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Central Charges in the Canonical Realization of Asymptotic Symmetries: An Example from Three Dimensional Gravity

J. D. Brown and Marc Henneaux*,**
Center for Theoretical Physics, The University of Texas at Austin, Austin, Texas 78712, USA

Abstract. It is shown that the global charges of a gauge theory may yield a nontrivial central extension of the asymptotic symmetry algebra already at the classical level. This is done by studying three dimensional gravity with a negative cosmological constant. The asymptotic symmetry group in that case is either $R \times SO(2)$ or the pseudo-conformal group in two dimensions, depending on the boundary conditions adopted at spatial infinity. In the latter situation, a nontrivial central charge appears in the algebra of the canonical generators, which turns out to be just the Virasoro central charge.

I. Introduction

In general relativity and in other gauge theories formulated on noncompact ("open") spaces, the concept of asymptotic symmetry, or "global symmetry," plays a fundamental role.

The asymptotic symmetries are by definition those gauge transformations which leave the field configurations under consideration asymptotically invariant, and recently, it has been explicitly shown that they are essential for a definition of the total ("global") charges of the theory [1, 2]. (For earlier connections between asymptotic symmetries and conserved quantities in the particular case of Einstein theory, see [3, 4] and references therein.)

The basic link between asymptotic symmetries and global charges has been emphasized again in recent papers dealing with the monopole sector of the SU(5) grand unified theory [5] and with $D = 3$ gravity and supergravity [6], where it is confirmed that the absence of asymptotic symmetries prohibits the definition of global charges. In the first instance, the unbroken symmetry group of the monopole solution is not contained in the set of asymptotic symmetries because of topological obstructions. This forbids the definition of meaningful global color charges

* Permanent address: Faculté des Sciences, Université Libre de Bruxelles, Campus Plaine C.P. 231, B-1050 Bruxelles, Belgium
** Chercheur qualifié du Fonds National Belge de la Recherche Scientifique

associated with the unbroken group. In the second case, the nontrivial global properties of the conic geometry, which describes the elementary solution of $D=3$ gravity, prevents the existence of well defined spatial translations and boosts, and hence, also of meaningful linear momentum and "Lorentz charge."

In the Hamiltonian formalism, the global charges appear as the canonical generators of the asymptotic symmetries of the theory: with each such infinitesimal symmetry ξ is associated a phase space function $H[\xi]$ which generates the corresponding transformation of the canonical variables. It is generally taken for granted that the Poisson bracket algebra of the charges $H[\xi]$ is just isomorphic to the Lie algebra of the infinitesimal asymptotic symmetries, i.e., that

$$\{H[\xi], H[\eta]\} = H[[\xi, \eta]]. \tag{1.1}$$

The purpose of this paper is to analyze this question in detail.

It turns out that, while (1.1) holds in many important examples, it is not true in the generic case. Rather, the global charges only yield a "projective" representation of the asymptotic symmetry group,

$$\{H[\xi], H[\eta]\} = H[[\xi, \eta]] + K[\xi, \eta]. \tag{1.2}$$

In (1.2), the "central charges" $K[\xi, \eta]$, which do not involve the canonical variables, are in general nontrivial, i.e., they cannot be eliminated by the addition of constants C_ξ to the generators $H[\xi]$.

The occurrence of classical central charges is by no means peculiar to general relativity and gauge theories, and naturally arises in Hamiltonian classical mechanics ([7] appendix 5). It results from the non-uniqueness of the canonical generator associated with a given (Hamiltonian) phase space vector field. Indeed, this generator is only determined up to the addition of a constant, which commutes with everything. Accordingly, the Poisson bracket of the generators of two given symmetries can differ by a constant from the generator associated with the Lie bracket of these symmetries.

A similar phenomenon occurs with asymptotic symmetries in gauge theories. In that case, the Hamiltonian generator $H[\xi]$ of a given asymptotic symmetry ξ^A differs from a linear combination of the constraints $\phi_A(x)$ of the canonical formalism by a surface term $J[\xi]$ which is such that $H[\xi]$ has well defined functional derivatives [8],

$$H[\xi] = \int d^n x \xi^A(x) \phi_A(x) + J[\xi]. \tag{1.3}$$

But this requirement fixes $J[\xi]$, and hence $H[\xi]$, only up to the addition of an arbitrary constant. This ambiguity signals the possibility of central charges.

Because the theory of central charges in classical mechanics is well known [7], we will only discuss here the aspects which are peculiar to gauge theories and asymptotic (as opposed to exact) symmetries. This will be done by treating three dimensional Einstein gravity with a negative cosmological constant Λ in detail. In that instance, we show that the asymptotic symmetry group is either $R \times SO(2)$, or the conformal group in two dimensions, depending on the boundary conditions adopted at spatial infinity. In the latter case, a nontrivial central charge—actually familiar from string theory [9]—appears in the Poisson bracket algebra of the canonical generators.

Three dimensional gravity with $\Lambda < 0$ is presented here primarily to provide an *example of central charges in the canonical realization of asymptotic symmetries.* However, the study of three dimensional gravity is not entirely academic and possesses some intrinsic interest apart from its connection with central charges. Indeed, previous experience with gauge theories has indicated that something can be learned from lower dimensional models about both the classical and quantum aspects of the more complicated four dimensional theory. In the gravitational case, three is the critical number of dimensions, since in fewer dimensions there is no Einstein theory of the usual type (i.e., with a local action principle involving only the pseudo-Riemannian metric). Thus, it is natural to turn to three dimensional models in an effort to better understand Einstein gravity in higher dimensions.

The discussion involves some subtleties because the constraint algebra of general relativity is not a true algebra, but rather, contains the canonical variables. This fact has two implications: (i) the algebra of the asymptotic symmetries is a true algebra only asymptotically; (ii) standard group theoretical arguments cannot be used in a straightforward way.

In the course of our study, we shall rely on a useful theorem which is proved in [10] and concerns Hamiltonian dynamics on infinite dimensional phase spaces. This theorem establishes, under appropriate conditions, that the Poisson bracket of two differentiable functionals contains no unwanted surface term in its variation, and therefore has well defined functional derivatives. This property is used to prove that the Poisson bracket of the asymptotic symmetry generators yields a (trivial or nontrivial) projective representation of the asymptotic symmetry group. It should be stressed that the techniques developed here in treating three dimensional gravity are quite general and can be applied, for instance, to four dimensional gravity to prove a similar representation theorem. Such a theorem has been implicitly used, but not explicitly demonstrated, for example in [8, 12].

The example of three dimensional gravity with a negative cosmological constant also demonstrates the key role played by boundary conditions, which determine the structure of the asymptotic symmetry group but are not entirely dictated by the theory. (This was also pointed out in [11].)

As a final point, let us note that the existence of a true central charge can be ruled out in the particular case when the asymptotic symmetries can be realized as exact symmetries of some background configuration. Indeed, in this situation the charges evaluated for that background are invariant under an asymptotic symmetry transformation, since the background itself is left unchanged. By adjusting the arbitrary constant in $H[\xi]$ so that $H[\xi]$ (background) $= 0$, Eq. (1.2) shows that $K[\xi, \eta]$ vanishes. However, the important case of "background symmetries" does not exhaust all interesting applications of the asymptotic symmetry concept. For example, the infinite dimensional B.M.S. group [3, 4] cannot be realized as the group of isometries of some four dimensional metric. This gives additional motivation for analyzing the canonical realization of the asymptotic symmetries on general grounds.

