_{\omega: 0 \rightarrow 0} \frac{|t|^{n(\omega)-2}}{\prod_{i=1}^{|\omega|} (n_i - 1)!} \frac{1}{(|\omega| - 2)!} &\leq \sum_{\omega: 0 \rightarrow 0} \frac{1}{(|\omega| - 2)!} \binom{n(\omega) - |\omega|}{n_1, \dots, n_{|\omega|}} \frac{|t|^{n(\omega)-2}}{(n(\omega) - |\omega|)!} \\ &\leq \sum_{\omega: 0 \rightarrow 0} \frac{|t|^{n(\omega)-2}}{(n(\omega) - |\omega|)!} \frac{|\omega|^{n(\omega) - |\omega|}}{(|\omega| - 2)!} \\ &\leq \sum_{|\omega|=1}^{\infty} \frac{1}{(|\omega| - 2)!} \sum_{N=|\omega|}^{\infty} \frac{(2\nu)^N |\omega|^{N-|\omega|} |t|^{N-2}}{(N - |\omega|)!} \\ &\leq \sum_{|\omega|=1}^{\infty} \frac{(2\nu|t|)^{|\omega|}}{(|\omega| - 2)!} \sum_{N=0}^{\infty} \frac{(2\nu|t||\omega|)^N}{N!} \\ &\leq \sum_{|\omega|=1}^{\infty} \frac{(2\nu|t|e^{2\nu|t|})^{|\omega|}}{(|\omega| - 2)!} < \infty. \end{aligned} \quad (3.13)$$

Here we have used the fact that the number of path ω having $\sum n_i(\omega) = N$ is bounded by $(2\nu)^N$. This completes the proof of the lemma. \square

4. Exponential Decay and Large Order Behaviour

The density of states $\varrho(E)$ is defined by

$$\varrho(E) = (2\pi i)^{-1} \{ \overline{G(E + i0; 0, 0)} - \overline{G(E - i0; 0, 0)} \}. \quad (4.1)$$

In this chapter we shall determine the asymptotic decay of $\varrho(E)$ as $E \rightarrow \infty$. Recall that $\overline{G(E)}$ is given as

$$\overline{G(-E)} = \int_0^{\infty} e^{-Et} (\hat{f}_{(0)}(t) + \hat{f}_{(1)}(t)) dt, \quad (4.2)$$

where $\hat{f}_{(0)}(t) = (1+t)^{-1}$ and $\hat{f}_{(1)}(t) = \hat{f}(t) - \hat{f}_{(0)}(t)$, and the representation (4.2) is valid for $|\arg(-E)| < \frac{\pi}{2}$. The contribution from $\hat{f}_{(0)}$ can be evaluated immediately either by using special functions or by rotating the contour of the Laplace integral

and taking into account

$$(1 + te^{i\pi})^{-1} - (1 + te^{-i\pi})^{-1} = \pi\delta(1 + t). \quad (4.3)$$

The contribution of $\hat{f}_{(0)}$ to ϱ is given by

$$\varrho_{(0)}(E) = e^{-E}. \quad (4.4)$$

Next we estimate the contribution $\varrho_{(1)}$ given by $\hat{f}_{(1)}$. To do this we recall that $\hat{f}_{(1)}(t)$ is of exponential type uniformly in any sector $\{t: |\arg t| < \pi - \varepsilon\}$. So in (4.2) we can rotate the contour of integration by an angle greater than $\frac{\pi}{2}$ and obtain a representation for $G(E)$ for $\arg(-E) = \pi$ [and $\arg(-E) = -\pi$ respectively]. So if we write $\overline{G_1(z)}$ for the Laplace transform of $\hat{f}_{(1)}(t)$ and recall that $z = -E$, we obtain the following representation ($\eta \in (0, \frac{\pi}{2})$):

$$\begin{aligned} \overline{G_1(ze^{i\pi})} &= \int_0^{e^{i(\pi/2 - \eta)\infty}} e^{-zt} \hat{f}_{(1)}(t) dt, \\ \overline{G_1(ze^{-i\pi})} &= \int_0^{e^{-i(\pi/2 + \eta)\infty}} e^{-zt} \hat{f}_{(1)}(t) dt, \end{aligned} \quad (4.5)$$

$$\varrho_{(1)}(E) = (2\pi i)^{-1} \int_C e^{Et} \hat{f}_{(1)}(t) dt, \quad (4.6)$$

where the contour C is depicted in Fig. 3.

