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The Distributional Borel Summability and the Large Coupling Φ^4 Lattice Fields

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Abstract. Following 't Hooft we extend the Borel sum and the Watson-Nevanlinna criterion by allowing distributional transforms. This enables us to prove that the characteristic function of the measure of any $g^{-2}\Phi^4$ finite lattice field is the sum of a power series expansion obtained by fixing exponentially small terms in the coefficients. The same result is obtained for the trace of the double well semigroup approximated by the n^{th} order Trotter formula.

1. Introduction

Borel summability has by now become an important tool largely applied in many fields of mathematical physics (e.g. see [7, 17]). On the other hand it has also become clear that certain problems do not fulfill all the requirements for Borel summability. The simplest and best known counterexample is probably the double well quantum mechanical model with Hamiltonian: $p^2+x^2(1-gx)^2$ [3, 8]. Other "non-Borel" series are expected in quantum field theory because of the presence of "renormalons" [12]. The lack of Borel summability of a real divergent series is evident when the coefficients have asymptotically constant sign [3, 17].

A few years ago 't Hooft showed [10] that a simplified double well model is Borel summable only in some generalized sense, allowing distributions in the transform. Following 't Hooft we define here a class of distributional Borel sums extending the ordinary Borel-Le Roy ones, and correspondingly we extend the Watson-Nevanlinna criterion. This way we prove that the characteristic function of the measure of a $g^{-2}\Phi^4$ finite volume field is the sum of its power series expansion obtained by fixing exponentially small terms in the coefficients. More precisely the characteristic functions have the following asymptotic expansions:

$$C(z,g)\sim\sum_k a_k(z,g^{-2})g^k, \quad \text{where} \quad a_k(z,\gamma)=\sum_{m=0}^{N} a_m^k(z)e^{-m\gamma},$$

and $C(z,\gamma,g)\sim\sum_k a_k(z,\gamma)g^k$ is the Borel sum of the asymptotic series for small g and any positive γ, in particular for $\gamma=g^{-2}$.

The same results are given for the trace of the double well semigroup approximated by the Trotter n^{th} order formula. We also notice that the original 't Hooft model coincides with the first order approximant.

Other interesting problems, such as the Lo Surdo-Stark effect resonances, the double well eigenvalues and the eigenvalues of the ionized molecule H_2^+, will be treated elsewhere. The Lo Surdo-Stark effect resonances are known to be limits of Borel sums from complex values to the real axis. We conjecture that the energy and the width of the resonances are directly given by the sum in the real direction.

The criterion is proved following the line of the classical one by Nevanlinna and Watson [9, 13, 18] which is thus extended to the distributional case. On the other hand, in this case the assumptions on the function cannot be reduced to the traditional ones, and the definition of the sum itself involves a subtle discussion on properties of distributions. The method of sum directly defines, besides the sum itself, an independent function, called discontinuity, whose asymptotic expansion is zero to all orders.

In Sect. 2 we define the distributional Borel-Le Roy sums of order (α, β) and we prove the corresponding criteria, extending the classical one by Nevanlinna and Watson, for $\alpha = p/q$ $(p, q \in \mathbb{N})$ and $\beta = 1$.

In Sect. 3 these criteria are applied to finite volume Φ^4 lattice fields and to the n^{th} order Trotter approximants for the trace of the semigroup of the double well Hamiltonian.

2. Distributional Borel Summability

We can define a distributional Borel summability in the following way.

Definition 1. Let $\varrho(t)dt$ be a measure with finite positive moments $\mu_k = \int\limits_0^\infty t^k \varrho(t)dt$.

We say that the formal series $\sum\limits_k a_k z^k (a_k \in \mathbb{R})$ is $\mu - \varrho$-Borel summable (in the distributional sense) if:

(1) $B(t) = \sum\limits_{k=0}^{\infty} (a_k/\mu_k) t^k$ converges in some disk of radius $d > 0$;

(2) $B(t)$ admits an analytic continuation to the intersection of some neighbourhood of $\mathbb{R}_+$ with $\mathbb{C}^+ = \{t \in \mathbb{C}/\operatorname{Im} t > 0\}$;

(3) $f(z) = z^{-1} \int\limits_0^\infty PP(B(t))\varrho(t/z)dt$, where $PP(B(t)) = (1/2)(B(t+i0) + \overline{B(t+i0)})$.

Then the distribution $PP(B(t))$ is called the "$\mu - \varrho$-Borel transform" of the series $\sum\limits_k a_k z^k$ and $f(z)$ is its "$\mu - \varrho$-Borel sum."

Examples. (a) $\mu_k = k!$, $\varrho(t) = e^{-t}$; (b) $\mu_k = \Gamma(\alpha k + \beta)$, $\varrho(t) = \alpha^{-1} \exp(-t^{1/\alpha}) t^{-1+\beta/\alpha}$ [distributional Borel-Le Roy summability of order (α, β)].

Of course the ordinary Borel summability (with $a_k \in \mathbb{R}$) is included in Definition 1: (2) is a fortiori verified if $B(t)$ has an analytic continuation to some neighbourhood of $\mathbb{R}_+$.

Remark 1. We notice that the class of distributions coinciding with $\sum\limits_k (a_k/\mu_k) t^k$ when localized in $(-d, d)$ and defined as a boundary value of $B(u)$ is:

$$\mathscr{B}_\lambda(t) = \lambda B(t+i0) + (1-\lambda)\overline{B(t+i0)}, \qquad 0 \leqq \lambda \leqq 1. \tag{2.1}$$

Hence, if we consider $\mathscr{B}_\lambda(t) = \mathrm{Re}\mathscr{B}_\lambda(t) + i\,\mathrm{Im}\mathscr{B}_\lambda(t)$, we have $\mathrm{Re}\mathscr{B}_\lambda(t) = \mathscr{B}_{1/2}(t)$ for all λ, while $\mathrm{Im}\mathscr{B}_\lambda(t)$ is the analytic continuation of $\mathrm{Im}\mathscr{B}_\lambda(t)$, when localized in $(-d, d)$, if and only if $\lambda = 1/2$, i.e. $\mathrm{Im}\mathscr{B}_\lambda(t) \equiv 0$. (It is known that the concept of analytic continuation can be extended to distributions, although this notion can be applied only to distributions which are represented by analytic functions [4].)