II. Solutions to 3-Dimensional Gravity with $\Lambda < 0$

This section provides a discussion of a solution to Einstein gravity in $2+1$ dimensions with a negative cosmological constant. This solution will help motivate our choice of appropriate boundary conditions to be imposed on the metric in general.

In three dimensions, the gravitational field contains no dynamical degrees of freedom, so that the spacetime away from sources is locally equivalent to the empty space solution of Einstein's equations, namely anti-de Sitter space when $\Lambda < 0$. This is demonstrated by noting that the full curvature tensor can be expressed in terms of the Einstein tensor, and where the empty space Einstein equations hold, the curvature tensor reduces to that of anti-de Sitter space.

Matter, which is assumed to be localized, has no influence on the local geometry of the source free regions, and therefore can only effect the global geometry of the spacetime. The basic solution which we consider then is locally anti-de Sitter space with radius of curvature $R = (-1/\Lambda)^{1/2}$,

$$dS^2 = -\left(\frac{\bar{r}^2}{R^2} + 1\right)d\bar{t}^2 + \left(\frac{\bar{r}^2}{R^2} + 1\right)^{-1} d\bar{r}^2 + \bar{r}^2 d\bar{\phi}^2, \tag{2.1}$$

but with an unusual identification of points which will alter the global geometry. By identifying the points $(\bar{t} = t', \bar{r} = r', \bar{\phi} = \phi')$ with the points $(\bar{t} = t' - 2\pi A, \bar{r} = r', \bar{\phi} = \phi' + 2\pi\alpha)$ for all t', r' and ϕ', this will have the effect of removing a "wedge" of coordinate angle $2\pi(1-\alpha)$ and introducing a "jump" of $2\pi A$ in coordinate time. Because the Ricci tensor is defined locally, it is not modified by this unusual identification except at the origin $\bar{r} = 0$. Hence, the vacuum Einstein equations will be satisfied everywhere except at the origin.

The motivation for considering the spacetime just described is that it is the analogue of the conic geometry for $2+1$ gravity with $\Lambda = 0$ [12], for which the wedge $\alpha \neq 1$ is related to total energy and the jump $A \neq 0$ is related to total angular momentum. It is also interesting to note that, just as in the de Sitter case [13], a wedge cut from anti-de Sitter space provides a solution to Einstein's equations with the stress-energy tensor of a point mass. The metric (2.1) can also be assumed to apply to the empty region exterior to a more general compact source distribution.

The geometrically invariant character of the wedge and the jump can be seen in the following way which does not depend on the details of the interior to the spacetime containing the source. First note that even though the spacetime is locally maximally symmetric, the only Killing vector fields consistent with the unusual identification of points are linear combinations of $d/d\bar{t}$ and $d/d\bar{\phi}$. The vectors $d/d\bar{t}$ and $d/d\bar{\phi}$ can be singled out uniquely (to within normalization constants) as the only two Killing vector fields which are everywhere orthogonal to one another. To within a normalization, $d/d\bar{r}$ is the unique vector field everywhere orthogonal to all Killing vector fields.

So the curves which serve as the $\bar{t}$, $\bar{r}$, $\bar{\phi}$ coordinate lines for the metric (2.1) can always be singled out. Furthermore, consider the proper length L of the curve of a trajectory of $d/d\bar{\phi}$ between points of intersection with a trajectory of $d/d\bar{t}$. The change dL as the curve is moved a proper distance dS along the direction $d/d\bar{r}$ equals

$$\frac{dL}{dS} = \left[\frac{L^2}{R^2} + (2\pi\alpha)^2\right]^{1/2}.$$

For $\alpha < 1$, the length L increases more slowly with proper distance than if the space were globally anti-de Sitter. Finally, the jump A is proportional to the proper time distance between points of intersection of the trajectories just considered.

From now on, it will be more convenient to write the metric (2.1) with a continuous time variable. The coordinate transformation $t = \bar{t} + (A/\alpha)\bar{\phi}$, $r = \bar{r}$, $\phi = (1/\alpha)\bar{\phi}$ yields

$$dS^2 = -\left(\frac{r^2}{R^2} + 1\right)(dt - A\,d\phi)^2 + \left(\frac{r^2}{R^2} + 1\right)^{-1} dr^2 + \alpha^2 r^2 d\phi^2, \tag{2.2}$$

where ϕ has period 2π, and there is no jump in the new time. The Killing vector fields in this coordinate system are linear combinations of d/dt and $d/d\phi$. Also note that the trajectories of $d/d\phi$ will form closed timelike lines unless $|A| < \alpha|R|$ and

$$r^2 > \frac{A^2R^2}{\alpha^2R^2 - A^2}.$$

As a result, the spacetime constructed represents a reasonable solution to Einstein gravity only for $|A| < \alpha|R|$ and large values of r; in particular it is valid in the asymptotic limit $r \to \infty$.

III. Global Charges and the $R \times SO(2)$ Asymptotic Symmetries

The procedure for obtaining the global charges of a gauge theory within the Hamiltonian formalism has been well established [8]. The first step is to define the boundary conditions at spatial infinity which the generic fields should obey, and then identify the asymptotic symmetries which preserve this asymptotic behavior. Of course, for gravity theories in particular, in order to continue with the Hamiltonian formulation, the boundary conditions on the spacetime metric must be converted into boundary conditions on the canonical variables g_{ij}, π^{ij}. Likewise, the asymptotic symmetries of the spacetime determine the allowed surface deformation vectors ξ^μ ($\mu = \perp, i$) for the space-like hypersurfaces under consideration.

Now, for the boundary conditions and asymptotic symmetries of a gravitation theory to be acceptable, it must be possible to write the Hamiltonian as the usual linear combination of constraints [14]

$$\int d^nx\, \xi^\mu(x) \mathscr{H}_\mu(x) \tag{3.1}$$

plus an appropriate surface term $J[\xi]$. This surface term $J[\xi]$, which will be referred to as the charge from now on, must have a variation which will cancel the unwanted surface terms in the variation of (3.1). Then the Hamiltonian,

$$H[\xi] = \int d^nx\, \xi^\mu(x) \mathscr{H}_\mu(x) + J[\xi], \tag{3.2}$$

will have well defined variational derivatives, and may be used as the generator of the allowed surface deformations.

In practice, the charges $J[\xi]$ are usually determined by looking at the surface terms coming from the variation of the "volume piece" (3.1) of the Hamiltonian, namely

$$-\lim_{r\to\infty}\oint d^{n-1}S_l\{G^{ijkl}[\xi^{\perp}\delta g_{ij;k}-\xi^{\perp}{}_{,k}\delta g_{ij}]+2\xi_i\delta\pi^{il}+(2\xi^i\pi^{kl}-\xi^l\pi^{ik})\delta g_{ik}\}, \tag{3.3}$$

where $G^{ijkl}=\frac{1}{2}g^{1/2}(g^{ik}g^{jl}+g^{il}g^{jk}-2g^{ij}g^{kl})$ and the semicolon denotes covariant differentiation within a spacelike hypersurface. Using the assumed asymptotic behavior of the fields g_{ij}, π^{ij} and vectors ξ^{μ}, this is rewritten as the total variation of a surface integral. Then the negative of this surface integral is, to within a constant, the charge $J[\xi]$. (As stated in the introduction, this constant represents the non-uniqueness of the canonical generators, and in Sect. V will be related to the possible existence of central charges in the algebra of these generators.)

