PACIFIC JOURNAL OF MATHEMATICS
Vol. 52, No. 2, 1974

PERTURBATIONS OF TYPE I VON NEUMANN ALGEBRAS

JOHN PHILLIPS

In a recent paper, R. V. Kadison and D. Kastler studied a certain metric on the family of von Neumann algebras defined on a fixed Hilbert space. The distance between two von Neumann algebras was defined to be the Hausdorff distance between their unit balls. They showed that if two von Neumann algebras were sufficiently close, then their central portions of type $K(K = I, I_n, II, II_1, II_\infty, III)$ were also close.

In the introduction to their paper, they conjectured that neighbouring von Neumann algebras must actually be unitarily equivalent. It is the purpose of this paper to prove this conjecture in the case that one of the algebras is of type I. The question of "inner" equivalence is left open. (Can the unitary equivalence be implemented by a unitary operator in the von Neumann algebra generated by the two neighbouring algebras?)

2. Notation and definitions. If $\mathscr{A}$ is a set of bounded linear operators on the Hilbert space $\mathscr{H}$, then we use $\mathscr{A}'$ to denote the set of all bounded linear operators on $\mathscr{H}$ which commute with every element of $\mathscr{A}$. We use the notation $\mathscr{A}_1$ to denote the set of all operators in $\mathscr{A}$ whose bound is less than or equal to 1. The algebra of all bounded operators on $\mathscr{H}$ is denoted by $\mathscr{B}(\mathscr{H})$. If $\mathscr{A}$ is an algebra of operators on $\mathscr{H}$ with identity, then the identity of $\mathscr{A}$ is denoted by $1_{\mathscr{A}}$. However, the identity operator on $\mathscr{H}$ will be denoted by 1. For each subset $\mathscr{F}$ of $\mathscr{B}(\mathscr{H})$ and bounded operator A on $\mathscr{H}$ we let

$$\|A - \mathscr{F}\| = \inf \{\|A - F\|: F \text{ in } \mathscr{F}\} .$$

DEFINITION 2.1. If $\mathscr{A}$ and $\mathscr{B}$ are linear subspaces of $\mathscr{B}(\mathscr{H})$,

$$\|\mathscr{A} - \mathscr{B}\| = \sup \{\|A - \mathscr{B}_1\|, \|B - \mathscr{A}_1\|: A \text{ in } \mathscr{A}_1, B \text{ in} \mathscr{B}_1\} .$$

If $\mathscr{S}$ is a subset of $\mathscr{B}(\mathscr{H})$, then the closure of $\mathscr{S}$ in the ultraweak topology is denoted by $\mathscr{S}^-$. Also, co $\mathscr{S}$ will denote the set of convex linear combinations of elements of $\mathscr{S}$.

As in [4], we make frequent use of the monotone increasing function α: $[0, 1/8] \rightarrow [0, 5/8]$ defined by $\alpha(a) = a + 1/2 - (1/4 - 2a)^{1/2}$. We also make use of the following estimates for α:

for $n \geqq 10$, $(n-4)^2 - 16 \geqq \left[(n-4) - \dfrac{16}{n}\right]^2$ so that

$$\alpha\left(\frac{1}{n}\right) = \frac{n + 2 - [(n-4)^2 - 16]^{1/2}}{2n} \leqq \frac{3}{n} + \frac{8}{n^2} < \frac{4}{n}\,.$$

DEFINITION 2.2. Let G be a topological group and denote by $BC(G)$ the Banach space of all bounded complex valued continuous functions on G with the supremum norm. A *right invariant mean* on G is a linear functional μ on $BC(G)$ such that

(i) $\mu(f) \geqq 0$ for $f \geqq 0$ and f in $BC(G)$,

(ii) $\mu(f_g) = \mu(f)$ for all f in $BC(G)$ and all g in G where $f_g(h) = f(hg)$ for h in G,

(iii) $\mu(1) = 1$.

