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SOME CONTENT MAXIMIZING PROPERTIES OF THE REGULAR SIMPLEX

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In this paper it is shown that the regular simplex maximizes the sum of the squared contents of all i-dimensional faces, for all $i = 2, \cdots, n$, when the sum of the one-dimensional squared contents is fixed. An immediate corollary is that the regular simplex has the largest total length of all joining lines, total area of all triangles, total volume of all tetrahedra, and so forth, for a fixed sum of squared line lengths. Some related unsolved conjectures are presented.

If a set of $n+1$ points in n-dimensional Euclidean space do not lie in an $(n-1)$-dimensional hyperplane, their convex hull is a simplex, the simplest of all n-dimensional polytopes. Any two points in the set determine a one-dimensional edge, or line, any three points determine a two-dimensional face, or triangle, and so forth. If all one-dimensional edges have the same length, the simplex is called regular. If for no other reason than pure symmetry, the regular simplex should seem to be the extremum in any number of constrained maximization problems involving simplices. In this paper we consider a class of problems for which it is indeed maximal.

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1. A symmetrization. Let S be the simplex in E^n with vertices $A_1, \cdots, A_{n+1}$ and let $V(i, j) = a_{ij} = |A_i - A_j|$ be a typical one-dimensional edge length. More generally, let $V(i_1, \cdots, i_s)$ be the content of the subsimplex formed by $A_{i_1}, \cdots, A_{i_s}$.

THEOREM. *For $\sum a_{ij}^2 > 0$ fixed, the content power sum*

$$\sum V^\lambda(i_1, \cdots, i_s) \quad 0 < \lambda \leqq 2 ,$$

taken over all s-tuples $i_1 < \cdots < i_s$, is maximal if and only if S is a regular simplex.

The proof is based on a symmetrization which can be applied to any nonregular simplex to increase each of the sums while maintaining the constraint. We will give two equivalent constructions for the symmetrization: The first gives insight into its general nature, whereas the second is more immediately useful to the proof.

Given $A_1, \cdots, A_{n+1}$, let J be the matrix whose (i, j)th component is $A_i \cdot A_j$, a vector inner product. If the simplex is not regular, we