For the case of $2+1$, $\Lambda<0$ gravity, the analogy with $2+1$, $\Lambda=0$ gravity [6, 12] suggests that we restrict the metric outside sources to the family of metrics defined by the two parameters α and A appearing in (2.2). This restriction serves as the boundary condition on the metric. Then the asymptotic symmetries coincide with the Killing vector fields d/dt and $d/d\phi$, and the asymptotic symmetry group associated with these boundary conditions is $R\times SO(2)$.

The values of the charges associated with d/dt and $d/d\phi$ for the metric (2.2) can be computed in the following way. Denote by ξ some linear combination of vectors $d/dt, d/d\phi$ with components ${}^{(3)}\xi^{\alpha}$, $\alpha=t,r,\phi$ in the spacetime coordinate system. Then the $\perp$, r, ϕ components ξ^{μ} of this vector describe an allowed deformation of the surface outside the source. They are related to the spacetime components by

$$\begin{aligned}\xi^{\perp}&=N{}^{(3)}\xi^{t},\\ \xi^{r}&={}^{(3)}\xi^{r}+N^{r}{}^{(3)}\xi^{t},\\ \xi^{\phi}&={}^{(3)}\xi^{\phi}+N^{\phi}{}^{(3)}\xi^{t},\end{aligned} \tag{3.4}$$

where N is the lapse and N^r, N^{ϕ} are the shifts for the spacetime coordinate system. The lapse and shifts and computed straightforwardly from (2.2); in particular,

$$\begin{aligned}N&=\alpha\left[\frac{r^2+R^2}{\alpha^2R^2-A^2}\right]^{1/2}\left[1-\frac{A^2R^2}{(\alpha^2R^2-A^2)r^2}\right]^{-1/2},\\ N^r&=0,\\ N^{\phi}&=\frac{A(r^2+R^2)}{r^2(\alpha^2R^2-A^2)-A^2R^2},\end{aligned}$$

and, since ${}^{(3)}\xi^r=0$, the component $\xi^r=0$ always.

The only nonzero components of the canonical variables needed for computing expression (3.3) are

$$\begin{aligned}g_{rr}&=\left(\frac{r^2}{R^2}+1\right)^{-1},\\ g_{\phi\phi}&=\alpha^2r^2-A^2\left(\frac{r^2}{R^2}+1\right),\\ \pi^r_{\phi}&=\alpha A,\end{aligned} \tag{3.5}$$

which gives

$$-\delta J[\xi] = 4\pi[{}^{(3)}\xi^t \delta\alpha - {}^{(3)}\xi^\phi \delta(\alpha A)].$$

Thus, the charges associated with the symmetries d/dt and $d/d\phi$ are, to within constants,

$$J[d/dt] = 4\pi(1-\alpha), \tag{3.6a}$$

$$J[d/d\phi] = 4\pi\alpha A. \tag{3.6b}$$

These are precisely the energy and angular momentum of locally flat 2 + 1 gravity [6, 12], so that the limit of these charges as $\Lambda \to 0$ is trivially correct.

IV. The Conformal Group of Asymptotic Symmetries

It is natural to question whether the restriction of the metric to the form (2.2) outside sources is too severe. Ideally, the boundary conditions could be weakened just enough so that the group of asymptotic symmetries is enlarged to the anti-de Sitter group in 2 + 1 dimensions, namely $O(2,2)$. This section addresses such a weakening of the boundary conditions, although the group of asymptotic symmetries which naturally arises is not $O(2,2)$, but the conformal group in two dimensions.

The inspiration for the weakened boundary conditions comes from rewriting the metric (2.2) by making the replacements

$$\begin{aligned} t &\to \frac{t}{\alpha}\left(\alpha^2 - \frac{A^2}{R^2}\right), \\ r &\to r\left(\alpha^2 - \frac{A^2}{R^2}\right)^{-1/2}, \\ \phi &\to \phi - \frac{A}{\alpha R^2} t, \end{aligned} \tag{4.1}$$

so that the metric now reads

$$dS^2 = -\left(\frac{r^2}{R^2} + \alpha^2\right)dt^2 + 2A\alpha\, dt\, d\phi + \left(\frac{r^2 - A^2}{R^2} + \alpha^2\right)^{-1} dr^2 + (r^2 - A^2)d\phi^2. \tag{4.2}$$

Notice that when $A = 0$, the dominant contributions in this metric and in a globally anti-de Sitter space coincide with one another, equaling

$$dS^2 \to -\left(\frac{r^2}{R^2}\right)dt^2 + \left(\frac{R^2}{r^2}\right)dr^2 + r^2 d\phi^2.$$

In this sense, it seems natural to consider the metric (4.2), at least when $A = 0$, to be "asymptotically anti-de Sitter."

The notion of "asymptotically anti-de Sitter" must be made precise by specifying the boundary conditions that the metric should satisfy. If the anti-de Sitter group is to be a part of the asymptotic symmetries preserving these conditions, then the metric obtained from an anti-de Sitter transformation acting on (4.2) must also be "asymptotically anti-de Sitter." By acting on (4.2) (with or without $A = 0$) with all possible anti-de Sitter group transformations, the following boundary conditions are generated:

$$g_{tt} = -\frac{r^2}{R^2} + O(1), \tag{4.3a}$$

$$g_{tr} = O(1/r^3), \tag{4.3b}$$

$$g_{t\phi} = O(1), \tag{4.3c}$$

$$g_{rr} = \frac{R^2}{r^2} + O(1/r^4), \tag{4.4a}$$

$$g_{r\phi} = O(1/r^3), \tag{4.4b}$$

$$g_{\phi\phi} = r^2 + O(1). \tag{4.4c}$$

It is interesting to compare the boundary conditions (4.3, 4.4) with the boundary conditions on the metric for gravity in $3+1$ dimensions with $\Lambda < 0$ [15]. By restricting the spatial sections in the $3+1$ case to two dimensions (for example, by $\theta = \pi/2$) this shows that the difference between the allowed metrics and anti-de Sitter space must fall off faster by one power of $1/r$ in $3+1$ dimensions than in $2+1$ dimensions.