A *mean* on $BC(G)$ is a linear functional on $BC(G)$ satisfying (i) and (iii) above. A *Dirac mean* on G is a linear functional on $BC(G)$ of the form $m_g(f) = f(g)$ for some fixed g in G and all f in $BC(G)$. If G has a right invariant mean we say that G is *amenable.*

3. The main theorem.

The crucial step in proving the main theorem is Lemma 3.2. The proof of Lemma 3.2 is really just an application of the fixed point property for amenable groups. (See Theorem 3.3.5 on p. 55 of [2].) However, since the fixed point property is usually stated for locally compact groups and the group we wish to apply it to is far from being locally compact, we have (essentially) incorporated the proof of the fixed point property into the proof of Lemma 3.2. Lemma 3.5 (whose proof is quite trivial) is inserted solely to be able to state the main theorem in its most general form.

LEMMA 3.1. *Convex combinations of Dirac means are weak-* dense in the set of all means in the Banach space dual of $BC(G)$ where G is any topological group.*

For a proof of this fact see §1.1 of [2].

LEMMA 3.2. *Let $\mathscr{A}$ and $\mathscr{B}$ be C^*-algebras with identity on a Hilbert space $\mathscr{H}$ such that $\|\mathscr{A} - \mathscr{B}\| < a$ ($\leqq 1/10$). Suppose that the unitary groups of $\mathscr{A}$ and $\mathscr{B}$ each contain an amenable subgroup* (given the norm topology) *whose linear span is ultraweakly dense in the algebra. Then $\|\mathscr{A}'1_{\mathscr{A}} - \mathscr{B}'1_{\mathscr{B}}\| < 4a + \alpha(a)$. If $\mathscr{A}$ and $\mathscr{B}$ each have the same identity,* 1, *then we have $\|\mathscr{A}'1 - \mathscr{B}'1\| < 4a$.*

Proof. Let T_0 be in $(\mathscr{A}'1_{\mathscr{A}})_1$ and let G be an amenable subgroup of the unitary group of $\mathscr{B}$ with the property described above. Let

$$\mathscr{S} = \mathrm{co}\{U^* T_0 U\colon U \text{ in } G\}^- .$$

Then $\mathscr{S}$ is an ultraweakly closed convex subset of the unit ball of $\mathscr{B}(\mathscr{H})$ and so is ultraweakly compact. Moreover, $\mathscr{S} \subseteqq 1_{\mathscr{B}}\mathscr{B}(\mathscr{H})1_{\mathscr{B}}$.

Now, let U in G be fixed and let $\|\mathscr{A} - \mathscr{B}\| < b < a$ so that there is an operator V in $\mathscr{A}_1$ such that $\|U - V\| < b < a$. Then,

$$\begin{aligned}\|U^*T_0U - T_0\| &\leqq \|U^*T_0U - V^*T_0U\| + \|V^*T_0U - V^*T_0V\| \\ &\quad + \|V^*VT_0 - T_0\| < b + b + \|V^*V - 1_{\mathscr{A}}\| \\ &\leqq 2b + \|V^*V - V^*U\| + \|V^*U - U^*U\| \\ &\quad + \|U^*U - 1_{\mathscr{A}}\| < 2b + b + b \\ &\quad + \|1_{\mathscr{B}} - 1_{\mathscr{A}}\| < 4b + \alpha(a) .\end{aligned}$$

Where $\|1_{\mathscr{B}} - 1_{\mathscr{A}}\| < \alpha(a)$ can be seen by the following argument. By Lemma 6 of [4] there is a central projection P in $\mathscr{B}$ such that $\|1_{\mathscr{A}} - P\| < \alpha(a)$. Also, there is an A in $\mathscr{A}_1$ such that $\|A - 1_{\mathscr{B}}\| < a$. So, we have $\|P - 1_{\mathscr{B}}\| \leqq \|P1_{\mathscr{B}} - 1_{\mathscr{A}}1_{\mathscr{B}}\| + \|1_{\mathscr{A}}1_{\mathscr{B}} - 1_{\mathscr{A}}A\| + \|A - 1_{\mathscr{B}}\| < \alpha(a) + 2a < 1$. Thus, $P = 1_{\mathscr{B}}$ and $\|1_{\mathscr{A}} - 1_{\mathscr{B}}\| < \alpha(a)$. Therefore, by the previous argument $\mathscr{S}$ is contained in the following set

$$\{T \text{ in } 1_{\mathscr{B}}\mathscr{B}(\mathscr{H})1_{\mathscr{B}}\colon \|T\| \leqq 1 \text{ and } \|T - T_0\| \leqq 4b + \alpha(a)\} .$$