In this sense we can say that the distribution $PP(B(t)) = \mathscr{B}_{1/2}(t)$ is uniquely determined by the Taylor expansion and by the analytic continuation procedure.

We also have: $f(z) = (\phi(z) + \bar{\phi}(\bar{z}))/2$, where $\phi(z) = z^{-1} \int_0^\infty B(t+i0)\,\varrho(t/z)\,dt$ is called "the upper sum," and $\bar{\phi}(\bar{z}) = z^{-1} \int_0^\infty \overline{B(t+i0)}\,\varrho(t/z)\,dt$ "the lower sum" of the series.

It is interesting to notice that with this method we can single out a unique function with zero asymptotic power series expansion, that is the "discontinuity" $d(z) = z^{-1} \int_0^\infty (B(t+i0) - \overline{B(t+i0)})\,\varrho(t/z)\,dt = \phi(z) - \bar{\phi}(\bar{z})$.

In the case $\mu_k = k!$ we can characterize a large class of functions which have Borel summable power series expansions in the distributional sense.

Theorem 1. *Let $f(z)$ be bounded and analytic in $C_R = \{z/\mathrm{Re}\,z^{-1} > R^{-1}\}$ and let $f(z) = (\phi(z) + \bar{\phi}(\bar{z}))/2$, with $\phi(z)$ analytic in C_R and such that*

$$\left|\phi(z) - \sum_{k=0}^{N-1} a_k z^k\right| \leqq c_0 c(\varepsilon)^N N!|z|^N \tag{2.2}$$

uniformly in $C_{R,\varepsilon} = \{z \in C_R/\arg z \geqq -\pi/2 + \varepsilon\}$, for any $\varepsilon > 0$. Then the series $\sum_{k=0}^{\infty} (a_k/k!)u^k$ is convergent for small $|u|$ and it admits an analytic continuation $B(u) = B_1(u) + B_2(u)$, where $B_1(u)$ is analytic in $S_\sigma^1 = \{u/\mathrm{dist}(u, \mathbb{R}_+) < \sigma^{-1}\}$, and $B_2(u)$ is analytic in $S_\sigma^2 = \{u/(\mathrm{Im}\,u > 0,\ \mathrm{Re}\,u > -\sigma^{-1})$ or $|u| < \sigma^{-1}\}$ for some $\sigma > 0$. $B(u)$ satisfies

$$|B(t + i\eta_0)| \leqq A'(\eta_0)^{-1} \exp(t/R) \tag{2.3}$$

uniformly for $t > 0$, for any η_0 such that $0 < \eta_0 < \sigma^{-1}$.

Setting $PP(B(t)) = (B(t+i0) + \overline{B(t+i0)})/2$, $f(z)$ admits the integral representation:

$$f(z) = z^{-1} \int_0^\infty PP(B(t))\, e^{-t/z}\, dt, \qquad z \in C_R. \tag{2.4}$$

Conversely, if $B(u) = \sum_{k=0}^{\infty} (a_k/k!)u^k$ is convergent for $|u| < \sigma^{-1}$ and admits the decomposition $B(u) = B_1(u) + B_2(u)$ with the above quoted properties, then the function defined by (2.4) *is real-analytic in C_R and $\phi(z) = z^{-1} \int_0^\infty B(t+i0)\, e^{-t/z}\, dt$ is analytic and satisfies* (2.2) *in C_R.*

Proof. Without loss of generality, in (2.2) we may assume $c(\varepsilon) = \varepsilon^{-1}$ if $0 < \varepsilon < \min(\pi/2, \sigma^{-1})$ and $c(\varepsilon) = \sigma$ otherwise (see Remark 2 and Proposition 2). By (2.2) the function $\tilde{\phi}(z)$ defined by

$$\tilde{\phi}(z) = \begin{cases} \phi(z), & z \in C_R, \mathrm{Im}\, z > 0 \\ \bar{\phi}(\bar{z}), & z \in C_R, \mathrm{Im}\, z \leqq 0 \end{cases}$$

satisfies

$$\left|\tilde{\phi}(z) - \sum_{k=0}^{N-1} a_k z^k\right| \leqq c_0 \sigma^N N!\,|z|^N \tag{2.5}$$

uniformly in C_R.

For $0<t\leqq\tau$, $\eta>0$, we consider the following expression independent of r, $0<r<R$:

$$B(t+i\eta t) - \sum_{k=0}^{N-1} (a_k/k!)(t+i\eta t)^k$$
$$=(2\pi i)^{-1} \oint_{\operatorname{Re} z^{-1}=r^{-1}} \left(\phi(z) - \sum_{k=0}^{N-1} a_k z^k\right) z^{-1} e^{(t+i\eta t)/z} dz =$$

[by choosing $r=\tau/N$, and noting that $\phi(z)-\tilde{\phi}(z)$ is zero for $\operatorname{Im} z>0$ and is equal to $\phi(z)-\bar{\phi}(\bar{z})$ for $\operatorname{Im} z\leqq 0$]

$$=(2\pi i)^{-1} \oint_{\operatorname{Re} z^{-1}=N/\tau} e^{(t+i\eta t)/z}\left[\tilde{\phi}(z) - \sum_{k=0}^{N-1} a_k z^k\right] z^{-1} dz$$
$$+(2\pi i)^{-1} \oint_{\Gamma_{\tau/N}} e^{(t+it\eta)/z}[\phi(z)-\bar{\phi}(\bar{z})] z^{-1} dz$$
$$\equiv R_N^{(1)}(t+i\eta t) + R_N^{(2)}(t+i\eta t), \tag{2.6}$$

where $\Gamma_{\tau/N}=\{z/\operatorname{Re} z^{-1}=(\tau/N)^{-1}, \operatorname{Im} z\leqq 0\}$ and $R_N^{(1)}$, $R_N^{(2)}$ are defined by the first and the second integral respectively.

Now, for $0<t\leqq\tau$, $\eta=0$, (2.5) implies the following estimates:

$$\left|\frac{d^m}{dt^m} R_N^{(1)}(t)\right| \leqq c_1(\sigma\tau)^N N^{1/2}(N/\tau)^{m+1}, \quad \text{if} \quad m<N, \tag{2.7}$$

$$\left|\frac{d^m}{dt^m} R_N^{(1)}(t)\right|$$
$$= \left|a_m + (2\pi i)^{-1} \oint_{\operatorname{Re} z^{-1}=N/\tau} e^{t/z} z^{-(m+1)}\left[\tilde{\phi}(z) - \sum_{k=0}^{m} a_k z^k\right] dz\right|$$
$$\leqq c_2 \sigma^{m+1}(m+1)!\, e^N \leqq c_2 \sigma^{m+1}(m+1)!\, e^{2m-N}, \quad \text{if} \quad m\geqq N. \tag{2.8}$$

By (2.7) and (2.8):

$$|R_N^{(1)}(t+i\eta t)| \leqq |R_N^{(1)}(t)| + \left|\sum_{m=1}^{\infty} (i\eta t)^m (m!)^{-1} \frac{d^m}{dt^m} R_N^{(1)}(t)\right|$$
$$\leqq c_3(\sigma\tau)^{N-1} N^{3/2} + \sum_{m=1}^{\infty} (\eta t)^m (m!)^{-1}[c_1(\sigma\tau)^N (N/\tau)^{m+1}$$
$$+ c_2 e^{-N}(e^2\sigma)^{m+1}(m+1)!] \leqq c_4[N^{3/2}(\sigma\tau e^{\eta})^N + e^{-N}] \xrightarrow{(N\to\infty)} 0 \tag{2.9}$$

for some fixed $\tau<\sigma^{-1}e^{-\eta-2}$.

In order to bound $R_N^{(2)}(t+i\eta t)$ in (2.6), we can replace $\Gamma_{\tau/N}$ by the path given by an arc of the circle tangent in 0 to the line $\operatorname{Im} z=\tan(\pi/2+\varepsilon)\operatorname{Re} z$ and containing the real point τ/N: it is the path from 0 to τ/N which lies in the lower half-plane, that is

$$\Gamma_{\tau/N,\varepsilon}=\{z=e^{i\varepsilon}\zeta/\operatorname{Re}\zeta^{-1}=(\tau/(N\cos\varepsilon))^{-1}, \operatorname{Im} z\leqq 0\}, \tag{2.10}$$

$0<\varepsilon<\pi/2$. Thus, choosing $\varepsilon=\arctan(\eta)$,

$$R_N^{(2)}(t+i\eta t)=(2\pi i)^{-1}\int_{\Gamma_{\tau/N,\varepsilon}}\exp(\zeta^{-1}|t+i\eta t|)\,[\phi(z)-\bar{\phi}(\bar{z})]z^{-1}dz\,, \tag{2.11}$$

where $\zeta=e^{-i\varepsilon}z$. Since $\mathrm{Re}\,\zeta^{-1}=(\tau/(N\cos\varepsilon))^{-1}\leqq N/\tau$, by (2.2) we have

$$|R_N^{(2)}(t+i\eta t)|\leqq c_5\exp(N(1+\eta^2)^{1/2}t\tau^{-1})\,c(\varepsilon)^N N!(\tau/(N\cos\varepsilon))^N\,, \tag{2.12}$$

which tends to zero as $N\to+\infty$, if τ is sufficiently small, for fixed $\eta>0$.

Therefore, for any $\eta>0$ there are small values of t such that $R_N(t+i\eta t)\to 0$ as $N\to\infty$. Hence the integral in (2.6) which defines $B(t+i\eta t)$ is uniquely determined by the convergent expansion $\sum_{k=0}^{\infty} a_k u^k(k!)^{-1}$, $|u|<\sigma^{-1}$. Let us now show that it can be analytically continued to $S_\sigma^1\cap S_\sigma^2$. Indeed, in the expression

$$\frac{d^m}{du^m}B(u)=a_m+(2\pi i)^{-1}\oint_{\mathrm{Re}\,z^{-1}=r^{-1}} e^{u/z}z^{-(m+1)}\left[\phi(z)-\sum_{k=0}^{m}a_k z^k\right]dz \tag{2.13}$$

$(0<r<R)$, we can separate the integral along the upper half-circle $\{z/\mathrm{Re}\,z^{-1}=r^{-1}, \mathrm{Im}\,z>0\}$ and the integral along the lower half-circle. The former, when considered for $u=t>0$, is bounded by $c_6e^{t/R}\sigma^{m+1}(m+1)!$ by means of (2.2). So it defines a convergent power series expansion near any $t>0$, and a unique analytic function $B_1(u)$ in the strip $S_\sigma^1=\{u/\mathrm{dist}(u,\mathbb{R}_+)<\sigma^{-1}\}$. The second integral, when considered for $u=t+i\eta t(t>0,\,\eta>0)$ can be performed on the equivalent path $\Gamma_{r,\varepsilon}$ defined in (2.10), where again $\varepsilon=\arctan\eta$. Hence, by (2.2), it is bounded by $c_7c(\varepsilon)^{m+1}(m+1)!\exp(tR^{-1}(1+\eta^2)^{1/2})$, and this second integral is the m^{th} derivative of a function $B_2(u)$ analytic at least for $\mathrm{Re}\,u>-\sigma^{-1}$, $\mathrm{Im}\,u>0$. The sum $B(u)$ of such two functions is analytic in $S_\sigma^1\cap S_\sigma^2$.

Setting $m=0$ and $\eta=\eta_0t^{-1}$ in the second bound (with η_0 sufficiently small), $B(u)$ satisfies (2.3) along any half-line $\mathrm{Im}\,u=\eta_0$ in $S_\sigma^1\cap S_\sigma^2$ as $\mathrm{Re}\,u\to\infty$. Hence, for $\mathrm{Re}\,z^{-1}>R^{-1}$, $z^{-1}\int_{\gamma(\eta_0)} e^{-u/z}B(u)\,du$ is absolutely convergent and independent of $\gamma(\eta_0)$, if $\gamma(\eta_0)$ is any path in $S_\sigma^1\cap S_\sigma^2$ with endpoints $u_1=0$ and $u_2=i\eta_0+\infty$. In fact it can be written as $z^{-1}\int_0^\infty e^{-t/z}B(t+i0)dt$, where the distribution $B(t+i0)$ is the boundary value of $B(u)$ as $\mathrm{Im}\,u\to+0$. Furthermore

$$\phi(z)=z^{-1}\int_0^\infty e^{-t/z}B(t+i0)\,dt \tag{2.14}$$

by (2.6) [which is the Riemann-Fourier inversion formula of the Laplace transformation, in z^{-1}, when applied to $z\phi(z)$ and calculated in $t+i\eta t$], and by the one-to-one relationship between half-plane holomorphic functions and a large class of distributions via the Laplace transformation (e.g. in [1, Theorem 2.1]). Since $f(z)=(\phi(z)+\bar{\phi}(\bar{z}))/2$, (2.14) implies (2.4) for $\mathrm{Re}\,z^{-1}>R^{-1}$ and the criterion is proved.