Having chosen boundary conditions for the metric, the asymptotic symmetries are described by vector fields which transform metrics of this form (4.3, 4.4) into themselves. Of course, these vector fields will include the anti-de Sitter group of symmetries, $O(2,2)$. Analysis of the Lie transformation equations for metrics (4.3, 4.4) shows that the spacetime components ${}^{(3)}\xi^\alpha$ of these vectors satisfy

$$\begin{aligned} {}^{(3)}\xi^t &= RT(t,\phi) + \frac{R^3}{r^2}\bar{T}(t,\phi) + O(1/r^4), \\ {}^{(3)}\xi^r &= rR(t,\phi) + O(1/r), \\ {}^{(3)}\xi^\phi &= \Phi(t,\phi) + \frac{R^2}{r^2}\bar{\Phi}(t,\phi) + O(1/r^4) \end{aligned} \tag{4.5}$$

with

$$\begin{aligned} RT_{,t}(t,\phi) &= \Phi_{,\phi}(t,\phi) = -R(t,\phi), \\ R\Phi_{,t}(t,\phi) &= T_{,\phi}(t,\phi), \\ \bar{T}(t,\phi) &= -\frac{R}{2}R_{,t}(t,\phi), \\ \bar{\Phi}(t,\phi) &= \tfrac{1}{2}R_{,\phi}(t,\phi). \end{aligned} \tag{4.6}$$

For the above vectors, the $O(1/r^4)$ terms in the t, ϕ components and $O(1/r)$ terms in the r components are arbitrary, and just represent the pure, or "proper" [16], gauge transformations. That is, consider any deformation vector whose t, ϕ components behave as $O(1/r^4)$ and r component behaves as $O(1/r)$. As will be shown below, such deformation vectors have no associated charge and the generators of these deformations vanish weakly. Then the transformations described by these vectors are pure gauge, producing effects which are not to be considered as physically meaningful. So to be precise, the asymptotic symmetry group will be defined as the factor group obtained by identifying all transformations described by vectors (4.5) which may differ by $O(1/r^4)$ terms in their t, ϕ components, or by $O(1/r)$ terms in their r components.

The asymptotic symmetry group defined above is isomorphic to the pseudo-conformal group in two dimensions. This may be seen from (4.6) by noticing that the functions $T(t, \phi)$ and $\Phi(t, \phi)$ satisfy the conformal Killing equations in two dimensions with an indefinite metric, and once a solution $T(t, \phi)$, $\Phi(t, \phi)$ has been selected, the remaining functions $R(t, \phi)$, $\bar{T}(t, \phi)$, and $\bar{\Phi}(t, \phi)$ are determined. We will often refer to the asymptotic symmetry group as simply the conformal group.

The conformal group also arises as the asymptotic symmetry group from a conformal analysis of infinity [17]. Denoting the metric (4.3, 4.4) by dS^2, the conformally related metric $d\bar{S}^2 = (1/r^2)dS^2$ has a surface at $r = \infty$ with induced metric

$$d\bar{S}_I^2 = -\frac{1}{R^2}dt^2 + d\phi^2.$$

The group of conformal motions on this surface is just the pseudo-conformal group in two dimensions.

Because of the periodicity conditions in the coordinate ϕ, the conformal Killing equations (4.6) can be Fourier analyzed. Then the asymptotic symmetries (4.5) may be written explicitly in terms of an integer n as

$$\begin{aligned}
A_n = A_{-n} = &\left[R\left(1 - \frac{n^2R^2}{2r^2}\right)\cos\frac{nt}{R}\cos n\phi + O(1/r^4)\right]d/dt \\
&-\left[\left(1 + \frac{n^2R^2}{2r^2}\right)\sin\frac{nt}{R}\sin n\phi + O(1/r^4)\right]d/d\phi \\
&+\left[rn\sin\frac{nt}{R}\cos n\phi + O(1/r)\right]d/dr, \\
B_n = B_{-n} = &\left[R\left(1 - \frac{n^2R^2}{2r^2}\right)\sin\frac{nt}{R}\sin n\phi + O(1/r^4)\right]d/dt \\
&-\left[\left(1 + \frac{n^2R^2}{2r^2}\right)\cos\frac{nt}{R}\cos n\phi + O(1/r^4)\right]d/d\phi \\
&+\left[rn\cos\frac{nt}{R}\sin n\phi + O(1/r)\right]d/dr, \\
C_n = -C_{-n} = &\left[R\left(1 - \frac{n^2R^2}{2r^2}\right)\sin\frac{nt}{R}\cos n\phi + O(1/r^4)\right]d/dt \\
&+\left[\left(1 + \frac{n^2R^2}{2r^2}\right)\cos\frac{nt}{R}\sin n\phi + O(1/r^4)\right]d/d\phi
\end{aligned}$$

$$-\left[rn\cos\frac{nt}{R}\cos n\phi + O(1/r)\right]d/dr,$$

$$D_n = -D_{-n} = \left[R\left(1 - \frac{n^2R^2}{2r^2}\right)\cos\frac{nt}{R}\sin n\phi + O(1/r^4)\right]d/dt$$
$$+\left[\left(1 + \frac{n^2R^2}{2r^2}\right)\sin\frac{nt}{R}\cos n\phi + O(1/r^4)\right]d/d\phi$$
$$+\left[rn\sin\frac{nt}{R}\sin n\phi + O(1/r)\right]d/dr. \tag{4.7}$$

The group algebra for the generators (4.7) may be written as follows:

$$[A_n, A_m] = \left(\frac{n-m}{2}\right)C_{n+m} + \left(\frac{n+m}{2}\right)C_{n-m},$$
$$[B_n, B_m] = \left(\frac{n-m}{2}\right)C_{n+m} + \left(\frac{n+m}{2}\right)C_{n-m},$$
$$[C_n, C_m] = -\left(\frac{n-m}{2}\right)C_{n+m} + \left(\frac{n+m}{2}\right)C_{n-m},$$
$$[D_n, D_m] = -\left(\frac{n-m}{2}\right)C_{n+m} + \left(\frac{n+m}{2}\right)C_{n-m},$$
$$[A_n, B_m] = -\left(\frac{n-m}{2}\right)D_{n+m} - \left(\frac{n+m}{2}\right)D_{n-m},$$
$$[A_n, C_m] = -\left(\frac{n-m}{2}\right)A_{n+m} + \left(\frac{n+m}{2}\right)A_{n-m},$$
$$[A_n, D_m] = \left(\frac{n-m}{2}\right)B_{n+m} - \left(\frac{n+m}{2}\right)B_{n-m},$$
$$[B_n, C_m] = -\left(\frac{n-m}{2}\right)B_{n+m} + \left(\frac{n+m}{2}\right)B_{n-m},$$
$$[B_n, D_m] = \left(\frac{n-m}{2}\right)A_{n+m} - \left(\frac{n+m}{2}\right)A_{n-m},$$
$$[C_n, D_m] = -\left(\frac{n-m}{2}\right)D_{n+m} + \left(\frac{n+m}{2}\right)D_{n-m}. \tag{4.8}$$

Notice that the anti-de Sitter group $O(2, 2)$ is the subgroup spanned by the vectors (4.7) with $n = 0, 1$. However, $O(2, 2)$ is not an invariant subgroup, so there is no obvious way to restrict the asymptotic symmetries to just the anti-de Sitter group. The situation here is similar to 3 + 1 asymptotically flat gravity which has the Spi group (similar to the BMS group) of asymptotic symmetries containing the Poincaré group as a sub-group. In contrast, the group of asymptotic symmetries for gravity in $3+1$ dimensions with $\Lambda < 0$ is precisely the anti-de Sitter group $O(2,3)$ [15].

To this point, the asymptotic symmetries have been treated as the group of vector fields preserving the spacetime metric (4.3, 4.4) under Lie transport. In the canonical formalism, these vector fields become the allowed asymptotic deformations of a spacelike surface which is described by the canonical variables g_{ij}, π^{ij}. From (4.3, 4.4), the lapse and shift are determined to be

$$N = \frac{r}{R} + O(1/r), \quad N^r = O(1/r), \quad N^\phi = O(1/r^2), \tag{4.9}$$

so that the asymptotic behavior of the canonical variables is given by Eqs. (4.4) along with

$$\pi^{rr} = O(1/r), \quad \pi^{r\phi} = O(1/r^2), \quad \pi^{\phi\phi} = O(1/r^5). \tag{4.10}$$

However, in the canonical formalism, the spacelike surfaces are evolved according to Hamiltonian evolution, which generally differs from Lie transport unless the spatial Einstein equations ${}^{(3)}G_{ij} = \Lambda g_{ij}$ hold. To insure that spacelike surfaces initially obeying the boundary conditions (4.4) and (4.10) will preserve these boundary conditions under deformations generated by the Hamiltonian, it is necessary to impose further restrictions on the canonical variables [15].

In the appendix, we show that when the Hamiltonian constraints $\mathscr{H}_\mu = 0$ hold in a neighborhood of infinity, then the boundary conditions (4.4, 4.10) are preserved under Hamiltonian evolution. The reason it is possible to formulate the extra conditions on the canonical variables in terms of the constraints is because the spatial part of the Einstein tensor ${}^{(3)}G_{ij}$, which determines the difference between Lie and Hamiltonian evolution, is related to the constraints $\mathscr{H}_\mu$ through the contracted Bianchi identities. In $2+1$ dimensions, no further conditions on the canonical variables are needed, because there are precisely three components ${}^{(3)}G_{ij}$ to be restricted by the three constraints $\mathscr{H}_\mu$. Of course, as long as the spacelike surface is imbedded in a spacetime which solves Einstein's equations, the constraints $\mathscr{H}_\mu = 0$ will be satisfied anyway, so these conditions have no serious consequences.

As described in Sect. III, the charge $J[\xi]$ may be found by taking into account the asymptotic behaviour of the canonical variables (4.4, 4.10) and deformation vectors (4.5) and rewriting the integral (3.3) as the total variation of a surface integral. The negative of this surface integral, actually a line integral for $2+1$ spacetime dimensions, determines the charge $J[\xi]$ to within a constant which will be adjusted so that $J[\xi]$ vanishes for globally anti-de Sitter space. Denoting the spatial metric for a globally anti-de Sitter spacetime by $\mathring{g}_{ij}$, the charge is

$$J[\xi] = \lim_{r\to\infty} \oint dS_l \{ \mathring{G}^{ijkl} [\xi^\perp g_{ij|k} - \xi^\perp{}_{,k}(g_{ij} - \mathring{g}_{ij})] + 2\xi^i \pi_i{}^l \}, \tag{4.11}$$

where the horizontal bar indicates covariant differentiation with respect to $\mathring{g}_{ij}$. This expression for $J[\xi]$ has the same form as the one obtained for $3+1$ dimensional gravity with $\Lambda < 0$ [15]. Also note that, as previously mentioned, the charge vanishes for any surface deformation which describes a pure gauge transformation.

For the spacetime (4.2), the only nonzero charges are those associated with $A_0 = R\, d/dt$ and $B_0 = -d/d\phi$, namely

$$J[A_0] = 2\pi R\left[1 - \alpha^2 - \frac{A^2}{R^2}\right], \tag{4.12a}$$

$$J[B_0] = -4\pi\alpha A. \tag{4.12b}$$

The two vectors A_0 and B_0 are essentially the generators of the asymptotic symmetry group $R \times \mathrm{SO}(2)$ treated in Sect. III, differing from those quantities only in their normalization. However, the "energy" $1/R\; J[A_0]$ obtained from (4.12a), in contrast with (3.6a), no longer has the desired limit as $R \to \infty$. This should not be too surprising, since the coordinate change (4.1) involved the "canonical variables" in the form of α and A. The coordinate t in (4.2) is no longer normalized to proper time in the $R \to \infty$ limit, and correspondingly, the normal components $A_0^{\perp}$ of the deformation vector A_0, used to determine the energy, is no longer normalized to unity in this limit.

Finally, it will be important for what remains to understand the asymptotic form of the $\perp$, r, ϕ components of the surface deformation vectors ξ. These components are given in Eqs. (3.4) in terms of the spacetime components of some conformal group vector ${}^{(3)}\xi^{\alpha}$ restricted to a $t = \text{constant}$ surface. The leading order terms in ξ^{μ} are completely determined once the spacetime components ${}^{(3)}\xi^{\alpha}$ are given. But to higher orders in $1/r$, $\xi^{\perp}$ and ξ^{ϕ} depend on the unspecified $O(1/r)$ term in the lapse N and on the shift N^{ϕ}. Then in Hamiltonian language, the asymptotic form of the surface deformation vectors depends on the canonical variables. (See the appendix for details.)

Actually, the dependence of ξ^{μ} on the canonical variables is not relevant in establishing (4.11) as the proper surface integral to appear in the Hamiltonian, or in evaluating the charges for a spacetime such as (4.2), because for these purposes, ξ^{μ} is only needed to leading order in $1/r$. However, more than just the leading order terms in ξ^{μ} are important for the requirement that the boundary conditions on the canonical variables be preserved under surface deformations.

V. The Canonical Realization of Asymptotic Symmetries

The primary goal of this article is to point out the possible existence of central charges in the canonical realization of asymptotic symmetries. In this section, we explicitly derive the Poisson bracket algebra of the Hamiltonian generators $H[\xi]$ for $2+1$, $\Lambda < 0$ gravity with the conformal group of asymptotic symmetries, and obtain such central charges. It should be clear from this example that for any gauge theory, the global charges may form a central extension of the asymptotic symmetry algebra with potentially non-trivial central charges.

The Hamiltonian generators for $2+1$, $\Lambda < 0$ gravity have the form

$$H[\xi] = \int d^2x\, \xi^{\mu}(x)\mathscr{H}_{\mu}(x) + J[\xi], \tag{5.1}$$

where $\mathscr{H}_{\mu}$ are the standard constraints for general relativity, and $J[\xi]$ are the charges. When the allowed deformations are defined by the conformal group of asymptotic symmetries, the charges $J[\xi]$ are given by the surface integral in Eq.

(4.11). These surface integrals are constructed in such a way that the Hamiltonian will have well defined variational derivatives, and as a result, will be a well defined generator of surface deformations through the Poisson bracket.

The asymptotic symmetries are canonically realized by the "factor group" of surface deformation generators, which is defined by identifying two Hamiltonian generators if they describe the same asymptotic (conformal group) deformation and differ only by a pure gauge deformation. It is in this sense that we shall loosely refer to the Hamiltonian generators $H[\xi]$ as providing a canonical realization, or else a central extension, of the conformal group algebra. On the other hand, fixing the gauge so that the constraints $\mathscr{H}_\mu = 0$ hold strongly is effectively the same as considering the factor group, because then the asymptotic part of the deformation vector ξ^μ determines the surface deformation everywhere, and the charges themselves become well defined as generators through the Dirac bracket [18]. The algebra of these charges is identical to the factor group algebra of the Hamiltonian generators, so that the charges $J[\xi]$ also realize the asymptotic symmetry group algebra.

In principle, the algebra of the generators $H[\xi]$ could be computed directly from the Poisson bracket. Such a calculation is typically very cumbersome, but for the case at hand, the situation is even worse because the deformation vector components ξ^μ depend on the canonical variables. This dependence was discussed at the end of Sect. IV, where it was also pointed out that ξ^μ does not depend on the canonical variables to leading order in $1/r$, and thus its asymptotic form can be completely determined once a conformal group vector is chosen. The derivation which we present here does not depend on any further details of ξ^μ, and it should also be emphasized that the dependence of ξ^μ on the canonical variables has no logical connection with the presence of central charges in the algebra of the generators.

