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A GEOMETRIC CONSTRUCTION OF GENERALIZED YOUDEN DESIGNS FOR v A POWER OF A PRIME

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A new method of construction of generalized Youden designs for $v = s^m$, s a power of a prime is introduced here. This generalizes the construction of Ruiz and Seiden which could be applied only to even powers of a prime. The number of experimental units required to carry out the design in the corresponding cases is the same. However, the present method can be used for construction of designs which could not be constructed previously even in the case of even powers. Moreover the present method presents a unified construction for even and odd powers of primes.

For a fixed value of a prime it is noticed here that one can construct an infinite number of designs. This provides the experimenter with a choice of designs which may prove very useful in applications.

A simpler method of construction is also presented. The price one has to pay for the simplicity is that more experimental units are required for carrying out the design.

1. Introduction. Statisticians were engaged in construction of new designs at a very early stage of the development of the theory of mathematical statistics. They were motivated by the desire to help the experimenters lay out their experiments in such a fashion that the differences between any two varieties under consideration should be estimated with the same precision. The statisticians also provided the experimenters with easily understood techniques of analyzing and interpreting the results of the designs constructed. The reward of the statisticians was that the problems of the experimenters were stimulating and their solutions were interesting per se in addition to their usefulness for applications. However, because of the limitations of the human mind the statisticians had to disappoint the experimenters occasionally by telling them, for example, that the construction of the design requires the number of treatments to be a power of a prime or that the parameters of the designs have to satisfy certain relations and cannot be varied independently.

In the later stage of the development when many designs were already known and tabulated Abraham Wald (1943) formulated a criterion of optimality and showed that the commonly used Latin square designs are in fact optimal. Kiefer since 1958 has continued and generalized this work. He formulated several criteria of optimality, investigated the relationships among them and then

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formulated a criterion of universal optimality. He showed that Latin square designs and Youden designs, if regular, are universally optimal.

While formulating the optimality proofs Kiefer was led to the concept of balanced block designs (BBD) which generalized BIBD, and then via this to generalized Youden designs (GYD). His published work on optimality criteria exhibited some examples of GYD to illustrate the theory. A 1975 paper of Kiefer includes the results of long term investigations of various methods of construction of GYD. The construction is named "patchwork" because it makes use of various combinations of known designs to obtain GYD.

Inspired and stimulated by Kiefer, Ruiz and Seiden (1974) used a geometric method to construct GYD for v an even power of a prime.

At that time the authors were not familiar with Kiefer's work on the theory of construction of GYD. The aim was to use some geometric methods for construction of new GYD. The motivation for the present work was to find a method of construction which does not require m to be even.

The reader familiar with Kiefer's theory on "patchwork" (1975) will have no difficulty in identifying the propositions of Kiefer's paper of which the described constructions can be considered as realizations.

We are presenting here a construction of GYD for $v = s^m$, s a power of a prime and m any integer $\geqq 2$. We also show that for fixed s and m there are infinite number of GYD because one can vary independently the number of rows and columns of the designs by using independent multiples of s. For $s = 2$, $m = 2$ the construction reduces to Kiefer's (1975) construction.

2. Construction. First we define the GYD as formulated by Kiefer (1975).

In the block design setting with v varieties and b blocks of size k a design is a $k \times b$ array with blocks as columns and n_{ij} denotes the number of appearances of variety i in block j. Let $r_i = \sum_j n_{ij}$ and $\lambda_{ih} = \sum_j n_{ij} n_{hj}$ denote the number of occurrences of variety i and the number of times varieties i and h occur in the same block.

DEFINITION 1. A BBD is a design with all r_i and λ_{ih} equal for $i < h$, $|n_{ij} - k/v| < 1$ for all i and j.

DEFINITION 2. A $b_1 \times b_2$ array of the symbols $1, 2, \cdots, v$ is a GYD if it is a BBD when each of {rows} and {columns} is considered as blocks.

Clearly the parameters of GYD are not independent and some necessary conditions for construction were stated by Ruiz and Seiden (1974) and Kiefer (1975).