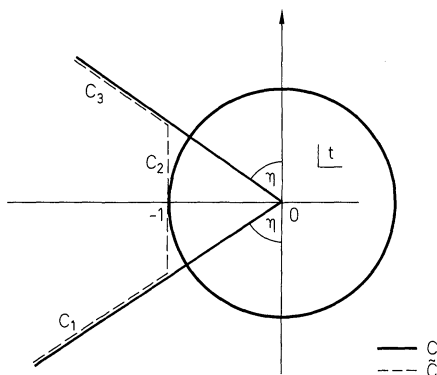


Fig. 3

Next we recall that $\hat{f}_{(1)}$ is analytic in $\mathbb{C} \setminus (-\infty, -1]$ and the divergence of $\hat{f}_{(1)}(t)$ as $t \rightarrow -1$ is of logarithmic order uniformly in $\{\operatorname{Re}(t+1) \geq 0\}$. So the contour can be deformed to pass through the singularity and

$$\varrho_{(1)}(E) = (2\pi i)^{-1} \int_{\tilde{C}} e^{Et} \hat{f}_{(1)}(t) dt \quad (4.7)$$

with \tilde{C} shown in Fig. 3. The contour \tilde{C} can be divided into three parts C_1 , C_2 , C_3 , as shown in Fig. 3, and we shall evaluate their contributions separately.

The integrals over C_1 and C_3 can be evaluated by using the Laplace method for complex contours [13]: For $|E|$ large enough the integrand in (4.7) taken along

$C_{1/3}$ decays exponentially and the main contribution comes from the points $-1 \pm iA$. We obtain

$$\int_{C_{1/3}} e^{Et} \hat{f}_{(1)}(t) dt = O(E^{-1} e^{-E}) \quad (E \rightarrow \infty). \quad (4.8)$$

For the contribution over C_2 we need the following:

Lemma 2. Let $\varphi \in L^1[0, A] \cap C^1(o, A]$. Furthermore assume that on $(o, A]$

$$|\varphi(t)| \leq \alpha_1 |\ln t| \quad (4.9)$$

and

$$\left| \frac{d}{dt} \varphi(t) \right| \leq \alpha_2 / t \quad (4.10)$$

Then

$$\int_0^A e^{\pm itx} \varphi(t) dt = O(x^{-1} \ln x) \quad (x \rightarrow \infty). \quad (4.11)$$

Proof.

$$\int_0^A e^{\pm itx} \varphi(t) dt = \int_{1/x}^A e^{\pm itx} \varphi(t) dt + \int_0^{1/x} e^{\pm itx} \varphi(t) dt =: I_1 + I_2,$$

$$I_1 = \frac{1}{\pm ix} e^{\pm itx} \varphi(t) \Big|_{1/x}^A - \frac{1}{\pm ix} \int_{1/x}^A e^{\pm itx} \varphi'(t) dt.$$

The first term is $O(x^{-1} \ln x)$ by (4.9) and the second is estimated by (4.10):

$$\frac{1}{x} \int_{1/x}^A |\varphi'(t)| dt \leq C_2/x \int_{1/x}^A 1/t dt = O(x^{-1} \ln x).$$

Furthermore a substitution $t \rightarrow t/x$ yields

$$|I_2| = \frac{1}{x} \left| \int_0^{1/x} e^{\pm it} \varphi(t/x) dt \right| \leq \frac{1}{x} \int_0^1 |\ln(t/x)| dt = O(x^{-1} \ln x),$$

which proves the lemma. \square

It is obvious how to apply Lemma 2 in our case. From Lemma 1 $\hat{f}_{(1)}$ obeys the conditions (4.9–10) and the integral over C_2 taken in parameter representation is just a Fourier integral of type (4.11). So

$$\int_{C_2} e^{tE} \hat{f}_{(1)}(t) dt = O(e^{-E} E^{-1} \ln E) \quad (E \rightarrow \infty). \quad (4.12)$$

Collecting (4.4), (4.8), and (4.12) we have proved the following:

Theorem 3. The density of states ϱ for the tight-binding model with exponential disorder (2.3) behaves asymptotically as

$$\varrho(E) = e^{-E} (1 + O(E^{-1} \ln E)) \quad (E \rightarrow \infty). \quad (4.13)$$