Our starting point for computing the algebra of the generators (5.1) is based on a theorem proved in [10]. The theorem is a completely natural one, stating that the Poisson bracket $\{H[\xi], H[\eta]\}$ of two well defined generators $H[\xi]$ and $H[\eta]$ is itself a well defined generator. As pointed out earlier, the charges $J[\xi]$ are only defined up to the addition of a constant, which has been adjusted in (4.11) so that a globally anti-de Sitter space has no charge. As a result, once it is shown that the volume integral part of $\{H[\xi], H[\eta]\}$ is of the same form as that of (5.1), it follows that the surface term which must occur in that Poisson bracket can at most differ from the charge of Eq. (4.11) by a constant $K[\xi, \eta]$, which depends only on the asymptotic form of the deformations ξ, η. Then given two generators $H[\xi]$ and $H[\eta]$ of the form (5.1), their Poisson bracket may be written as

$$\{H[\xi], H[\eta]\} = H[\zeta] + K[\xi, \eta], \tag{5.2}$$

where $H[\zeta]$ is also a well defined generator of the form (5.1).

In order to demonstrate that (5.2) is a central extension of the conformal group algebra, it must be shown that the asymptotic form of the deformation vector ζ is given by the Lie bracket $[\xi, \eta]$. Of course, this still leaves open the possibility that the constants $K[\xi, \eta] = 0$, so that the central extension is trivial. We will wait until the end to compute the constants K explicitly and show that they cannot be absorbed into a redefinition of the canonical generators.

The critical step in this analysis is to recognize that the "volume" term of the Poisson bracket (5.2) may be calculated by assuming that ξ and η are pure gauge, in which case the charges vanish. Indeed, the Poisson bracket is defined in terms of variational derivatives of the Hamiltonian generators. The definition of these generators includes the addition of surface integral charges in just such a way that variations will yield the "right-hand side" of the Hamiltonian equations, which are local in the canonical variables and deformation vectors, regardless of the asymptotic behavior of the deformation vectors. Then the generator obtained by computing the Poisson bracket under the assumption that ξ and η describe pure gauge deformations can only differ from the generator which would be obtained without this assumption by terms which vanish when ξ and η are pure gauge. Furthermore, these additional terms occurring in the Poisson bracket $\{H[\xi], H[\eta]\}$ are just surface terms arising from integration by parts. In view of the above mentioned theorem, they must be precisely those surface integrals necessary to make $\{H[\xi], H[\eta]\}$ a well defined Hamiltonian generator when the deformation vectors are allowed to describe conformal group transformations at infinity.

So by assuming ξ and η to be pure gauge, the charges vanish and the Poisson bracket can be computed as

$$\begin{aligned}\{H[\xi], H[\eta]\} &= \int d^2x d^2y \{\xi^\mu(x)\mathscr{H}_\mu(x),\, \eta^\nu(y)\mathscr{H}_\nu(y)\} \\ &= \int d^2x d^2y \xi^\mu(x)\eta^\nu(y)\{\mathscr{H}_\mu(x), \mathscr{H}_\nu(y)\} \\ &\quad + \int d^2x \{\xi^\mu(x), H[\eta]\}\mathscr{H}_\mu(x) \\ &\quad + \int d^2y \{H[\xi], \eta^\nu(y)\}\mathscr{H}_\nu(y) \\ &\quad - \int d^2x d^2y \{\xi^\mu(x), \eta^\nu(y)\}\mathscr{H}_\mu(x)\mathscr{H}_\nu(y) \\ &= \int d^2x [[\xi, \eta]^\mu_{SD} + \delta_\eta \xi^\mu - \delta_\xi \eta^\mu \\ &\quad - \int d^2y \{\xi^\mu(x), \eta^\nu(y)\}\mathscr{H}_\nu(y)]\mathscr{H}_\mu(x).\end{aligned}$$

Here, $[\xi, \eta]^\mu_{SD}$ is given by the usual surface deformation algebra for the vectors ξ^μ, η^ν [19], and $\delta_\eta \xi^\mu$ represents the change in the vector components ξ^μ under the surface deformations generated by $H[\eta]$. Also, Poisson brackets such as $\{\mathscr{H}_\mu(x), \mathscr{H}_\nu(y)\}$ must be computed by taking into account the integration over $\xi^\mu(x)$ and $\eta^\nu(y)$, since these are pure gauge deformation vectors and vanish sufficiently rapidly at infinity to insure that the variational derivatives of $\mathscr{H}_\mu(x)$ and $\mathscr{H}_\nu(y)$ can be well defined. Then by the arguments above, the Poisson bracket must generally have the form (5.2), where

$$\zeta^\mu(x) = [\xi, \eta]^\mu_{SD} + \delta_\eta \xi^\mu - \delta_\xi \eta^\mu - \int d^2y \{\xi^\mu(x), \eta^\nu(y)\}\mathscr{H}_\nu(y), \tag{5.3}$$

even when ξ and η are conformal group vectors.

In order to recognize ζ as a particular conformal group vector (4.5), recall that any such vector is uniquely determined, up to gauge terms, by its leading order contributions in $1/r$. Since the leading order terms of all conformal group vectors are independent of the canonical variables, it follows that $\delta_\eta \xi^\mu$ and $\delta_\xi \eta^\mu$ make only higher order contributions to ζ^μ in Eq. (5.3). The last term in (5.3) also will not contribute to the leading order of ζ^μ, because it is a linear combination of constraints and their derivatives, which must decrease faster than any power of $1/r$ (see the appendix). As a result, ζ^μ can be written to leading order as

$$\zeta \to [\xi, \eta]_{SD}.$$

Furthermore, the fact that the right-hand side of (5.2) is a well defined generator insures that the non-leading order terms in Eq. (5.3) must work out in such a way that ζ^μ meets the requirements of a conformal group vector to all necessary orders in $1/r$.

The final step in the demonstration that (5.2) is a central extension of the conformal group algebra is to show that, to leading order in $1/r$, the surface deformation algebra $[\xi, \eta]_{SD}$ coincides with the Lie algebra $[\xi, \eta]$ for conformal group vectors ξ and η. This can be done by first writing the surface deformation algebra in spacetime coordinates in terms of the spacetime components of the deformation vectors (where the superscript (3), previously used to denote spacetime components, has been dropped):

$$[\xi, \eta]^t_{SD} = (\xi^i + N^i\xi^t)\eta^t{}_{,i} + \frac{N_{,i}}{N}\xi^i\eta^t - (\xi \leftrightarrow \eta),$$

$$[\xi, \eta]^i_{SD} = g^{ij}(N)^2\xi^t\eta^t{}_{,j} + \left(N^i{}_{,j} - \frac{N^i N_{,j}}{N}\right)\xi^j\eta^t + (\xi^j + N^j\xi^t)\eta^i{}_{,j} - (\xi \leftrightarrow \eta).$$

These expressions are simplified by using the asymptotic forms for the spatial metric g_{ij} (4.4), lapse N and shifts N^i (4.9), and by using Eqs. (4.6) to relate the leading order terms in the components of the conformal group vectors. Then the surface deformation algebra is seen to coincide with the Lie algebra to leading order in $1/r$, and

$$\zeta \to [\xi, \eta]. \tag{5.4}$$

The preceding arguments show that the conformal group content of the deformation vector ζ—that is, the part which is not pure gauge—is given by the Lie bracket $[\xi, \eta]$. As a result, Eq. (5.2) states that the Hamiltonian generators form a central extension of the conformal group algebra. We will now compute the central charges $K[\xi, \eta]$ explicitly, and then show that the central extension is not trivial because the central charges cannot be absorbed into a redefinition of the generators.

