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General relativity for mathematicians, by R. K. Sachs and H. Wu, Graduate Texts in Mathematics, Springer-Verlag, New York, Heidelberg, Berlin, 1977, xii + 291 pp., $19.80.

The theory of general relativity is now more than sixty years old. During its formative years, the theory was a constant source of constructive interaction between mathematicians and physicists. Physicists drew heavily upon recent developments in differential geometry, and much research in differential geometry was stimulated by problems in general relativity. Many of the great physicists of the day (e.g., Einstein, Lorentz, Poincaré) were making fundamental contributions to mathematics, and many of the great mathematicians of the day (e.g., Cartan, Hilbert, Weyl) were making important contributions to physics. The years 1900-1930 marked a golden period in math-physics cooperation.

This pleasant state of affairs deteriorated, however, in the late 30s-early 40s. Mathematicians were moving toward a global viewpoint (manifolds, fiber bundles, cohomology) whereas physicists were content to work locally, doing all computations in a single coordinate system. Except for certain rather conjectural cosmological ideas, all the interesting physics (e.g. bending of light, perihelion precession, gravitational redshift, expansion of the universe) could be dealt with adequately without manifold theory. As the interests of geometers and relativists diverged, so of course did their languages. Differential geometers began to use invariant tensor notation and differential forms whereas physicists were content with classical tensor analysis, a language in which they were extremely fluent and which was quite adequate for the computations of interest to them. The years 1940-1970 marked a period of (comparatively) little interaction between geometers and physicists.

Now, in the 70s, the pendulum is swinging back. This is due largely to the fact that physicists have recently been applying global geometric techniques to obtain results of indisputable importance. One set of results (Hawking-Penrose [4]; see Penrose [6] for an account written for mathematicians) says that in any spacetime satisfying certain physically reasonable conditions

(conditions which are also geometrically fairly natural) there must exist an incomplete inextendible causal geodesic. Physically this says that some freely falling particle or photon will, after finite time, run out of the universe, or dually, that some particle or photon has a history which terminates a finite time into the past, e.g., in the big bang. Another set of results (see, e.g., Hawking [**1**]) describes the nature of black holes. (Typical theorems: the surface area of a black hole increases monotonically with time[1]; black holes cannot bifurcate; static black holes must have the geometry of a Schwarzschild or a Reissner-Nordström metric.) These black hole theorems depend heavily on a detailed analysis of the large scale conformal structure of spacetime. Still another set of results (see, e.g., Wu-Yang [**8**]) uses fiber bundles and characteristic classes to describe magnetic monopoles and gauge fields.

So the time has arrived when mathematicians should be (and are) taking a closer look at recent developments in the geometry of relativity. But this is a nontrivial task. One difficulty is that geometers of the current generation are untrained in the language of classical tensor analysis and hence are poorly equipped to read the physics literature. Another difficulty is more serious: mathematicians generally find the physics literature difficult because of the lack of mathematical precision to be found there and because of the difference between the mathematician's and the physicist's conception of what constitutes a proof. Quoting from the book under review (p. xii):

> In a serious physics text basic physical quantities are almost never explicitly defined. The reason is that the primary definitions are actually obtained by showing photographs, by pointing out of the window, or by manipulating laboratory equipment. The more mathematically explicit a definition, the less accurate it tends to be in this primary sense.

And, again (p. 72),

> Within physics, [the] *definition* and the *motivation* below would count as a *proposition* and a *proof*, rather than a definition and a motivation. (Italics added by reviewer.)

The book under review will ease the way for any mathematician who wants to learn about (and perhaps contribute to) modern developments in the geometry of general relativity. It is a timely collaboration between a physicist (Sachs) and a mathematician (Wu). They have taken great care to develop the subject with mathematical precision. The language used is almost exclusively that of modern differential geometry. Although the book does not contain the deeper theorems of the subject (singularity theorems, black hole theorems, etc.), it does develop sufficient background to enable the reader to study the standard books on the subject written by physicists [**3**], [**5**], [**7**]. There is an especially fine chapter on electromagnetism and matter. According to the authors (p. 60), "Mathematicians notoriously find discussions of matter

[1]This theorem, although valid within the framework of general relativity, must be modified in the presence of quantum corrections; see Hawking's excellent survey article [**2**].

difficult." Indeed, we do. After studying this chapter, mathematicians will find these discussions less difficult.

Prerequisites for profitable reading of this book consist of a knowledge of basic graduate level differential geometry and a knowledge of Newtonian physics at least equivalent to a good freshman level course in the subject. The presence of many exercises make the book appropriate as a text, perhaps for example as the second semester of a first graduate course in differential geometry. The first half of the book should already sufficiently prepare the reader for entry into the physics literature with a minimum of trauma.

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Stochastic integration and generalized martingales, by A. U. Kussmaul, Pitman Publishing, London, San Francisco, Melbourne, 1977, 163 pp., $14.00.

Modern stochastic integration began in 1828 when the English botanist Robert Brown observed the motion of pollen grains in a glass of water. Bachelier (1900, [2]) and Einstein (1905, [7]) studied the mathematical modelling of the motion of the grains, now called *Brownian motion*. But it was Wiener (1923, [25]), making use of the ideas of Borel and Lebesgue, who created the modern rigorous mathematical model of Brownian motion, the *Wiener process*.

The Wiener process has three properties which make it of fundamental importance to the theory of stochastic processes: it is a Gaussian process, it is a strong Markov process, and it is a martingale. Let $W = (W(t, \omega))_{t \geq 0}$ denote a Wiener process, in which t is the time and each ω is a particle; then $W(t, \omega)$ represents the position of that particle at time t. One can show that except on a set of probability zero, every sample path (i.e., $W(t, \omega)$ as a function of t for fixed ω) is continuous but is of unbounded variation on every compact time set. Since the sample paths are nowhere differentiable, the Brownian particle cannot have an instantaneous velocity. This bizarre property may be viewed as a consequence of the Markov property; that is: