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ON AN INVERSION THEOREM FOR THE GENERAL MEHLER-FOCK TRANSFORM PAIR

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Let $P_m^k(y)$ be the Legendre function of the first kind and let $\Gamma(z)$ be the Gamma function. Then the general Mehler-Fock transform of complex order k of a function $g(y)$ is defined by the equation

$$f(x) = L_2(g) = \pi^{-1}x \sin h(\pi x)\Gamma\left(\frac{1}{2} - k - ix\right) \times \Gamma\left(\frac{1}{2} - k + ix\right)\int_1^\infty g(y)P_{ix-1/2}^k(y)dy ,$$

the inversion theorem states

$$g(y)=L_1(f) = \int_0^\infty f(x)P_{ix-1/2}^k(y)dx .$$

It is stated on page 416 of I. N. Sneddon's book 'The Use of Integral Transforms, (1972) that apparently a class of functions $g(y)$ for which this result is valid is not yet clearly defined. The purpose of this paper is to define a class of functions $g(y)$ as well as a class $f(x)$ and give proofs that the above inversion formula hold for these classes.

Introduction. The theorem and proofs presented in the paper are basically a generalization of those in a paper of V. Fock [4] who treated the case $k = 0$, the Mehler-Fock transform. Some applications of the Mehler-Fock transform and general Mehler-Fock transform are given in [7], [8]. Tables of these transforms are given in [6].

All integrals are taken in the improper (complex) Riemann sense. $x \sim +\infty$ means x positive and sufficiently large, $x \sim +1 \equiv$ sufficiently close to 1, $x > 1$.

THEOREM 1. *Let G be the class of complex valued functions such that $g \in G$ if and only if*

1. $g(y) = (y-1)^{-k/2}g_1(y)$, $y > 1$, $g_1(y)$ *is twice differentiable and continuous for* $y \geqq 1$, *the real and imaginary parts of* $g_1''(y)$ *are of bounded variation on any closed and bounded interval contained in* $\infty > y \geqq 1$.

2. $d^ng_1/dy^n = O(y^{-(1/2)-n+(k/2)-\varepsilon})$, $y \geqq 1$, $1/4 > \varepsilon > 0$, $0 \equiv$ *large order relation*, $n = 0, 1, 2$ (the case $n = 0$ means g_1).

Then $L_1(L_2(g)) = g$, $y > 1$, $|\mathrm{Re}\, k| < 1/4$.

Proof of Theorem 1.

LEMMA 1. *Let*

$$g \in G,\ h(t) = \int_0^t p(t, q)dq,\ p = (\sinh q)^{1-k}(\cosh t - \cosh q)^{-1/2+k} g(\cosh q)\ ,$$
$$f(x) = \int_0^\infty \cos(xt)h'(t)dt,\ |\mathrm{Re}\, k| < \frac{1}{4}\,.$$

Then

1. $f(x) = 0(x^{-2}),\ x \sim +\infty,\ \int_0^\infty |f(x)|dx < \infty.$

2a. *$h'(t)$ is continuous for $t \geqq 0$.*

2b. *$h'(t)$ satisfies the conditions of a Fourier inversion theorem* [9, p. 13], *h', h'' are both absolutely integrable over the infinite interval $\infty \geqq t \geqq 0$, $\lim_{t\to+0,+\infty} h = 0$, $\lim_{t\to+\infty} h' = 0$.*

3. $\int_0^\infty \left(\int_0^t |p|\, dq\right)dt < \infty.$

Proof of Lemma 1. Let $s = \cosh t$, $r = \cosh q$, $r = (s-1)w + 1$. Then

$$p = (s-1)^{(1+k)/2}((s-1)w+2)^{-k/2} g((s-1)w+1)c(w)\ ,$$
$$c(w) = (1-w)^{-(1/2+k)} w^{-k/2}\ .$$

Hence there exists $c_n(w)$ independent of t such that

$$\left|\frac{\partial^n p}{\partial t^n}\right| \leqq e^{-\varepsilon t}|c_n(w)|,\ t \sim +\infty,\ \int_0^1 |c_n|\, dw < \infty,\ \frac{1}{4} > \varepsilon > 0,\ n$$
$$= 0, 1, 2, |\mathrm{Re}\, k| < \frac{1}{4}\,.$$

Again by dominated convergence we conclude $d^n h/dt^n = \int_0^1 (\partial^n p/\partial t^n)dw$, $\infty > t \geqq 0$, $n = 1, 2$, $|\mathrm{Re}\, k| < 1/4$. Hence parts 2, 3 of Lemma 1 hold. We are now permitted to integrate by parts with respect to t the right-hand side of the defining formula for $f(x)$ in the hypothesis of Lemma 1 to conclude $f(x) = x^{-1}F(x)$, $F(x) = \int_0^\infty \sin(xt)h''(t)dt$. Since $h''(t) = O(e^{-\varepsilon t})$, $t \sim +\infty$, $1/4 > \varepsilon > 0$, we conclude the real and imaginary parts of $h''(t)$ are of bounded variation in the infinite interval $\infty \geqq t \geqq 0$ (see I.P. Natanson "Theory of Functions of a Real Variable", p. 238, for definitions and theorem). This implies $F(x) = O(x^{-1})$, $x \sim +\infty$. This completes the proof of Lemma 1.