The central charges may be evaluated directly by recognizing that the Dirac bracket $\{J[\xi], J[\eta]\}^*$ is interpreted as the change in the charge $J[\xi]$ under the surface deformation of unit magnitude generated by $J[\eta]$, so that

$$\delta_\eta J[\xi] = \{J[\xi], J[\eta]\}^*.$$

On the other hand, since the charges $J[\xi]$ form a central extension of the conformal group algebra,

$$\delta_\eta J[\xi] = J[[\xi, \eta]] + K[\xi, \eta]. \tag{5.5}$$

The central charges $K[\xi, \eta]$ may be obtained from Eq. (5.5), which is most easily evaluated on the $t = 0$ surface of a globally anti-de Sitter spacetime, $\mathring{g}_{\mu\nu}$. Since the charge (4.11) has been chosen so that it vanishes for a globally anti-de Sitter spacetime, then $J[[\xi,\eta]]=0$, and the charge $J[\xi]$, before the surface is deformed, is also zero. In this case, the central charge $K[\xi,\eta]$ reduces to the value of the charge $J[\xi]$ on the surface deformed by η.

To evaluate $J[\xi]$ on the deformed surface, the expression (4.11) can be greatly simplified by specializing to r, ϕ coordinates and using the known asymptotic form of the canonical variables. This gives

$$J[\xi]=\lim_{r\to\infty}\int d\phi\left\{\frac{R}{r}\xi^{\perp}+\frac{r^3}{R^3}\xi^{\perp}\left(g_{rr}-\frac{R^2}{r^2}\right)+\frac{1}{R}\left(\frac{1}{r}\xi^{\perp}+\xi^{\perp}{}_{,r}\right)(g_{\phi\phi}-r^2)+2\xi^{\phi}\pi^{r}_{\phi}\right\},$$

which may be simplified even further by recognizing that, to leading order in $1/r$, $\pi^{r}{}_{\phi}\sim g_{t\phi}$. Then all that is needed are the metric components g_{rr}, $g_{\phi\phi}$, $g_{t\phi}$ at $t=0$, which may be easily computed from the deformed anti-de Sitter spacetime as

$$g_{\mu\nu}=\mathring{g}_{\mu\nu}+\mathscr{L}_{\eta}\mathring{g}_{\mu\nu}.$$

Carrying out the above for ξ, η equaling all possible combinations of A_n, B_n, C_n, D_n (Eqs. 4.7), the only non-zero central charges are found to be

$$K[A_n,C_m]=2\pi Rm(m^2-1)\delta_{|n||m|},$$
$$K[B_n,D_m]=-2\pi Rm(m^2-1)\delta_{|n||m|}.$$

(Incidentally, if either ξ or η are pure gauge deformations, then the above argument shows that the associated central charge vanishes, as it should [19]. When ξ is pure guage, this is so because $J[\xi]$ vanishes for all admissible field configurations. Likewise, because $\{J[\xi],J[\eta]\}^*$ may be interpreted as $-\delta_{\xi}J[\eta]$, this shows that the central charge vanishes whenever η is pure gauge.)

The Dirac bracket algebra of the charges can now be written as follows:

$$\begin{aligned}
&\{J[A_n],J[A_m]\}^*=J[[A_n,A_m]],\\
&\{J[B_n],J[B_m]\}^*=J[[B_n,B_m]],\\
&\{J[C_n],J[C_m]\}^*=J[[C_n,C_m]],\\
&\{J[D_n],J[D_m]\}^*=J[[D_n,D_m]],\\
&\{J[A_n],J[B_m]\}^*=J[[A_n,B_m]],\\
&\{J[A_n],J[C_m]\}^*=J[[A_n,C_m]]+2\pi Rm(m^2-1)\delta_{|n||m|},\\
&\{J[A_n],J[D_m]\}^*=J[[A_n,D_m]],\\
&\{J[B_n],J[C_m]\}^*=J[[B_n,C_m]],\\
&\{J[B_n],J[D_m]\}^*=J[[B_n,D_m]]-2\pi Rm(m^2-1)\delta_{|n||m|},\\
&\{J[C_n],J[D_m]\}^*=J[[C_n,D_m]].
\end{aligned}\tag{5.6}$$

It should be clear from this calculation that if the asymptotic symmetries were all exact symmetries of anti-de Sitter space, then the central charges would vanish. As pointed out in the introduction, for any theory whose asymptotic symmetries are exact symmetries of some background field configuration, the central charges can be arranged to vanish simply by adjusting the charges to zero on this background.

In the present case where the asymptotic symmetries cannot be realized as exact symmetries of some background, it is easy to see that the central charges are not trivial. For instance, the Lie bracket $[A_n, C_n]$ from Eqs. (4.8) is realized in (5.6) as

$$\{J[A_n], J[C_n]\}^* = nJ[A_0] + 2\pi Rn(n^2 - 1). \tag{5.7}$$

If the charges are redefined by $J[A_n] \to J[A_n] + a_n$ and $J[C_n] \to J[C_n] + c_n$, then (5.7) becomes

$$\{J[A_n], J[C_n]\}^* = nJ[A_0] + n[a_0 + 2\pi R(n^2 - 1)].$$

It is clear that the constants a_n, c_n can never be chosen so that the central charges are eliminated for all values of n.

It is interesting to note that the algebra (5.6) is actually a direct sum of two Virasoro algebras. The change of basis

$$L_n = \frac{i\sigma}{2} A_n + \frac{i\sigma}{2} B_n - \tfrac{1}{2} C_n + \tfrac{1}{2} D_n,$$

$$K_n = \frac{i\sigma}{2} A_n - \frac{i\sigma}{2} B_n - \tfrac{1}{2} C_n - \tfrac{1}{2} D_n,$$

is invertible for A_n, B_n, C_n, D_n in terms of L_n, L_{-n}, K_n, K_{-n}, and the algebra of the associated charges becomes

$$\begin{aligned}
&\{J[L_n], J[L_m]\}^* = (n - m)J[L_{n+m}] + 2\pi i\sigma Rn(n^2 - 1)\delta_{n,-m},\\
&\{J[K_n], J[K_m]\}^* = (n - m)J[K_{n+m}] + 2\pi i\sigma Rn(n^2 - 1)\delta_{n,-m},\\
&\{J[L_n], J[K_m]\}^* = 0.
\end{aligned}$$

This is just the familiar algebra for the canonical generators of string theory [9].

As a final comment, we briefly point out some analogies with four dimensional gravity in the asymptotically flat case. The asymptotic symmetry group is the infinite dimensional "Spi group" [4] as long as the behavior of the gravitational variables at spatial infinity is not restricted by means of parity conditions as in [8]. Then it turns out that a "central charge" appears in the canonical realization of these symmetries, in the sense that the Spi generators transform inhomogeneously under an asymptotic Spi transformation. However, the homogeneous part of the Poisson bracket algebra of the generators does not yield a representation of the Spi algebra (the bracket of two boosts contains an unwanted metric dependent, angle dependent transformation) [20], so that the situation is actually much worse in this case. This gives an additional motivation for imposing extra boundary conditions to eliminate the supertranslation ambiguities [8, 4].