To prove the necessary condition, let $B(u)=B_1(u)+B_2(u)$, where $B_1(u)$ and $B_2(u)$ are analytic in $S_{\sigma'}^1(\sigma'<\sigma)$ and $S_{\sigma'}^2$ respectively, and let (2.3) be verified. By Cauchy's integral formula

$$|B_1^{(m)}(t)|\leqq c_1(m+1)!\,\sigma^{m+1}e^{t/R}\,, \tag{2.15a}$$

$$|B_2^{(m)}(u)|\leqq c_2(m+1)!\,\eta_0^{-m-1}\exp(\mathrm{Re}\,u/R),\,0<\eta_0\leqq\sigma^{-1}\,, \tag{2.15b}$$

along some regular path $\gamma(\eta_0)$ in $S_\sigma^1 \cap S_\sigma^2$ starting from the origin and defined by $\operatorname{Im} u = \eta_0$ for large $\operatorname{Re} u$. Then $\phi(z) = z^{-1} \int_{\gamma(\eta_0)} e^{-u/z} B(u)\, du$ is finite and analytic for $\operatorname{Re} z^{-1} > R^{-1}$ and, integrating by parts N times, one checks:

$$\begin{aligned}
\left|\phi(z) - \sum_{k=0}^{N-1} a_k z^k\right| &= \left|z^{N-1} \int_{\gamma(\eta_0)} e^{-u/z} B^{(N)}(u)\, du\right| \\
&\leqq c_8 |z|^N (N+1)!\, \sigma^{N+1} \int_0^\infty \exp(tR^{-1} - tz^{-1})\, dt + c_9 \eta_0^{-N-1} (N+1)!\, |z|^N \\
&\quad \cdot \left(c_{10} + \int_1^\infty \exp(-t \operatorname{Re} z^{-1}(1 - t^{-1}\eta_0 \tan(\arg z^{-1}))) e^{t/R}\, dt\right) \\
&\leqq c_{11} (N+1)!\, (\max(\eta_0^{-1}, \sigma))^{N+1} |z|^N
\end{aligned} \tag{2.16}$$

if $\operatorname{Re} z^{-1} > R^{-1}$ and $|\tan(\arg z)| < \eta_0^{-1}$. If $\arg z = -\pi/2 + \varepsilon$, then $\tan(\arg z^{-1}) \sim -\varepsilon^{-1}$ as $\varepsilon \to +0$: hence an estimate of the type (2.2) is verified for $\operatorname{Re} z^{-1} > R^{-1}$, $-\pi/2 + \varepsilon \leqq \arg z < \pi/2$, by choosing $\eta_0 = \varepsilon$. Thus the necessary condition is proved with $\phi(z) = z^{-1} \int_0^\infty e^{-t/z} B(t + i0)\, dt$, $f(z) = (\phi(z) + \bar{\phi}(\bar{z}))/2$.

Remark 2. If hypothesis (2.2) holds then $c(\varepsilon) = O(\varepsilon^{-1})$ as $\varepsilon \to 0$ by a Phrägmen-Lindelöf type argument, as explained in the following Proposition.

Proposition 2. *Let $d(z)$ be analytic for $\operatorname{Re} z^{-1} > r^{-1}$, continuous and bounded for $\operatorname{Re} z^{-1} = r^{-1}$, and let $|d(z)| \leqq \sigma^n a(\varepsilon)^n n!\, |z|^n$ for $|\arg z| \leqq \pi/2 - \varepsilon$, for any $n \in \mathbb{N}_0$ and $\varepsilon > 0$. Then one can choose $a(\varepsilon) \leqq M\varepsilon^{-1}$ for some M independent of ε.*

Proof. For some $M > 0$, $\sigma_1 > 0$, $|d(z) e^{1/z}| \leqq M$ if $\operatorname{Re} z^{-1} = r^{-1}$ and $|d(z)| \leqq (\sigma_1 a(\varepsilon) e^{-1} n)^n |z|^n$ if $\operatorname{Re} z^{-1} > r^{-1}$, $|\arg z| \leqq \pi/2 - \varepsilon$. Since $(\sigma_1 a(\varepsilon) e^{-1} n)^n$ is the maximum value of $|z|^{-n} \exp(-(\sigma_1 a(\varepsilon)\, |z|)^{-1})$ for each $n \in \mathbb{N}$, we have:

$$|d(z) e^{1/z}| \leqq \exp(-(\sigma_1 a(\varepsilon)\, |z|)^{-1}) e^{1/|z|} \leqq e^{1/|z|} \tag{2.17}$$

for a sequence of values $z = z_n$, $n \in \mathbb{N}$, such that $|z_n| \equiv (\sigma_1 a(\varepsilon) n)^{-1} \to 0$ as $n \to \infty$, uniformly with respect to the phase of z. Hence the function $\Phi(u) = d((u + r^{-1})^{-1}) \exp(u + r^{-1})$ is analytic for $\operatorname{Re} u > 0$, continuous and bounded for $\operatorname{Re} u = 0$. If we consider that function only for $\operatorname{Re} u \geqq 0$, $\operatorname{Im} u \geqq 0$, setting $u = w^{1/2} e^{i\pi/4}$, the function $\Psi(w) = \Phi(w^{1/2} e^{i\pi/4})$ is analytic for $\operatorname{Re} w \geqq 0$ and continuous for $\operatorname{Re} w = 0$, since $\Phi(u)$ is of course continuous and bounded for $\operatorname{Im} u = 0$. Moreover $|\Psi(w)| \leqq \exp(|r^{-1} + w^{1/2} e^{i\pi/4}|) \leqq \exp(r^{-1} + |w|^{1/2})$ by (2.17), for a sequence of values $w = w_n$, $n \in \mathbb{N}$, such that $|w_n| \to +\infty$, uniformly with respect to the phase of w.