Now, for any x, y in $\mathscr{H}$ define a continuous bounded function f_{xy} on G via, $f_{xy}(U) = (U^*T_0Ux, y)$ for U in G. Let $\sum_{k=1}^{n} \lambda_k m_{U_k}$ (where U_k is in G for $k = 1, \cdots, n$) be a convex combination of Dirac means. Then, for any x, y in $\mathscr{H}$ we have

$$m(f_{xy}) = \sum_{k=1}^{n} \lambda_k m_{U_k}(f_{xy}) = \sum_{k=1}^{n} \lambda_k f_{xy}(U_k) = \sum_{k=1}^{n} \lambda_k (U_k^* T_0 U_k x, y) = (Tx, y)$$

where

$$T = \sum_{k=1}^{n} \lambda_k U_k^* T_0 U_k \text{ is in } \mathscr{S} .$$

Let μ be a right invariant mean on G and let $\{m_\delta\}$ be a net of means each of which is a convex combination of Dirac means and such that $\{m_\delta\}$ converges to μ in the weak-* topology on the dual of $BC(G)$. For each δ, let T_δ be in $\mathscr{S}$ such that $m_\delta(f_{xy}) = (T_\delta x, y)$ for all x and y in $\mathscr{H}$. Since $\mathscr{S}$ is ultraweakly compact we can choose a subnet $\{T_{\delta'}\}$ which converges ultraweakly to an operator T in $\mathscr{S}$. Then, for all x and y in $\mathscr{H}$ and each U in G we have

$$\begin{aligned}(U^*TUx, y) &= (TUx, Uy) = \lim_{\delta'}(T_{\delta'}Ux, Uy)\\ &= \lim_{\delta' m_{\delta'}}(f_{UxUy}) = \mu(f_{UxUy}) = \mu((f_{xy})_U) = \mu(f_{xy})\\ &= \lim_{\delta' m_{\delta'}}(f_{xy}) = \lim_{\delta'}(T_{\delta'}x, y) = (Tx, y)\,.\end{aligned}$$

Since the linear span of G is ultraweakly dense in $\mathscr{B}$, we have that T is in $\mathscr{B}'$ and since T is in $\mathscr{S}$ we have that T is in $\mathscr{B}'1_{\mathscr{B}}$, $\|T\| \leqq 1$ and $\|T - T_0\| \leqq 4b + \alpha(a)$. Similarly, if we start with an S_0 in $(\mathscr{B}'1_{\mathscr{B}})_1$ we can find an S in $(\mathscr{A}'1_{\mathscr{A}})_1$ such that $\|S - S_0\| \leqq 4b + \alpha(a)$. Therefore, $\|\mathscr{A}'1_{\mathscr{A}} - \mathscr{B}'1_{\mathscr{B}}\| \leqq 4b + \alpha(a) < 4a + \alpha(a)$. Clearly, if $\mathscr{A}$ and $\mathscr{B}$ have the same identity, 1, the above proof shows that $\|\mathscr{A}'1 - \mathscr{B}'1\| < 4a$.

LEMMA 3.3. *Let $\mathscr{A}$ and $\mathscr{B}$ be von Neumann algebras on a Hilbert space $\mathscr{H}$ such that $\|\mathscr{A} - \mathscr{B}\| < a\,(\leqq 1/12)$. Then the centre of $\mathscr{A}$ is *-isomorphic to the centre of $\mathscr{B}$.*

Proof. Let P be a projection in the centre of $\mathscr{A}$. Then by Lemma 6 of [4], there is a central projection Q in $\mathscr{B}$ such that $\|P - Q\| < \alpha(a) < 1/3$. If there were another central projection E in $\mathscr{B}$ such that $\|P - E\| < \alpha(a)$ then $\|E - Q\| < 2\alpha(a) < 2/3$ so that $E = Q$. Let φ be the map from the central projections of $\mathscr{A}$ to the central projections of $\mathscr{B}$ defined by $\|\varphi(P) - P\| < \alpha(a)$ for all central projections P in $\mathscr{A}$. By the symmetry of the situation it is easily seen that φ is one-to-one and onto the central projections of $\mathscr{B}$.