Appendix: The Initial Value Problem

In the main text, we have shown that a spacetime metric obeying the boundary conditions (4.3, 4.4) is asymptotically invariant under spacetime changes of coordinates (or "diffeomorphisms") which become asymptotically elements of the two dimensional conformal group in the sense of (4.5). We have also shown that in such a spacetime, the spatial metric and its canonical momentum fall off as in (4.4, 4.10) on the appropriate spacelike sections.

Then consider the following initial value problem: suppose that on an initial surface $t = 0$, data (g_{ij}, π^{ij}) are given which have the asymptotic behavior (4.4, 4.10). Can appropriate lapse and shift functions be found such that these initial data can be developed, by means of Hamilton equations, into a spacetime metric obeying (4.3, 4.4)?

This question is not the true converse of the analysis of Sect. IV, because Lie and Hamiltonian transports only coincide on shell. The difference between them is measured by the dynamical components ${}^{(3)}G_{ij}$ of the Einstein tensor. These components turn out to decrease too slowly at infinity, so that they can only be neglected under stronger boundary conditions on the initial data (see below). This phenomenon also occurs in $3+1$ gravity [15].

In order to derive these stronger conditions, first note that the initial data cannot simply be propagated by means of the generator $H[\xi^{\perp}_{(\mathrm{ad}S)}, \xi^{k}_{(\mathrm{ad}S)}]$, since this generator does not preserve the boundary conditions. Here $\xi^{\perp}_{(\mathrm{ad}S)}$ and $\xi^{k}_{(\mathrm{ad}S)}$ are the components in the ad$S$ orthogonal frame adapted to the surfaces $t = \text{constant}$ of a generic "conformal vector field,"

$$\xi^{\perp}_{(\mathrm{ad}S)} = N^{\perp}_{(\mathrm{ad}S)}\,{}^{(3)}\xi^{t}, \quad \xi^{k}_{(\mathrm{ad}S)} = {}^{(3)}\xi^{k}. \tag{A.1}$$

To preserve the boundary conditions, the deformation vector components $\xi^{\perp}$ and ξ^{k} must include "correction terms" of order

$$\xi^{\perp} - \xi^{\perp}_{(\mathrm{ad}S)} = O(1/r), \tag{A.2}$$

$$\xi^{r} - \xi^{r}_{(\mathrm{ad}S)} = O(1/r), \tag{A.3}$$

$$\xi^{\phi} - \xi^{\phi}_{(\mathrm{ad}S)} = O(1/r^{2}), \tag{A.4}$$

and these are not "pure gauge" (except $\xi^{r} - \xi^{r}_{(\mathrm{ad}S)}$ which will no longer be of interest). From the spacetime point of view, the necessity of (A.2, A.4) could have been anticipated by noticing that such terms are precisely induced by taking into account the difference between the actual lapse and shift, and the anti-de Sitter ones in the formulas

$$\xi^{\perp} = N^{\perp}\,{}^{(3)}\xi^{t}, \quad \xi^{\phi} = {}^{(3)}\xi^{\phi} + N^{\phi}\,{}^{(3)}\xi^{t}. \tag{A.5}$$

(See the discussion at the end of Sect. IV.)

For definiteness, consider the case of an asymptotic time translation (${}^{(3)}\xi^{t} = 1$, ${}^{(3)}\xi^{k} = 0$). It is easy to see that $\xi^{\phi} - \xi^{\phi}_{(\mathrm{ad}S)}$ is entirely determined up to the appropriate order by the condition that $\{g_{r\phi}, H[\xi]\}$ be of the same order as $g_{r\phi}$ (i.e., $O(1/r^{3})$). Once this is done, all the brackets $\{g_{ij}, H[\xi]\}$ behave correctly at infinity so that only the $\dot{\pi}^{ij}$ equations remain to be analyzed.

Elementary computations show that $\{\pi^{r\phi}, H[\xi]\}$ is of the same order as $\pi^{r\phi}$, but that $\{\pi^{rr}, H[\xi]\}$ and $\{\pi^{\phi\phi}, H[\xi]\}$ generically decrease too slowly unless $\xi^{\perp}$ is properly adjusted. By using the Ricci identity for second covariant derivatives of vectors, then the two conditions

$$\{\pi^{rr}, H[\xi]\} = O(1/r), \tag{A.6a}$$

$$\{\pi^{\phi\phi}, H[\xi]\} = O(1/r^{5}) \tag{A.6b}$$

admit a solution for $\xi^{\perp} - \xi^{\perp}_{(\mathrm{ad}S)}$ if and only if the curvature ${}^{(2)}R$ of the spatial sections approaches 2Λ at infinity as

$$ {}^{(2)}R - 2\Lambda = O(1/r^4). \tag{A.7} $$

What would naively be expected from (4.4) is ${}^{(2)}R - 2\Lambda = O(1/r^2)$, which is the reason why the boundary conditions (4.4, 4.10) at infinity must be strengthened.

When (A.7) holds, the general solution to Eqs. (A.6) is given by

$$ \xi^{\perp} - \xi^{\perp}_{(\mathrm{ad}S)} = f(g_{ij}, g_{ij,k}, g_{ij,kl}) + O(1/r^3), \tag{A.8} $$

where $f = O(1/r)$ is a given local function of the metric and its derivatives, and whose explicit form will not be of interest here. The $O(1/r^3)$ term is arbitrary and corresponds to a pure gauge transformation.

This is not the end of the story, for the compatibility condition (A.7) must be preserved in time by the Hamiltonian equations. This problem is most conveniently analyzed by noticing that (A.7) is equivalent to

$$ \mathscr{H}_{\perp} = O(1/r^4), \tag{A.9} $$

whose bracket with the generator $H[\xi]$ is easy to evaluate. This naturally leads to the additional conditions that the constraint functions should decrease faster than any power of $1/r$.

$$ \mathscr{H}_{\mu} = O(1/r^n) \quad \text{for all } n. \tag{A.10} $$

These conditions are obviously preserved under asymptotic transformations by the conformal group and hence, form a closed set. Accordingly, when the initial data obey (4.4, 4.10) and solve the constraints in the neighborhood of the surface at infinity, they can be propagated in a manner compatible with the requirement that the resulting spacetime be asymptotic to anti-de Sitter. This answers the question raised in the beginning of the appendix. Also note that under these conditions, Lie and Hamiltonian evolution are equivalent and the spacetime evolved from the initial data obeys all of Einstein's equations in the vicinity of infinity.

As a final point, we remark that the ξ^{μ} dependence on the canonical variables has no influence on the expression of the charges (which follows from varying $H[\xi]$). This is because the surface term which arises upon taking (A.8) into account is equal to zero, since it is proportional to the constraint functions $\mathscr{H}_{\mu}$.

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Note added. Cocycles have recently become very popular in view of their connection with anomalies [21]. Cocycles also appear in the field of a monopole [22], and arise in other areas of physics as well [23]. Our paper shows the existence of possibly non-trivial two-cocycles (central charges) in the canonical realization of the asymptotic symmetry algebra.

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