Thus, by a standard Phrägmen-Lindelöf theorem [2, Theorem 1.4.1, p. 3] $|\Psi(w)| \leqq M$ uniformly for $\operatorname{Re} w \geqq 0$. Since the same argument holds if we consider $\Phi(u)$ only for $\operatorname{Re} u \geqq 0$, $\operatorname{Im} u \leqq 0$, we have $|\Phi(u)| \leqq M$ for $\operatorname{Re} u \geqq 0$, that is $|d(z)| \leqq M |e^{-1/z}|$ uniformly for $\operatorname{Re} z^{-1} \geqq r^{-1}$. Now, for $|\arg z| = \pi/2 - \varepsilon$, $|e^{-1/z}| = \exp(-|z|^{-1} \sin \varepsilon) \leqq |z|^n (\sin \varepsilon)^{-n} n! \sim |z|^n \varepsilon^{-n} n!$ as $\varepsilon \to +0$. Hence the choice $a(\varepsilon) = \varepsilon^{-1}$ is always possible and the lemma is proved.

Remark 3. By Remark 2 a larger class of functions $f(z)$ can be considered, which are distributional Borel sums according to Definition 1, and such that the corresponding $d(z)$ (with zero expansion coefficients) is not bounded as $\arg z \to \pm\pi/2$ in C_R. For example, if $f(z) = z^{-1} \int_0^\infty e^{-t/z} PP(1-t)^{-1}\,dt$, then $d(z) = -i\pi z^{-1} e^{-1/z}$. In such cases it will be sufficient to verify the hypotheses of Theorem 1 for $g(z) \equiv z^m f(z)$, for some $m \in \mathbb{N}$.

Remark 4. For $\alpha = p/q$, with $p, q \in \mathbb{N}$, the distributional Borel-Le Roy summability of order $(\alpha, 1)$ (defined by $\mu_k = \Gamma(\alpha k + 1)$, $\varrho(t) = \alpha^{-1} \exp(-t^{1/\alpha}) t^{1/\alpha - 1}$) admits a criterion similar to the one contained in Theorem 1. The required condition is analyticity for $\mathrm{Re}\, z^{-1/\alpha} > R^{-1}$ and the estimate (2.2) in that region, with $N!$ replaced by $\Gamma(\alpha N + 1)$. Indeed, for $\alpha \in \mathbb{N}$, it turns out that $\Phi(z) = f(z^\alpha)$ satisfies the hypotheses of Theorem 1, so that $f(z) = \Phi(z^{1/\alpha})$ has the required integral representation. For non-integer values of α, the same arguments of Theorem 1 can be adapted only if we know a Watson-Nevanlinna criterion for the standard Borel-Le Roy summability of order $(\alpha, 1)$ (for the case $\alpha = 1/2$ see [7]). For rational (positive) α, let us prove the following:

Theorem 3. *Let $\alpha = p/q$, with $p, q \in \mathbb{N}$. Let f be analytic in $C_R = \{z / \mathrm{Re}\, z^{-1/\alpha} > R^{-1}\}$ and such that*

$$|R_N(z)| \equiv \left| f(z) - \sum_{k=0}^{N-1} a_k z^k \right| \leqq A \sigma^N \Gamma(\alpha N + 1) |z|^N \tag{2.18}$$

uniformly in C_R. Then the formal expansion $\sum_{k=0}^\infty a_k \tau^{k\alpha} / \Gamma(\alpha k + 1)$ is convergent for small $|\tau|$ and determines a function $B(\tau^\alpha)$ analytic in $S_\varrho = \{\tau / |\arg \tau| < \pi$, $\mathrm{dist}(\tau, \mathbb{R}_+) < \varrho\}$ for some $\varrho > 0$, and such that

$$|B(\tau^\alpha)| \leqq K e^{|\tau|/R} \tag{2.19}$$

uniformly in any S_{ϱ_1}, with $\varrho_1 < \varrho$. Moreover, setting $t = \tau^\alpha$,

$$f(z) = \alpha^{-1} z^{-1} \int_0^\infty B(t) e^{-(t/z)^{1/\alpha}} (t/z)^{1/\alpha - 1}\,dt \tag{2.20}$$

for all $z \in C_R$. Conversely, if $f(z)$ is given by (2.20), with the above properties for $B(\tau^\alpha)$, then it satisfies remainder estimates of the type (2.18) uniformly in any C_r, $0 < r < R$.

Proof. Notice that the above stated condition on $f(z)$ for Borel-Le Roy summability of order $(p\alpha, 1)$, with $p \in \mathbb{N}$, implies the analogous condition on $\Phi(z) = f(z^p)$ for Borel-Le Roy summability of order $(\alpha, 1)$. Hence it is sufficient to prove the theorem for $\alpha = 1/q$, $q \in \mathbb{N}$.

Fixing $\alpha = 1/q$, and regarding (2.20) as a Laplace transform in the variable z^{-q}, let us consider the integral formally obtained from (2.20) by the Riemann-Fourier inversion formula, and its formal derivatives with respect to $\tau \equiv t^q$:

$$\begin{aligned} \frac{d^m}{d\tau^m} B(\tau^{1/q}) = (2\pi i)^{-1} & \oint_{\mathrm{Re}\, z^{-q} = r^{-1}} e^{\tau z^{-q}} q z^{-1-mq} \left[f(z) - \sum_{k=0}^{mq} a_k z^k \right] dz \\ & + (2\pi i)^{-1} \oint_{\mathrm{Re}\, z^{-q} = r^{-1}} e^{\tau z^{-q}} q z^{-1-mq} \sum_{k=0}^{mq} a_k z^k \,dz\,. \end{aligned} \tag{2.21}$$