LEMMA 2. *Let $g \in G$. Then*

$$\lim_{A\to+\infty}\int_0^{A>0}\left(\int_0^t \hat{f}dq\right)dt = \lim_{A\to+\infty}\int_0^A\left(\int_q^A \hat{f}dt\right)dq = \int_0^\infty\left(\int_q^\infty \hat{f}dt\right)dq\ ,$$
$$\hat{f} = p\sin(xt),\ x \geqq 0,\ |\operatorname{Re} k| < \frac{1}{4}\ .$$

(See Lemma 1 for the definition of p.)

Proof of Lemma 2. Since $g \in G$, the iterated integrals in Lemma 2 are equal for finite A. Part 3 of Lemma 1 implies absolute integrability of the first iterated integral in Lemma 2. Hence we satisfy Fubini's theorem which implies Lemma 2.

LEMMA 3. *Let*

$$F(v) = \int_1^v (v-s)^{-1/2+k} r ds,\ r = (s^2-1)^{-k/2} g(s),\ g \in G\ .$$

Then

$$\frac{d}{dt}\int_1^t (t-v)^{-1/2-k}F(v)dv = \int_1^t (t-v)^{-1/2-k}\frac{dF}{dv}dv,\ |\operatorname{Re} k| < \frac{1}{4}\ .$$

Proof of Lemma 3. Part 2 of Lemma 1 implies $F(v)$, $F'(v)$ are both continuous for $v > 1$, $\lim_{v\to+1} F(v) = 0$. Hence we satisfy a theorem (relating to the Abel integral equation) [1, p. 5] (this theorem can be modified to include singularities of the type $(x-1)^a$, $x \sim +1$, $\operatorname{Re} a > -1$, our case, see [1, p. 6]), which implies the conclusion of Lemma 3.

The rest of the proof of Theorem 1 consists mainly in applying the above lemmas to show that all the operations we use to show that (2) is a solution to (1) are valid.

Using the integral representation for $P^k_{ix-1/2}$ from [5, p. 165], we obtain from (2), the iterated integral,

$$(3)\qquad f(x) = a(k)x\int_0^\infty\left(\int_t^\infty p\sin(xs)ds\right)dt$$

(see Lemma 1 for the definition of p)

$$a(k) = 2^{1/2}\pi^{-3/2}\Gamma\left(\frac{1}{2}-k\right)\sin\left(\left(\frac{1}{2}+k\right)\pi\right),\ x \geqq 0,\ |\operatorname{Re} k| < \frac{1}{4}\ .$$

(We note (3) is valid by Lemma 2.)

We now apply to the right-hand side of (3) the following operations in this order,

1. integration over a triangular domain (see Lemma 2),
2. integration by parts with respect to s,
3. the Fourier cosine transform.

Since operations 1, 2, 3 are now permissible by Lemmas 1, 2 ($g \in G$), we obtain from (3) the valid identity

$$\int_0^\infty \cos(tx) f(x) dx = a_1(k) \frac{dh}{dt} \quad \text{(see Lemma 1 for definition of } h)$$

$$(4) \qquad a_1(k) = (2\pi)^{-1/2} \Gamma\left(\frac{1}{2} - k\right) \sin\left(\left(\frac{1}{2} + k\right)\pi\right),$$

$$t > 0, \ |\mathrm{Re}\, k| < \frac{1}{4}.$$

Lemma 3 implies all the operations (those indicated in Lemma 3) to show the right-hand side of (4) is a solution to an Abel integral equation are now permissible [1, p. 9]. (Again we note only real k are treated on p. 9, but the theory can be extended to complex k, our case.) Hence applying these operations (those indicated in Lemma 3 to the right-hand side of (4), we obtain the valid identity

$$(5) \qquad \begin{aligned} &g(\cosh t) = \int_0^t \left(\int_0^\infty u\, dx\right) ds, \ u = a_2(k)(\sinh t)^k (\cosh t - \cosh s)^{-1/2-k} \\ &\cos(sx) f(x), \ a_2(k) = (2^{-1}\pi)^{-1/2} \left(\Gamma\left(\frac{1}{2} - k\right)\right)^{-1}, \ t > 0, \ |\mathrm{Re}\, k| < \frac{1}{4}. \end{aligned}$$

Interchanging the order of integration of the iterated integral on the right-hand side of (5) (which is now permissible by part 1 of Lemma 1), then using the integral representation for $P^k_{ix-1/2}$ from [2, p. 156], we obtain the valid identity $L_1(L_2(g)) = g$, $t > 0$, $|\mathrm{Re}\, k| < 1/4$. This completes the proof of Theorem 1.