Finally we want to discuss the relation between (4.13) and the large n behaviour of perturbation theory. From the fact that $f(z) = \bar{G}(-z)$ is Laplace transform of an

analytic function we know by Watson's lemma [13] that $f(z)$ has an asymptotic expansion

$$f(z) \sim \sum_{n=0}^{\infty} a_n z^{-(n+1)} \quad (z \rightarrow \infty), \quad (4.14)$$

valid in the region $\{z: \operatorname{Re} z > z_0\}$. Since G is a Green's function we have another representation

$$f(z) = \int_0^{\infty} (x+z)^{-1} dk(x), \quad (4.15)$$

where $dk(x)$ is the integrated-density-of-states(ids)-measure on $\operatorname{spec} H$. A formal large- z -expansion is given by

$$f(z) \sim \sum_{n=0}^{\infty} (-1)^n \mu_n z^{-(n+1)} \quad (z \rightarrow \infty), \quad (4.16)$$

where the μ_n are the moments of dk ,

$$\mu_n = \int_0^{\infty} x^n dk(x). \quad (4.17)$$

We shall prove

Theorem 4. *The coefficients a_n of (4.14) obey*

$$a_n = (-1)^n n! (1 + O(1/n)) \quad (n \rightarrow \infty). \quad (4.18)$$

Proof. This is an easy consequence of (4.17) and (4.13). Choose $x_0 \geq 1$ so large that $dk(x) = \varrho(x) dx$ and $|\varrho(x) - e^{-x}| \leq C e^{-x} x^{-1} \ln x$ for all $x \geq x_0$. Then

$$\begin{aligned} (-1)^n a_n &= \int_0^{x_0} x^n dk(x) + \int_{x_0}^{\infty} x^n(x) dx = \int_0^{x_0} x^n dk(x) + \int_{x_0}^{\infty} x^n e^{-x} dx + \int_{x_0}^{\infty} x^n (\varrho(x) - e^{-x}) dx \\ &= \int_0^{\infty} x^n e^{-x} dx + \int_0^{x_0} x^n (dk(x) - e^{-x} dx) + \int_{x_0}^{\infty} x^n (\varrho(x) - e^{-x}) dx. \end{aligned}$$

The first integral gives the leading behaviour $n!$, the second is $O(\text{const}^n)$, and the third can be estimated by $\int_1^{\infty} x^{n-1} e^{-x} \ln x dx$, which can be evaluated by iterated partial integration and gives a contribution $O((n-1)!) (n \rightarrow \infty)$. This completes the proof. \square

5. Conclusions and Discussion

Using the random path expansion for lattice resolvents we have analyzed the instanton structure of the averaged Green's function for a tight binding model with exponential disorder. This enabled us to find the exponential decay of the density of states and the large- n behaviour of the moments of the density of states. For the model under consideration our results are better than those obtained in [9, 14].

We have obtained an explicit series representation for the Borel transform, and this enabled us to work outside the circle of convergence of the series (1.3). Our method offers some perspectives for other models:

– the tight binding model with Gaussian disorder. Here we should consider the Green's function as a function of the variable $z=E^2$. According to [9] we expect Gaussian decay of the density of states. In the symmetric case the present methods enable us to go further and establish a result of the form (4.13) in the variable E^2 .

– exponential decay for small negative coupling constant as well as large order behaviour of the perturbation expansion for the pressure or the two-point functions of $\lambda P(\varphi)$ -models on the lattice (for rigorous work see [16]).

– life-time of resonances of the anharmonic oscillator with negative coupling and large order behaviour of the coefficients of the Rayleigh-Schrödinger series (see also [16]).

A remark about Lifshitz tails [8] for disordered systems is in order here. There is much work in physics on extracting Lifshitz tails of the density of states for the electronic problem or Griffith singularities of the free energy in disordered spin-systems from the first instanton singularity. In the present form our method is not applicable to this problem. We think that a different series expansion in connection with renormalisation group ideas should be better suited for applying our method to the problem of Lifshitz tails and Griffith singularities.

On the other hand from

$$(z+x)^{-1} = \int_0^{\infty} e^{-tz} e^{-tx} dt$$

it follows that our Borel transform is just the averaged diagonal element of the semi-group e^{-tH} ,

$$\hat{f}(t) = \overline{\langle 0 | e^{-tH} | 0 \rangle}.$$

Fukushima [8] has derived a Feynman-Kac formula for $\hat{f}(t)$. This enabled him to study the large- t behaviour of \hat{f} and via a Tauberian theorem the small- E behaviour of the integrated density of states $k(E)$. Again it seems to be difficult to extract from our expansion (2.15) the large- t behaviour of $\hat{f}(t)$ needed in the Tauberian theorem.

We finish by remarking that for the case of random Schrödinger operators on the lattice our method in the present form can be applied to study the exact asymptotic behaviour of tails of the density of states at infinity (including preexponential factors).

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