Here $\tau > 0$, $m \in \mathbb{N}_0$, and $0 < r < R$. Now, the first integral exists and is independent of r by the remainder estimate (2.18). The second integral is well-defined, too, and turns out to be equal to $\frac{d^m}{d\tau^m} \sum_{k=0}^{mq} a_k \tau^{k/q}/\Gamma(k/q+1)$ by a simple computation of the direct transform. Moreover, for all $\tau > 0$ and $m \in \mathbb{N}_0$:

$$\left|\frac{d^m}{d\tau^m} B(\tau^{1/q})\right| \leqq A_1 \left[\sigma^{mq+1} e^{\tau/R} \Gamma(m+1+1/q) + \sigma^{mq+1} \Gamma(m+1) \sum_{K=0}^{mq} \tau^{-m+k/q}\right]. \tag{2.22}$$

For $m=0$ and $N \in \mathbb{N}$:

$$B(\tau^{1/q}) = \sum_{k=0}^{N-1} a_k \tau^{k/q}/\Gamma(k/q+1) + (2\pi i)^{-1} \oint_{\operatorname{Re} z^{-q} = r^{-1}} \exp(\tau z^{-q}) q z^{-1} R_N(z)\, dz\,. \tag{2.23}$$

For $\operatorname{Re} z^{-q} = r^{-1}$, $|z| \leqq r^{1/q}$. Thus, by choosing $r = q/N$ and using (2.18), the remainder term in (2.23) is bounded by $A_2(N/q)^{1/2}(\sigma\tau^{1/q})^N$, which tends to zero as $N \to \infty$ if $0 < \tau^{1/q} \equiv t < \sigma^{-1}$. Thus $B(\tau^{1/q})$ is identified with the convergent Puiseux expansion in (2.23) for $|\tau| < \sigma^{-q}$. On the other hand, if $b_m(\tau_0) = \frac{d^m}{d\tau^m} B(\tau^{1/q})|_{\tau=\tau_0}$, for any fixed $\tau_0 > 0$, $B(\tau^{1/q}) = \sum_{m=0}^{\infty} b_m(\tau_0)\,(m!)^{-1}\,(\tau-\tau_0)^m$ in some neighbourhood of τ_0, by (2.22).

Therefore $B(\tau^{1/q})$ is uniquely determined as an analytic function of τ in some neighbourhood of $\mathbb{R}_+$ on the first sheet of the Riemann surface of $\tau^{1/q}$. On such a region, $B(\tau^{1/q})$ satisfies (2.19), again by (2.22). Hence $B(\tau^{1/q})$ admits a Laplace transform, which coincides with $zf(z)$ by (2.21), so that (2.20) is proved.

Conversely, let $B(\tau^{1/q})$ be analytic in some neighbourhood of $\mathbb{R}_+$ on the first sheet of the Riemann surface of $\tau^{1/q}$, with the bound (2.19). Then $f(z)$ defined by (2.20) is analytic for $\operatorname{Re} z^{-q} > R^{-1}$. By Cauchy's integral formula $|d^m/d\tau^m(B(\tau^{1/q}))| \leqq A_3 \sigma_0^{m+1} m!\, e^{\tau/R}$ for $\tau > 0$, for some A_3, $\sigma_0 > 0$. Integrating by parts (2.20) N times, an estimate of the type (2.18) follows and the theorem is proved.

On the basis of Theorem 3, closely following the arguments used in the proof of Theorem 1, one checks the explicit criterion for distributional Borel-Le Roy summability of order $(\alpha, 1)$, $\alpha \in \mathbb{Q}_+$.

Theorem 4. *Let α and C_R be defined as in Theorem 3. Let $f(z) = (\phi(z) + \bar{\phi}(\bar{z}))/2$, with $\phi(z)$ analytic in C_R and satisfying a remainder estimate of type* (2.18) *for $z \in C_R$, $-\pi/2 + \varepsilon \leqq \arg z < \pi/2$, for any fixed $\varepsilon > 0$. Then $f(z)$ is the Borel-Le Roy sum of order* $(\alpha, 1)$ *of its asymptotic series in the distributional sense of Definition 1. The stated hypothesis is essentially a necessary condition, too.*

3. Applications

(A) An example of distributional Borel summability can be given in the context of $\lambda\Phi^4$ lattice fields, which are defined in terms of measures in $\mathbb{R}^{Z^d}$ ($d \in \mathbb{N}$). For any finite subset $L \subset \mathbb{R}^{Z^d}$, let $\mu_L^\lambda(\cdot)$ be the measure which is concentrated on the cylinder sets with basis $\mathbb{R}^L$ and such that

$$\mu_L^\lambda(A) = \int_A \exp\left\{H_L(x_1, \ldots, x_{|L|}) - \lambda \sum_{p \in L} (x_p^2 - 1)^2\right\} dx_1 \ldots dx_{|L|}\,, \tag{3.1}$$

where $\lambda > 0$ and

$$H_L(x_1, \ldots, x_{|L|}) = -\beta \sum_{\substack{|p-q|=1 \\ p,q \in L}} x_p x_q + (d\beta + m_0^2) \sum_{p \in L} x_p^2 - h \sum_{p \in L} x_p . \tag{3.2}$$

Here the inverse temperature β and the magnetic field h are fixed. Notice that such models can be regarded, for large λ, as perturbations of the corresponding Ising models ($\lambda = +\infty$) in both finite and infinite volume [15]. Fix $L \subset Z^d$, $n = |L|$, and set $\mu_L^\lambda = \mu^\lambda$. Denoting the scalar product in $\mathbb{R}^n$ by $\langle \cdot, \cdot \rangle$, let us consider the characteristic function of μ^λ:

$$C_\lambda(z) = \int_{\mathbb{R}^n} e^{i\langle z, x\rangle} d\mu^\lambda(x) \tag{3.3}$$

for any fixed $z \in \mathbb{R}^n$.