To start the construction we split the $s(s^m - 1)/(s - 1)(m - 1)$ flats in $EG(m, s)$ into m sets as follows: Let G_p be the sets of $(m - 1)$ flats satisfying the equations

$$a_1 x_1 + a_2 x_2 + \cdots + a_{p-1} x_{p-1} + x_p = a \qquad p = 1, \cdots, m$$

where x_i denotes the ith coordinate in $EG(m, s)$ and $a, a_i, i = 1, \cdots, p^{-1}$ belong to $GF(s)$.

Keeping the a_i's fixed and letting "a" assume all values in $GF(s)$ we obtain a parallel pencil of s flats. Hence each set G_p comprises s^p flats and the totality of them accounts for all $s(s^m - 1)/(s - 1)$ flats of $EG(m, s)$.

Let $\alpha_0, \alpha_1, \cdots, \alpha_{s-1}$ be the s elements of $GF(s)$ in an arbitrarily defined order. This order will be used to order the s^m points of $EG(m, s)$ as follows: Consider two distinct points $A = \{a_1, a_2, \cdots, a_m\}$ and $B = \{b_1, b_2, \cdots, b_m\}$. Suppose that the first coordinate in which they differ is the ith coordinate.

$A < B$ (A precedes B) if and only if $a_i < b_i$.

We are ready now to describe the construction and shall do it in three steps.

STEP 1. $p = 1, 2, 3, \cdots, m - 1$.

Let l_i, $i = 1, 2, 3, \cdots, s^{p-1}$, be pencils in G_p, and L_i be $s \times s^{m-1}$ matrix with the flats in l_i as its rows. Note that the order of the flats in L_i is arbitrary.

We define $T_p = (L_1, L_2, \cdots, L_{sp-1})'$ and $\xi = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ where I is the identity matrix of size $s^{m-1} - 1$. It is clear that T_p is a $s^p \times s^{m-1}$ matrix. Let $T^* = (L_1, L_2\xi^s, L_3\xi^{2s}, \cdots, L_{sp-1}\xi^{(s^{p-1}-1)s})^T$. $T^{**} = (T_p{}^*, T_p{}^*\xi^{2s^p}, \cdots, T_p{}^*\xi^{s^{m+1-s^p}})^T$.

Since $T_p{}^{**}$ is a $s^{m-1} \times s^{m-1}$ matrix and the point on the $(n + us)$th position of any flat in any pencil l_i, where $1 \leqq n \leqq s$, $0 \leqq u \leqq s^{m-2} - 1$, is on the flat $x_m = \alpha_{n-1}$, the elements of the $(n - us)$th column of $T_p{}^{**}$ form the flat $x_m = \alpha_{n-1}$.

Let $E = \left(\begin{smallmatrix} 0 & 1 \\ I & 0 \end{smallmatrix}\right)$ where I is of size $(s - 1)$ and η a $s^{m-1} \times s^{m-1}$ matrix which has E as its diagonal elements.

Consider the matrix $D_p = (T_p{}^{**}, T_p{}^{**}\eta, \cdots, T_p{}^{**}\eta^{s-1})^T$. Multiplying $T_p{}^{**}$ on the right by η gives a permutation of each row in which each s-tuple within the $(us + 1)$th and $(u + 1)s$th columns is permuted cyclically, $0 \leqq u \leqq s^{m-2} - 1$. Moreover since the $(n + us)$th column of $T_p{}^{**}$ is the flat $x_m = \alpha_{n-1}$, each column of D_p is therefore a permutation of all s^m points of $EG(m, s)$.

Note that each flat in G_p as a row of $T_p{}^{**}$ has been repeated s^{m-p-1} times in $T_p{}^{**}$, and thus each flat in G_p as a row of D_p has been repeated s^{m-p} times in D_p.

STEP 2. $p = m$.

There are s^{m-1} pencils l_i in G_m, $i = 1, 2, 3, \cdots, s^{m-1}$. Let L_i be $s \times s^{m-1}$ matrix with the flats in l_i as its rows and ξ be defined as in Step 1.

Consider the following matrix:

$$D_m = (L_1, L_2\xi, L_3\xi^2, \cdots, L_{s^{m-1}}\xi^{(s^{m-1}-1)})^T .$$

It follows that each column of D_m is a permutation of s^m points in $EG(m, s)$.

Interchange the coordinates using the permutations (x_i, x_{m-i+1}) and order the points as before using the new coordinates.

Let $G_p{}'$ be the set of $s^p(m - 1)$ flats in terms of the new coordinates. Thus $G_p{}' = X_{m-p+1} + a_{p-1}X_{m-p+2} + \cdots + a_1X_m = a, a, a_i \in GF(s)$. Repeat Steps 1 and 2 using $G_p{}''$s instead of $G_p{}'$s.

NOTATION. Let every variety occur m_r or $m_r + 1$ in each row and n_c or $n_c + 1$ times in each column of the design.

Let every two distinct varieties occur together λ_1 times in each row and λ_2 times in the same column.

With this notation the above construction yields the following main results.

THEOREM 1. *There exist* GYD *with parameters* $v = s^m$, $m \geqq 2$, $b_1 = (t_1 s + 1)s^{m-1}(s^m - 1)/(s - 1)$, $b_2 = (t_2 s + 1)s^{m-1}(s^m - 1)/(s - 1)$, *where* $t_1, t_2 = 0, 1, 2, 3, \cdots$. *The other parameters are*

$$
\begin{aligned}
r &= (t_1 s + 1)(t_2 s + 1)s^{m-2}(s^m - 1)^2/(s - 1)^2 \\
m_r &= (t_1 s + 1)(s^{m-1} - 1)/(s - 1) + t_1 \\
n_c &= (t_2 s + 1)(s^{m-1} - 1)/(s - 1) + t_2 \\
\lambda_1 &= [(t_1 s + 1)(s^{m-1} - 1)/(s - 1) + t_1](t_2 s + 1)s^{m-2}(s^m - 1)/(s - 1) \\
&\qquad \times [(t_1 s + 1)(s^m - 1)/(s - 1) + 1] + (t_2 s + 1)s^{m-2}(s^{m-1} - 1)/(s - 1) \\
\lambda_2 &= [(t_2 s + 1)(s^{m-1} - 1)/(s - 1) + t_2](t_1 s + 1)s^{m-2}(s^m - 1)/(s - 1) \\
&\qquad \times [(t_2 s + 1)(s^m - 1)/(s - 1) + 1] + (t_1 s + 1)s^{m-2}(s^{m-1} - 1)/(s - 1) .
\end{aligned}
$$

PROOF. For the sake of convenience, for a matrix M and a positive integer n, the matrix

$$[\overbrace{M M \cdots M}^{n}]$$

is denoted $M^{[n]}$.

Let

$$
\begin{aligned}
A_{(i,t_2)} &= [D_i^T]^{[(t_2 s+1)s^{i-2}]} , && i = 2, 3, \cdots, m-1 , \\
A'_{(i,t_1)} &= [(D_i')^T]^{[(t_1 s+1)s^{i-2}]} , && i = 2, 3, \cdots, m-1 , \\
A &= [[D_1^T]^{[t_2]}\, [D_m^T]^{[(t_2 s+1)s^{m-2}]}\, A_{(m-1,t_2)}\, A_{(m-2,t_2)} \cdots A_{(2,t_2)}]
\end{aligned}
$$

and

$$A' = [[(D_1')^T]^{[t_1]}\, [D_m'^T]^{[(t_1 s+1)s^{m-2}]}\, A'_{(m-1,t_1)}\, A'_{(m-2,t_1)} \cdots A'_{(2,t_1)}] .$$

Then the desired GYD is

$$G = \begin{bmatrix} T_1^{**} & A' \\ A^T & (L)_\beta{}^\alpha \end{bmatrix}$$

where $\alpha = (t_1 + 1)(s^{m-1} - 1)/(s - 1) + t_1 s^{m-1}$, $\beta = (t_2 + 1)(s^{m-1} - 1)/(s - 1) + t_2 s^{m-1}$ and $(L)_\beta{}^\alpha$ is a $\beta \times \alpha$ matrix with each entry a Latin square of order s^m.

It follows from the above construction that each flat in $EG(m, s)$ appears $(t_1 s + 1)s^{m-2}$ times as columns in the upper part from row 1 to row s^{m-1} of G. Similarly each flat appears $(t_2 s + 1)s^{m-2}$ times as rows in the left-hand part from column 1 to column s^{m-1} of G. Moreover D_i, D_i', $i = 1, 2, 3, \cdots, m-1, m$, are all arranged in such a way that each column is a permutation of s^m points. Since the number of flats in $EG(m, s)$ which contains a prescribed point (pair of points) are all the same for any two points (pairs of points) in $EG(m, s)$, therefore G must be a GYD.

NOTATION. The remainder of the division of an integer a by b will be written as $a_{(b)}$.

REMARK 1. For $s = m = 2$, Theorem 1 asserts the existence of GYD stated by Kiefer (1975) in Example 4.1.1. We wish to add at this point that $b_1 = 6(2j_1 - 1)$ and $b_2 = 6(2j_2 - 1)$ are the only possibilities for b_1, b_2 when $v = 4$. For $v = 4$ if the GYD is nonregular $b_{1(v)}$ and $b_{2(v)}$ are either 1, 2 or 3. Ruiz and Seiden (1974) stated that for GYD

(i) $vr_{(b_1)} = b_1 b_{2(v)}$, $b_2 r_{(b_1)} = rb_{2(v)}$
(ii) $r_{(b_1)}(b_{2(v)} - 1)/(v - 1)$ and $r_{(b_2)}(b_{1(v)} - 1)/(v - 1)$

must be integers.

It follows from (*i*) that $r_{(b_1)}$ cannot be an integer unless $b_{1(v)}$ and $b_{2(v)}$ are equal to 2 and from (ii) $r_{(b_1)}$ and $r_{(b_2)}$ must be divisible by 3. Hence b_1 and b_2 must be multiplies of 6.

We shall illustrate the construction stated in Theorem 1 by the following example.

EXAMPLE 1.

$$s = 2\,, \quad m = 3\,, \quad t_1 = t_2 = 0\,.$$
$$v = s^m = 2^3 = 8\,. \qquad b = 28\,, \quad k = 28\,, \quad r = 98\,, \quad m_r = 3\,, \quad n_e = 3$$
$$\lambda_1 = \lambda_2 = 342.$$

Let

$$0 = (0,0,0)\,, \quad 1 = (0,0,1)\,, \quad 2 = (0,1,0)\,, \quad 3 = (0,1,1)\,, \quad 4 = (1,0,0)$$
$$5 = (1,0,1)\,, \quad 6 = (1,1,0)\,, \quad 7 = (1,1,1)\,.$$

$$G_1\colon \quad x_1 = 0\,, \quad x_1 = 1$$
$$G_2\colon \quad x_2 = 0\,, \quad x_2 = 1\,; \quad x_1 + x_2 = 0\,, \quad x_1 + x_2 = 1$$
$$G_3\colon \quad x_3 = 0\,, \quad x_3 = 1\,; \quad x_2 + x_3 = 0\,, \quad x_2 + x_3 = 1\,; \quad x_1 + x_2 + x_3 = 0\,,$$
$$x_1 + x_2 + x_3 = 1\,; \quad x_1 + x_3 = 0\,, \quad x_1 + x_3 = 1\,.$$

Then

$$T_1 = \begin{matrix} 0 & 1 & 2 & 3 \\ 4 & 5 & 6 & 7 \end{matrix}\,, \qquad T_2 = \begin{matrix} 0 & 1 & 4 & 5 \\ 2 & 3 & 6 & 7 \\ 0 & 1 & 6 & 7 \\ 2 & 3 & 4 & 5 \end{matrix}$$

$$T_1{}^* = \begin{matrix} 0 & 1 & 2 & 3 \\ 4 & 5 & 6 & 7 \end{matrix}\,, \qquad T_2{}^* = T_2{}^{**} = \begin{matrix} 0 & 1 & 4 & 5 \\ 2 & 3 & 6 & 7 \\ 6 & 7 & 0 & 1 \\ 4 & 5 & 2 & 3 \end{matrix}$$

$$T_1{}^{**} = \begin{matrix} 0 & 1 & 2 & 3 \\ 4 & 5 & 6 & 7 \\ 2 & 3 & 0 & 1 \\ 6 & 7 & 4 & 5 \end{matrix}\,,$$

$$D_2 = \begin{matrix} 0 & 1 & 4 & 5 \\ 2 & 3 & 6 & 7 \\ 6 & 7 & 0 & 1 \\ 4 & 5 & 2 & 3 \\ 1 & 0 & 5 & 4 \\ 3 & 2 & 7 & 6 \\ 7 & 6 & 1 & 0 \\ 5 & 4 & 3 & 2 \end{matrix}, \qquad D_3 = \begin{matrix} 0 & 2 & 4 & 6 \\ 1 & 3 & 5 & 7 \\ 3 & 4 & 7 & 0 \\ 2 & 5 & 6 & 1 \\ 5 & 6 & 0 & 3 \\ 4 & 7 & 1 & 2 \\ 7 & 0 & 2 & 5 \\ 6 & 1 & 3 & 4 \end{matrix}$$

G_1': $x_3 = 0\,, \quad x_3 = 1$

G_2': $x_2 = 0\,, \quad x_2 = 1\,; \quad x_3 + x_2 = 0\,, \quad x_3 + x_2 = 1\,,$

G_3': $x_1 = 0\,, \quad x_1 = 1\,; \quad x_2 + x_1 = 0\,, \quad x_2 + x_1 = 1\,; \quad x_3 + x_2 + x_1 = 0\,,$
$x_3 + x_2 + x_1 = 1\,; \quad x_3 + x_1 = 0\,, \quad x_3 + x_1 = 1\,.$

$$T_2' = \begin{matrix} 0 & 4 & 1 & 5 \\ 2 & 6 & 3 & 7 \\ 0 & 4 & 3 & 7 \\ 2 & 6 & 1 & 5 \end{matrix}, \qquad T_2'^{*} = T_2'^{**} = \begin{matrix} 0 & 4 & 1 & 5 \\ 2 & 6 & 3 & 7 \\ 3 & 7 & 0 & 4 \\ 1 & 5 & 2 & 6 \end{matrix}$$

$$D_2' = \begin{matrix} 0 & 4 & 1 & 5 \\ 2 & 6 & 3 & 7 \\ 3 & 7 & 0 & 4 \\ 1 & 5 & 2 & 6 \\ 4 & 0 & 5 & 1 \\ 6 & 2 & 7 & 3 \\ 7 & 3 & 4 & 0 \\ 5 & 1 & 6 & 2 \end{matrix}, \qquad D_3' = \begin{matrix} 0 & 2 & 1 & 3 \\ 4 & 6 & 5 & 7 \\ 6 & 1 & 7 & 0 \\ 2 & 5 & 3 & 4 \\ 5 & 3 & 0 & 6 \\ 1 & 7 & 4 & 2 \\ 7 & 0 & 2 & 5 \\ 3 & 4 & 6 & 1 \end{matrix}$$

$$A_{(2,0)} = [D_2^T] = \begin{matrix} 0 & 2 & 6 & 4 & 1 & 3 & 7 & 5 \\ 1 & 3 & 7 & 5 & 0 & 2 & 6 & 4 \\ 4 & 6 & 0 & 2 & 5 & 7 & 1 & 3 \\ 5 & 7 & 1 & 3 & 4 & 6 & 0 & 2 \end{matrix}$$

$$A'_{(2,0)} = D_2'^T = \begin{matrix} 0 & 2 & 3 & 1 & 4 & 6 & 7 & 5 \\ 4 & 6 & 7 & 5 & 0 & 2 & 3 & 1 \\ 1 & 3 & 0 & 2 & 5 & 7 & 4 & 6 \\ 5 & 7 & 4 & 6 & 1 & 3 & 0 & 2 \end{matrix}$$

$$A = [[D_3^T]^{[2]} A_{(2,0)}]\,, \qquad A' = [[D_3'^T]^{[2]}[A'_{(2,0)}]]$$

$G = T_1^{**} D_3'^T D_3'^T D_2'^T$

$D_3 \quad (L)_3^3$

D_3 where L is a Latin square of order 8×8

D_2 .

PROPOSITION 1. *Each of the* D_p *of matrices* $p = 1, \cdots, m-1$ *and* D_m *defined in Steps* 1 *and* 2 *of the construction can be used to construct Latin squares of order*

s^m which can be split into s groups of s^{m-1} columns in such a way that every row in each group is a flat in $EG(m, s)$ construction:

$$L_p = [D_p, \eta D_p, \cdots, \eta^{s-1} D_p] \qquad p = 1, \cdots, m-1$$
$$L_m = [D_m, ED_m, \cdots, E^{s-1} D_m] .$$

Here $\eta^i D_p$, $i = 0, \cdots, s-1$ means that the matrices $T_p{}^{**}\eta^j$, $j = 0, \cdots, s-1$ forming the matrix D_p are premultiplied by η^i. Similarily the matrices $L_j \xi^{j-1}$, $j = 1, \cdots, s^{m-1}$ are premultiplied by E^i, $i = 0, \cdots, s^{-1}$.

We shall presently use the construction which led to Theorem 1 and Proposition 1 to establish the existence of two more classes of GYD.

First we define m new matrices M_p, $p = 1, \cdots, m-1$ and M_m obtained from L_p and L_m respectively by deleting the columns of D_p and D_m respectively. These matrices have the form M_p, M_m. Let

$$M_p = [\eta D_p, \cdots, \eta^{s-1} D_p] , \qquad p = 1, 2, \cdots, m-1 .$$
$$M_m = [ED_m, \cdots, E^{s-1} D_m] .$$

It is easy to see that each row of M_p and M_m is a complement of a $(m-1)$ flat in $EG(m, s)$ with respect to the whole space $EG(m, s)$. Hence the M matrices are of order $s^m \times (s^m - s^{m-1})$. Moreover from the definition of the D matrices it follows that each complement will be repeated s^{m-p} times in M_p.

Let U_1 be the matrix obtained from M_1 by deleting the first s^{m-1} rows. Then U_1 is a $(s^m - s^{m-1}) \times (s^m - s^{m-1})$ matrix in which the complement of each flat of G_1 is repeated $(s-1)s^{m-2}$ times in the corresponding row. Analogously to Step 3 we construct the M_q' matrices $q = 1, \cdots, m$.

We now state the following theorem.

THEOREM 2. *There exists* GYD *with parameters $v = s^m$, $m \geqq 2$,*

$$b = t_1(s-1)s^{m-1}(s^m - 1)/(s-1) , \qquad k = (t_2 s - 1)s^{m-1}(s^m - 1)/(s-1) ,$$

where $t_1, t_2 = 1, 2, 3, \cdots$.

CONSTRUCTION. The desired GYD is

$$G = \begin{bmatrix} U_1 & B' \\ B^\tau & (L)_\beta{}^\alpha \end{bmatrix}$$

where

$$B = [[M_1{}^T]^{[t_2-1]} [M_m{}^T]^{[(t_2 s-1)s^{m-2}]} [M_{m-1}^T]^{[(t_2 s-1)s^{m-3}]} \cdots [M_2{}^T]^{[t_2 s-1]}]$$
$$B' = [[M_1{}'^T]^{[t_1-1]} [M_m{}'^T]^{[(t_1 s-1)s^{m-2}]} [M_{m-1}'^T]^{[(t_1 s-1)s^{m-3}]} \cdots [M_2{}'^T]^{[t_1 s-1]}]$$
$$\alpha = (t_1 s - 1)(s^{m-2} + s^{m-3} + \cdots + s + 1) + t_1 - 1$$
$$\beta = (t_2 s - 1)(s^{m-2} + s^{m-3} + \cdots + s + 1) + t_2 - 1 .$$

Combining the construction of Theorem 1 and Theorem 2 we get

THEOREM 3. *There exists* GYD *with parameters $v = s^m$, $m \geqq 2$, $b_2 = (t_1 s - 1)s^{m-1}(s^m - 1)/(s-1)$*

$$b_1 = (t_2 s + 1)(s^m - 1)s^{m-1}/(s-1) , \qquad t_1 = 1, 2, 3, \cdots, t_2 = 0, 1, 2, 3, \cdots .$$

