PACIFIC JOURNAL OF MATHEMATICS
Vol. 41, No. 1, 1972

A COMBINATORIAL PROBLEM; STABILITY AND ORDER FOR MODELS AND THEORIES IN INFINITARY LANGUAGES

SAHARON SHELAH

Some infinite combinatorial problems of Erdös and Makkai are solved, and we use them to investigate the connection between unstability and the existence of ordered sets; we also prove the existence of indiscernible sets under suitable conditions.

0. **Introduction.** In § 1 we deal with combinatorial problems raised by Erdös and Makkai in [5] (they appear later in Erdös and Hajnal [3], [18] Problem 71).

Let us define: $P2(\lambda, \mu, \alpha)$ holds when for every set A of cardinality μ, and family S of subsets of A of cardinality λ, there are $a_k \in A$, $X_k \in S$ for $k < \alpha$, such that either $k, l < \alpha$ implies $a_k \in X_l \Leftrightarrow k < l$ or $k, l < \alpha$ implies $a_k \in X_l \Leftrightarrow l \leqq k$.

Erdös and Makkai proved in [5] that if $\lambda > \mu \geqq \aleph_0$, then $P2(\lambda, \mu, \omega)$ holds. Assuming G.C.H. for simlicity only, our theorems imply $P2(\aleph_{\beta+2}, \aleph_{\beta+1}, \aleph_\beta)$ holds for every β.

In § 2 we mainly generalize results on stability from Morley [9] and Shelah [12] to models, and theories of infinitary languages. We first deal with stable models. Let M be a model, L the first-order language associated with it, Δ a set of formulas of $L_{\lambda^+,\omega}$ (for any λ) each with finite number of free variables. We shall assume Δ is closed under some simple operations. M is (Δ, λ)-stable, if for each $A \subset |M|$, $|A| \leqq \lambda$, the elements of M realize over A no more than λ different Δ-types. Let $\lambda \in Od_\Delta(M)$ if there is $\varphi(\bar{x}, \bar{y}) \in \Delta$ and sequences $\bar{a}^k$, $k < \lambda$, of elements of M such that for every $k, l < \lambda$, $M \models \varphi[\bar{a}^k, \bar{a}^l]$ if and only if $k < l$.

By Theorem 2.1, if M is not (Δ, κ)-stable $\kappa^{|\Delta|} = \kappa$, $\kappa = \sum_{\mu<\lambda}(\kappa^\mu + 2^{2^\mu})$, then $\lambda \in Od_\Delta(M)$. Theorem 2.2 says that if M is (Δ, λ)-stable, $\lambda \notin Od_\Delta(M)$, $\|M\| > \lambda$, $A \subset |M|$, $|A| \leqq \lambda$, and the cofinality of λ is $> |\Delta|$, then in M there is an indiscernible set over A of cardinality $> \lambda$. This generalizes Theorem 4.6 of Morley [9] for models of totally transcendental theories.

A theory T, $T \subset L_{\lambda^+,\omega}$ for some λ, is (Δ, μ)-stable, if every model of T is (Δ, μ)-stable. By Theorem 2.4, if $T, \Delta \subset L_{\lambda^+,\omega}$ $|T| \leqq \lambda$, and $\mu(\lambda) \in Od_\Delta(M)$ for some model M of T, *then* for every κ, T is not (Δ, κ)-stable. This is a converse of Theorem 2.1. (Morley [9] proved a particular case of this theorem (3.9) that if T is a first-order, counta-