COROLLARY 1. *Let $g_1, g_2 \in G$ such that $L_2(g_1) = L_2(g_2)$, then $g_1(t) = g_2(t)$, $t > 0$, $|\mathrm{Re}\, k| < 1/4$.*

Proof. Let $u = g_1 - g_2$. Then $u \in G$. Hence $L_2(u) = 0$ by linearity of L_2. Hence $f(x)$ (of (3)) $= 0$, $x \geqq 0$. We then obtain from (5) the conclusion of Corollary 1.

THEOREM 2. *Let F be the class of real valued functions such that $f \in F$ if and only if*

1. *$f(x) = x^2 f_1(x)$, $f_1'(x)$ is continuous for $x \geqq 0$, and of bounded variation on any closed and bounded interval contained in $\infty > x \geqq 0$.*
2. *$f, f' = O(x^{-1-\varepsilon})$, $x \sim +\infty$, $\varepsilon > 0$.*

Then $L_2(L_1(f)) = f$, $x \geqq 0$, $|\mathrm{Re}\, k| < 1/2$.

Proof of Theorem 2.

LEMMA 4. *Let $f \in F$, $g = L_1(f)$, then*

1. *$\int_1^A |g(x)| dy$ exists for any $A > 1$.*

2. $g = 0((\cosh^{-1} y)^{-2}(y^2 - 1)^{-1/4})$, $y \sim +\infty$, *providing* $|\mathrm{Re}\, k| < 1/2$.

Proof of Lemma 4. From formula 26 [2, p. 129],

(a) $P^k_{ix-1/2}(\cosh t) = (2\pi \sinh t)^{-1/2}(e^{-itx} f_1 + e^{itx} f_2)$,

$$f_1 = \frac{\Gamma(-ix)}{\Gamma\left(\frac{1}{2} - k - ix\right)} f_3,\ f_3 = F\left(\frac{1}{2}+k,\ \frac{1}{2}-k,\ 1+ix;\ -\frac{1}{2}e^{-t}\cosh t\right),$$

$$f_2(x) = f_1(-x),\ F(a, b, c; z) = M\int_0^1 w ds,\ w = s^{b-1}(1 - s)^{c-b-1}(1 - zs)^{-a},$$

$\mathrm{Re}\, b$, $\mathrm{Re}\,(c - b) > 0$, $|z| < 1$, M independent of z[2, p. 59].

(b) $z^{b-a}(\Gamma(z + a)/\Gamma(z + b)) \sim a_1 + a_2 z^{-1} + \cdots$ (an asymptotic series), $|z| \sim +\infty$ uniformly for $|\arg z| \leqq \pi - \varepsilon$, $\pi/2 > \varepsilon > 0$ [2, p. 47], so differentiation of the right-hand side of (b) is permissible [3, p. 21]. From (a) we conclude $(1 + x)^{-1/2+k} f_3'(x)$, $(1 + x)^{-1/2+k} f_3''(x)$ are uniformly bounded for $x \geqq 0$ and $t \geqq 1$, providing $|\mathrm{Re}\, k| < 1/2$. In (1) we now use the integral representation from (a), then integrate by parts with respect to x, which is permissible ($f \in F$) to conclude $g^{(j)}(y) = (\cosh^{-1} y)^{-1}(y^2 - 1)^{-1/4}\int_0^\infty e^{\pm itx} c^{(j)}(y, x, k) dx$, $y \geqq 2$, $|\mathrm{Re}\, k| < 1/2$, further the real and imaginary parts $c^{(j)}$ are of bounded variation in x on the infinite interval $\infty \geqq x \geqq 0$, $y \geqq 2$, $|\mathrm{Re}\, k| < 1/2$. Hence the real and imaginary parts of $c^{(j)}$ can each be written as the difference of two monotonically decreasing functions $c_n^{(j)}(x)$, $x \geqq 0$, $\lim_{x\to+\infty} c_n^{(j)}(x) = 0$ uniformly in $y \geqq 2$, $c_n^{(j)}$ are uniformly bounded, $x \geqq 0$, $y \geqq 2$, $|\mathrm{Re}\, k| < 1/2$, $n = 1, 2$, $j = 1, 2$, since $f(x) = O(x^{-1-\varepsilon})$, $x \sim +\infty$. Also $g(y) = O((y - 1)^{-1/4})$, $2 > y > 1$, $|\mathrm{Re}\, k| < 1/2$, by (5) (in the proof of Theorem 1), $f \in F$. Hence Lemma 4 holds.