Proposition 5. *Set $g = \lambda^{-1/2}$, $g > 0$. Then* (3.3) *admits a series expansion in powers of g with coefficients given by polynomials in $e^{-g^{-2}}$. Fixing $e^{-g^{-2}} = e^{-\gamma}$, $\gamma > 0$, in the coefficients, one obtains a power series in g, whose real and imaginary parts satisfy Theorem 4 and are the distributional Borel-Le Roy sums of order* 1/2 *of their series for all $g > 0$. In particular, for $g = \gamma^{-1/2}$ such sums coincide with the real and the imaginary parts of $C_\gamma(z)$.*

Proof. Setting $F(x) = (\cos\langle z, x\rangle)\exp(H(x))$ [or $F(x) = (\sin\langle z, x\rangle)\exp(H(x))$, respectively], where $H(x)$ is given by (3.2), let us first consider the case $n = 1$:

$$\begin{aligned} C_{g^{-2}} &\equiv \int_{-\infty}^{+\infty} F(x) e^{-g^{-2}(x^2-1)^2} dx \quad [\text{setting } \varrho = x^2 - 1,\ x^\pm(\varrho) = \pm\sqrt{1+\varrho}] \\ &= \int_0^1 \{F(x^-(-\varrho)) + F(x^+(-\varrho))\} e^{-g^{-2}\varrho^2} 2^{-1}(1-\varrho)^{-1/2} d\varrho \\ &\quad + \int_0^\infty \{F(x^-(\varrho)) + F(x^+(\varrho))\} e^{-g^{-2}\varrho^2} 2^{-1}(1-\varrho)^{-1/2} d\varrho \\ &\equiv I_1(g) + I_2(g) . \end{aligned} \tag{3.4}$$

Now $I_1(g)$ can be written as an integral from 0 to ∞: setting

$$G(\varrho) = 2^{-1}(1-\varrho)^{-1/2}\{F(x^-(-\varrho)) + F(x^+(-\varrho))\} , \tag{3.5}$$

we have

$$\begin{aligned} I_1(g) &= 2^{-1} \int_0^\infty \{G(\varrho + i0) + G(\varrho - i0)\} e^{-g^{-2}\varrho^2} d\varrho - 2^{-1} e^{-g^{-2}} \\ &\quad \cdot \int_0^\infty \{G((1+\tau^2)^{1/2} + i0) + G((1+\tau^2)^{1/2} - i0)\} \\ &\quad \cdot \tau e^{-g^{-2}\tau^2}(1+\tau^2)^{-1/2} d\tau , \end{aligned} \tag{3.6}$$

where the substitution $\varrho = (1+\tau^2)^{1/2}$ gives rise to the exponential factor $e^{-g^{-2}}$. This suggests that we define:

$$\begin{aligned} \phi_1(\gamma, g) &= \int_0^\infty G(\varrho + i0) e^{-g^{-2}\varrho^2} d\varrho \\ &\quad - e^{-\gamma} \int_0^\infty G((1+\tau^2)^{1/2} + i0) e^{-g^{-2}\tau^2} \tau (1+\tau^2)^{-1/2} d\tau , \end{aligned} \tag{3.7a}$$

$$f_1(\gamma, g) = 2^{-1}\{\phi_1(\gamma, g) + \bar{\phi}_1(\gamma, \bar{g})\} \tag{3.7b}$$

for any $\gamma>0$. This implies $I_1(g)=f_1(\gamma, g)$, if $\gamma=g^{-2}$. Now it turns out that $f_1(\gamma, g)$ satisfies Theorem 4 with $\alpha=1/2$. Indeed, from (3.7a) analyticity follows for $\mathrm{Re}\, g^{-2}>0$. By rescaling $\varrho\to|g|\varrho e^{i\varepsilon}$, $\tau\to|g|\tau e^{i\varepsilon}$, for any small $\varepsilon>0$, and setting $g=|g|e^{i\theta}$, $\phi_1(\gamma, g)$ has the form:

$$\begin{aligned}\phi_1(\gamma, g)= & \int_0^\infty G(|g|\varrho e^{i\varepsilon}) \exp(-e^{-2i\theta+2i\varepsilon}\varrho^2)\,|g|e^{i\varepsilon}\,d\varrho-e^{-\gamma}\\ & \cdot\int_0^\infty G((1+|g|^2\tau^2 e^{2i\varepsilon})^{1/2})\,|g|^2\tau e^{2i\varepsilon}(1+|g|^2\tau^2 e^{2i\varepsilon})^{-1/2}\\ & \cdot\exp(-e^{-2i\theta+2i\varepsilon}\tau^2)\,d\tau\,. \end{aligned} \tag{3.8}$$

Hence

$$\left|\frac{d^m}{dg^m}\phi_1(\gamma, g)\right| \leqq B_1 B_2(\varepsilon)^m m!\,\Gamma(1+m/2) \tag{3.9}$$

uniformly for $-\pi/4+\varepsilon<\arg(g)<\pi/4$, because $\left|\frac{d^m}{d|g|^m}G(|g|u)\right|\leqq m!\,|u|^m B_3^m$ if u is contained in some analyticity region for G, and this is the case for any fixed $\varepsilon>0$, by (3.5). From (3.9) the required remainder estimates follow and $f_1(\gamma, g)$ satisfies the conditions of Theorem 4 for $\alpha=1/2$. Since in (3.4) $I_2(g)$ is a usual Borel-Le Roy sum of order 1/2 by Theorem 3, the assertion is proved for $n=1$.

For $n>1$ we have to consider

$$C_{g^{-2}}\equiv\int_{\mathbb{R}^N} F(x_1, \dots, x_n)\exp\left\{-g^{-2}\sum_{h=1}^n (x_h^2-1)^2\right\}dx_1\dots dx_n$$

[setting $\varrho_h=(x_h)^2-1$, $x_h^1=(1+\varrho_h)^{1/2}$, $x_h^{-1}=-(1+\varrho_h)^{1/2}$, $\alpha=(\alpha_1, \dots, \alpha_n)$ with either $\alpha_h=1$ or $\alpha_h=-1$]

$$\begin{aligned} &=\sum_{\beta\in\{-1,1\}^n}\sum_{\alpha\in\{-1,1\}^n}\int_{B_1}\dots\int_{B_n} F(x_1^{\alpha_1}(\beta_1\varrho_1), \dots, x_n^{\alpha_n}(\beta_n\varrho_n))\\ &\quad\cdot\prod_{h=1}^n \exp(-g^{-2}\varrho_h^2)\alpha_h 2^{-1}(1+\beta_h\varrho_h)^{-1/2}\,d\varrho_h\,, \end{aligned} \tag{3.10}$$