PROOF. Let $\bar{T}_1 = [T_1^{**}(\eta T_1^{**})(\eta^2 T_1^{**}) \cdots (\eta^{s-2} T_1^{**})]$.
The following design G is a desired design:

$$G = \begin{bmatrix} \bar{T}_1 & A'' \\ (B'')^T & (L)_\beta{}^\alpha \end{bmatrix}$$

where

$$\begin{aligned} A'' &= [(D_1')^T]^{[t_1-1]}\,[(D_m')^T]^{[(t_1s-1)s^{m-2}]}\,A'_{(m-1)}(t_1 s - 1) \cdots A'_{(2)}(t_1 s - 1) \\ B'' &= [M_1{}^T]^{[t_2]}\,[M_m{}^T]^{[(t_2s+1)s^{m-2}]}\,[M^T_{m-1}]^{[(t_2s+1)s^{m-3}]} \cdots \\ &\quad [M_p{}^T]^{[(t_2s+1)s^{p-2}]} \cdots [M_2{}^T]^{[(t_2s+1)]} \end{aligned}$$

where

$$\alpha = [t_1 s - 1)(s^{m-2} + s^{m-3} + \cdots + 1) + t_1 - 1$$

$$\beta = [t_2 s + 1)(s^{m-2} + s^{m-3} + \cdots + s + 1) + t_2 .$$

$$A'_{(i)}(t) = [(D_i')^T]^{[ts^{1-2}]} , \qquad i = 2, 3, \cdots, m-1 .$$

Therefore

$$A'_{(i)}(t_1 s - 1) = [(D_i')^T]^{[(t_is-1)s^{i-2}]} , \qquad i = 2, 3, \cdots, m-1 .$$

3. Discussion of other possibilities of construction of GYD. For $v = s^m$, s a power of a prime it is easy to construct designs which satisfy Proposition 6 of Kiefer's (1975) paper. To facilitate the reading of the paper we quote Proposition 6.

PROPOSITION 6 (Kiefer, 1975). *Assume* $v \mid c_1 c_2$. *Suppose* (i) *there are* BBD'*s* $B_i = (v, b_t, c_{3-i})$ *with* $v \mid b_i - c_i$ *and* (ii) *there are* c_2 *blocks of* B_2, *whose union contains each variety exactly* t *times, and such that* (iii) *the* $c_1 \times c_2$ *array formed with these blocks as columns has rows which are blocks of* B_1. *Then there is a* $b_1 \times b_2$ GYD. *Moreover,* (iii) *is satisfied if* (ii) *is satisfied and* (iv) B_1 is *composed of all the blocks of at least* c_1 BBD'*s*; *and* (ii) *and* (iii) *are satisfied if, in addition to* (iv), B_2 *is composed of all the blocks of at least* c_2 BBD'*s*.

The s^{m-k} column (s^k rows) of the chosen pencils belong to a BIB design found by all k (all $(m-k)$) dimensional flats in $EG(m, s)$. The number of blocks in these BIB designs is respectively $s^{m-k}\varphi(m-1, k-1, s)$ and $s^k\varphi(m-1, m-k-1, s)$ where $\varphi(m-1, k-1, s)$ and $\varphi(m-1, m-k-1, s)$ denote the numbers of $(k-1)$ dimensional flats and $(m-k-1)$ dimensional flats in the $PG(m-1, s)$ respectively. It remains to be shown that one can choose two numbers, say, x and y such that $[x\varphi(m-1, k-1, s) - 1]$ and $[y\varphi(m-1, m-k-1, s) - 1]$ are divisible by s^k and s^{m-k} respectively. However, it is clear that this can always be achieved since $\varphi(N, l, s)$ and s are relatively prime for any integers $N > l$. Filling the empty space with a suitable number of Latin squares of order s^m will complete the construction. Clearly one has to arrange the points in the flats to get BBD's in both directions but this can be easily carried out either using known theorems on systems of distinct representatives or making use of the permutations which were applied for the constructions in this paper.

We wish to remark however that the price which has to be paid for the simplicity of construction is that the number of experimental units required will increase considerably. Using this method for $s = 2$, $m = 3$ with $k = 2$ and $m - k = 1$ we would get a GYD with the 28×42 units while Example 1 required 28×28 units.

In conclusion we wish to remark that the constructions described here yield the same result as obtained by Ruiz and Seiden (1974) for $v = s^{2m}$ when s^m is replaced by s. Clearly the method described here has the advantage that it is not limited to even powers. It should be mentioned however that applying the method described here to s^{2m} will result in a larger design if we do not make the substitution to which the method of Ruiz and Seiden could be applied.

The two constructions yield for $v = s^{2m}$ different GYD's with the same parameters. This raises again the question whether one should search for additional optimality criteria to distinguish designs which are undistinguishable by the present criteria although their structure seems to be essentially different.

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