LEMMA 5. *The* g *of Lemma* 4 *implies* $\int_0^\infty\left(\int_q^\infty |\hat{f}|\, dt\right) dq < \infty$, $x \geqq 0$, $|\mathrm{Re}\, k| < 1/2$ (see Lemma 2 of Theorem 1 for the definition of $\hat{f}$).

Proof. Using the change of variable $(\cosh t - \cosh q) = (\cosh q + 1)w$, we conclude $\int_q^\infty |\hat{f}|\, dt \leqq M(\sinh q/2)^{-1}|(\sinh q)^{1-k}(\cosh q)^k g(\cosh q)|$, $q > 0$, $x \geqq 0$, M a constant, $|\mathrm{Re}\, k| < 1/2$. Hence the conclusion of Lemma 5 follows.

The rest of the proof of Theorem 2 consists mainly in justifying in reverse order all the formulas arising from the solution of the integral equation $L_1(f) = g$ in the proof of Theorem 1. Hence we will point only where the rest of the proof of Theorem 2 must be modified from that of Theorem 1.

REMARK 1. The inversion theorem for the solution to the Abel integral equation [1, p. 9] appealed to in the proof of Theorem 1 has been modified to include functions which have singularities of the type $(x-1)^a$, $x \sim +1$, $\operatorname{Re} a > -1$. Hence this modified form of the theorem applies again to our case (see (5) in the proof of Theorem 1) since we have a singularity of this type when we use the change of variable $s = \cosh q$.

REMARK 2. Lemma 5, $f \in F$ imply the sum $\hat{h}(+\infty) - \hat{h}(+0)$, $x \geqq 0$, $|\operatorname{Re} k| < 1/2$, of the upper and lower limits (both are finite) (arising when one does an integration by parts, i.e., the reverse operation corresponding to the one of part 2 of (3) in the proof of Theorem 1) is zero.

REMARK 3. Lemma 5 implies the g of Lemma 4 satisfies the conclusion of Lemma 2 of Theorem 1. Hence the reverse operation of integrating over a triangular domain (see Lemma 2 of Theorem 1) is now permissible. Hence we conclude all the reverse formulas are valid. This completes the proof of Theorem 2.

COROLLARY 2. *Let $f_1, f_2 \in F$ such that $L_1(f_1) = L_1(f_2)$. Then $f_1(x) = f_2(x)$, $x \geqq 0$, $|\operatorname{Re} k| < 1/2$.*

Proof. Let $r = f_1 - f_2$. Then $r \in F$. Hence by linearity $L_1(r) = 0$. Then by (3) of Theorem 1 (see also Lemma 5 of Theorem 2) we obtain the conclusion of Corollary 2.

We note in closing, using the change of variable $(\cosh t - \cosh q) = (\cosh q + \cos a)s$, the integral representations for $P^k_{ix-1/2}$ in Theorem 1 and [5], we obtain a pair of reciprocal transforms

1. $g(\cosh q) = \sin a(\cosh q + \cos a)^{-3/2+k}(\sinh q)^{-k}$, $|a| < \pi/2$,

2. $f(x) = 2^{1/2}\pi^{-1/2}(\Gamma(1/2-k))^{-1}\beta(1/2-k, 1)x\Gamma(1/2-k+ix)\Gamma(1/2-k-ix)\sinh ax$, $|\operatorname{Re} k| < 1/2$. (The case $k=0$ specializes to the example in [4].) $\beta \equiv$ Beta function. Further, $g \in G$ of Theorem 1 and $f \in F$ of Theorem 2.

If in Theorem 1, part 1, we now assume g_1 is analytic for $y \geqq 1$, $\operatorname{Re} k < 1/2$, in 2 we assume $n \geqq 0$ and arbitrary, then by the methods in the proofs of Theorems 1 and 2 (we use the integral representation for $P^k_{ix-1/2}$ from (5) in L_2), we conclude $c(k) = L_1(L_2(g))$ is an analytic function in k for $\operatorname{Re} k < 1/2$, $y > 1$. Hence by analytic continuation, Theorem 1 and Corollary 1 are now valid for $\operatorname{Re} k < 1/2$.

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