Now, let P and Q be central projections in $\mathscr{A}$. Then,

$$\begin{aligned}&\|\varphi(PQ) - \varphi(P)\varphi(Q)\| \leqq \|\varphi(PQ) - PQ\| + \|PQ - \varphi(P)Q\|\\ &\quad + \|\varphi(P)Q - \varphi(P)\varphi(Q)\| < 3\alpha(a) < 1, \text{ and so } \varphi(PQ) = \varphi(P)\varphi(Q)\,.\end{aligned}$$

Since $P \leqq Q$ if and only if $PQ = P$; $\varphi(P) \leqq \varphi(Q)$ if and only if $P \leqq Q$. Hence, φ is a lattice isomorphism from the central projections of $\mathscr{A}$ onto the central projections of $\mathscr{B}$. Therefore, φ extends to a *-isomorphism (also denoted by φ) of the algebra generated by the central projections in $\mathscr{A}$ to the algebra generated by the central projections in $\mathscr{B}$. By inspection, φ is an is isometry and so extends to a *-isomorphism of the centre of $\mathscr{A}$ onto the centre of $\mathscr{B}$.

The following lemma is due to F. Riesz (see [1], p. 148). A proof is provided here for the sake of completeness.

LEMMA 3.4. *Let X be a normed linear space and let Y be a closed proper subspace of X. For any t, $0 < t < 1$ there is an x_t in X, $\|x_t\| = 1$ such that*

$$\inf \{\|x_t - y\| : y \text{ in } Y\} \geqq t\,.$$

Proof. Let x_1 be in X but not in Y. Then, $\inf\{\|x_1 - y\| : y$ in $Y\} = d > 0$. Choose y_0 in Y such that

$$\|x_1 - y_0\| \leqq \frac{d}{t}\,. \text{ Let } x_t = \frac{x_1 - y_0}{\|x_1 - y_0\|}\,.$$

Let y be any vector in Y, then

$$\begin{aligned}\|x_t - y\| &= \left\|\frac{x_1 - y_0}{\|x_1 - y_0\|} - y\right\| \\ &= \frac{1}{\|x_1 - y_0\|}\|x_1 - (y_0 + \|x_1 - y_0\|y)\| \geqq \frac{1}{\|x_1 - y_0\|}d \geqq t\,.\end{aligned}$$

LEMMA 3.5. *Let $\mathscr{A}$ and $\mathscr{B}$ be C^*-algebras on a Hilbert space $\mathscr{H}$ such that $\|\mathscr{A} - \mathscr{B}\| < 1/2$. If $\mathscr{A}$ is a von Neumann algebra, then $\mathscr{B}$ is a von Neumann algebra.*

Proof. By Lemma 5 of [4] we have $\|\mathscr{A} - \mathscr{B}^-\| < 1/2$. Therefore $\|\mathscr{B} - \mathscr{B}^-\| < 1$. But if $\mathscr{B} \neq \mathscr{B}^-$ then by Lemma 3.4 $\|\mathscr{B} - \mathscr{B}^-\| \geqq 1$. Hence, $\mathscr{B} = \mathscr{B}^-$.

THEOREM 3.6. *Let $\mathscr{A}$ and $\mathscr{B}$ be C^*-algebras on a Hilbert space $\mathscr{H}$ such that $\|\mathscr{A} - \mathscr{B}\| < a\ (\leqq 1/25618)$. If $\mathscr{A}$ is a type* I *von Neumann algebra then $\mathscr{B}$ is unitarily equivalent to $\mathscr{A}$.*

Proof. Let $P_n(Q_n)$ be the largest projection in the centre of $\mathscr{A}(\mathscr{B})$ such that $P_n\mathscr{A}(Q_n\mathscr{B})$ is of type I_n or 0 and let $P'_m(Q'_m)$ be the largest projection in the centre of $\mathscr{A}'1_{\mathscr{A}}(\mathscr{B}'1_{\mathscr{B}})$ such that $P'_m\mathscr{A}'1_{\mathscr{A}}(Q'_m\mathscr{B}'1_{\mathscr{B}})$ is of type I_m or 0.

Since $\mathscr{A}$ and $\mathscr{B}$ are both von Neumann algebras by Lemma 3.5 and are both type I by Lemma 10 of [4] we have by the proof of Corollary 6.5 of [3] that the unitary groups of $\mathscr{A}$ and $\mathscr{B}$ each have amenable subgroups of the type described in Lemma 3.2. Hence, by Lemma 3.2 $\|\mathscr{A}'1_{\mathscr{A}} - \mathscr{B}'1_{\mathscr{B}}\| < 4a + \alpha(a) < 1/3660$. Thus, by Lemma 15 of [4], not only do we have $\|P_n - Q_n\| < \alpha(a)$ but also $\|P'_m - Q'_m\| < \alpha(4a + \alpha(a))$. So, we have $\|P'_mP_n - Q'_mQ_n\| < \alpha(a) + \alpha(4a + \alpha(a))$. If we let $P_{mn} = P'_mP_n$ and $Q_{mn} = Q'_mQ_n$ then we have $\|P_{mn}\mathscr{A} - Q_{mn}\mathscr{B}\| < a + \alpha(a) + \alpha(4a + \alpha(a)) < 1/12$. By Lemma 3.3, the centres of these two algebras are * isomorphic. Therefore, by Theorem 8, p. 90 of [6] $\mathscr{A}$ is unitarily equivalent to $\mathscr{B}$.

THEOREM 3.7. *Let $\mathscr{H}$ be a separable Hilbert space and let $\mathscr{A}$ and $\mathscr{B}$ be C^*-algebras on $\mathscr{H}$ each containing the identity of $\mathscr{H}$*

such that $\|\mathscr{A} - \mathscr{B}\| < a\ (\leqq 1/32)$. *If* $\mathscr{A}$ *is a von Neumann algebra which is a type* I *factor, then* $\mathscr{B}$ *is unitarily equivalent to* $\mathscr{A}$.

Proof. If $\mathscr{A}$ is a factor of type I_n, then so is $\mathscr{B}$ by Theorem B of [4]. If $\mathscr{A}'$ is a factor of type I_m, then $\|\mathscr{A}' - \mathscr{B}'\| < 4a \leqq 1/8$ so again by Theorem B of [4], $\mathscr{B}'$ is a factor of type I_m. Hence, $\mathscr{A}$ and $\mathscr{B}$ are unitarily equivalent.

REMARKS 3.8. By Theorem 3.6, if $\mathscr{A}$ is maximal abelian on its essential subspace, then $\mathscr{B}$ is maximal abelian on its essential subspace and so the *-isomorphism between $\mathscr{A}$ and $\mathscr{B}$ defined in Lemma 3.3 is actually spatial. Using the fact that $\mathscr{A}'1_{\mathscr{A}} = \mathscr{A}$ and $\mathscr{B}'1_{\mathscr{B}} = \mathscr{B}$ it is not too hard to show that any unitary which implements this particular isomorphism can be extended to a unitary on $\mathscr{H}$ which is in the von Neumann algebra generated by $\mathscr{A}$ and $\mathscr{B}$. Thus, it seems highly likely that neighbouring type I von Neumann algebras are actually "inner" equivalent.

As noted on p. 421 of [5], hyperfinite factors satisfy the hypotheses of Lemma 3.2 and so Lemma 3.2 may be of some help in analysing the hyperfinite case.

ACKNOWLEDGMENTS. I would like to thank Professors R. V. Kadison and J. R. Ringrose for their advice and encouragement while I was working on this paper. In particular, I would like to thank Professor Ringrose for pointing out that all type I von Neumann algebras satisfy the hypotheses of Lemma 3.2 and not just abelian von Neumann algebras as I had originally thought.

Added in proof. In a paper entitled *Perturbations of Type I von Neumann algebras* to appear in J. London Math. Soc., E. Christensen proves a much sharper version of Theorem 3.6 which answers the question of "inner" equivalence affirmatively. The methods are quite different from those of this paper.

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Received June 28, 1973 and in revised form March 13, 1974. This research was supported by a National Research Council of Canada postdoctoral fellowship.

THE UNIVERSITY, NEWCASTLE, UPON TYNE, ENGLAND

Present address: Department of Mathematics
Queen's University,
Kingston, Ontario, Canada