where $B_h=(0,\infty)$ if $\beta_h=1$, and $B_h=(0,1)$ if $\beta_h=-1$ $(h=1, \dots, n)$. Notice that singularities in (3.10) can arise from the square roots $x^{\alpha_h}(\beta_h\varrho_h)\equiv\alpha_h(1+\beta_h\varrho_h)^{1/2}$ when $\beta_h=-1$, i.e. when the integration interval is $B_h=(0,1)$. Thus, setting

$$\begin{aligned} G^{\alpha,\beta}(\varrho_1, \dots, \varrho_n)&\equiv F(x_1^{\alpha_1}(\varrho_1), \dots)\prod_{h=1}^n \alpha_h 2^{-1}(1+\beta_h\varrho_h)^{-1/2}\\ &=2^{-1}\{G^{\alpha,\beta}(\varrho_1+i0, \dots, \varrho_n+i0)+G^{\alpha,\beta}(\varrho_1-i0, \dots, \varrho_n-i0)\} \end{aligned} \tag{3.11}$$

for any α and β, for $\varrho_h\in B_h$ $(h=1, \dots, n)$, we can write each integral on $(0,1)$ as a sum of two integrals on $(0,\infty)$ exactly as in (3.5). This produces exponential factors $e^{-g^{-2}}$ to be fixed $(g^{-2}=\gamma)$ in order to obtain a power series expansion $\sum_{k=0}^\infty a_k(\gamma)g^k$, with coefficients depending on $e^{-\gamma}$. Therefore (3.10) can be written as a finite sum of integrals of the form

$$2^{-1}e^{-r\gamma}\int_0^\infty\dots\int_0^\infty\{L(\sigma_1+i0, \dots, \sigma_n+i0)+L(\sigma_1-i0, \dots, \sigma_n-i0)\}\prod_{h=1}^n e^{-g^{-2}\sigma_h^2}\,d\sigma_h\,, \tag{3.12}$$

where $r \in \{0, 1, \ldots, n\}$ and L has a singular dependence on σ_h either through $(1-\sigma_h)^{1/2}$ or through $(1-(1+\sigma_h^2)^{1/2})^{1/2}$. Therefore by scaling $\sigma_h \to |g|\sigma_h e^{i\varepsilon}$, with $\varepsilon > 0$, (3.12) is a sum $2^{-1}\{\phi(\gamma, g) + \bar{\phi}(\gamma, \bar{g})\}$, where

$$\phi(\gamma, g) = e^{-r\gamma} \int_0^\infty \ldots \int_0^\infty L(|g|\sigma_1 e^{i\varepsilon}, \ldots, |g|\sigma_n e^{i\varepsilon}) \prod_{h=1}^n (\exp(-e^{-2i\theta+2i\varepsilon}\sigma_h^2\, d\sigma_h)\,. \tag{3.13}$$

Thus we obtain the required bound $\left|\frac{d^m}{dg^m}\phi(\gamma, g)\right| \leqq B_1 B_2(\varepsilon)^m \Gamma(1+3m/2)$ uniformly for $-\pi/4+\varepsilon \leqq \arg(g) \leqq \pi/4$ by evaluating $\frac{d^m}{d|g|^m} L(|g|u_1, \ldots, |g|u_n)$ with $u_1, \ldots, u_n$ contained in analyticity domains for L, and Proposition 5 is proved.

(B) Let $H(g) = 2^{-1}p^2 + x^2(1-gx)^2$, $g > 0$, be the double-well Hamiltonian, as a self-adjoint operator in $L^2(\mathbb{R})$. One can approximate the trace $\mathrm{Tr}(e^{-tH(g)})$, $t > 0$, by means of the Trotter formula and the corresponding integral kernels $K^{(n)}(x, y)$. Setting $t = g^{-2}\tau$, $\mathrm{Tr}(e^{-g^{-2}\tau H(g)})$ is approximated by

$$\begin{aligned} F_n(\tau, g) &= \int_{\mathbb{R}} K^{(n)}(x, x; -i\tau g^{-2})\, dx \\ &= (2\pi g^{-2}\tau/n)^{-n/2} \int_{\mathbb{R}^n} \exp\left\{-(2\tau)^{-1} n(x_1 - x_n)^2 \right. \\ &\quad \left. - \sum_{h=2}^n 2^{-1} n(x_{h-1} - x_h)^2 \tau^{-1} - g^{-4}\tau n^{-1} \sum_{h=1}^n x_h^2(1-x_h)^2 \right\} \\ &\quad \cdot g^{-n}\, dx_1 \ldots dx_n\,, \end{aligned} \tag{3.14}$$

where the $K^{(n)}$'s are defined in [16, p. 6] and a scaling $x_h \to g^{-1}x_h$ $(h = 1, \ldots, n)$ has been performed in the integral.

Now, for fixed τ and n, $F_n(\tau, g)$ has essentially the same form of the function (3.3) discussed in (A). In fact, by a translation $x_h \to (x_h + 1/2)$, $x_h^2(1-x_h)^2$ is replaced by $(x_h^2 - 1/4)^2$ $(h = 1, \ldots, n)$ and the same arguments of (A) allow us to prove the following analogous result.

Proposition 6. *Any approximant* (3.14) *of* $\mathrm{Tr}(e^{-g^{-2}\tau H(g)})$, *where* $H(g)$ *is the double-well Schrödinger operator, as a function of* g^2, *is a distributional Borel-Le Roy sum of order* 1/2 *of its power series expansion, obtained by fixing exponentially small terms in the coefficients, as in Proposition 5.*

Notice that the case $n = 1$ (i.e. the original model proposed by 't Hooft [11]) admits a remarkable simplification due to the fact that the "kinetic" part in (3.14) disappears. In this case we can directly consider

$$\int_{\mathbb{R}} K^{(1)}(x, x; -it)\, dx = \int_{\mathbb{R}} e^{-tx^2(1-gx)^2}\, dx \tag{3.15}$$

and sum its series expansion in powers of g (with coefficients independent of g) by the distributional Borel-Le Roy method of order 1/2.

Indeed, by the substitutions $\xi = g^{-1}x$ and $u = \xi(\xi - 1)$, and noting that $(1-4(u+i0))^{-1/2} + (1-4(u-i0))^{-1/2} = 0$ for $u > 1/4$, (3.15) turns out to be a sum of the required type $(\phi(g) + \bar{\phi}(\bar{g}))/2$, and Theorem 4 applies with $\alpha = 1/2$.

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