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The Principle of Symmetric Criticality

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Abstract. It is frequently explicitly or implicitly assumed that if a variational principle is invariant under some symmetry group **G**, then to test whether a symmetric field configuration φ is an extremal, it suffices to check the vanishing of the first variation of the action corresponding to variations $\varphi + \delta\varphi$ that are also symmetric. We show by example that this is not valid in complete generality (and in certain cases its meaning may not even be clear), and on the other hand prove some theorems which validate its use under fairly general circumstances (in particular if **G** is a group of Riemannian isometries, or if it is compact, or with some restrictions if it is semi-simple).

0. Introduction

What we call the Principle of Symmetric Criticality, abbreviated herein to "the Principle", states in brief that critical symmetric points are symmetric critical points.

In more detail, let **M** be a smooth (i.e., C^∞) manifold on which a group **G** acts by diffeomorphisms (a "smooth **G**-manifold") and let $f:\mathbf{M}\to\mathbf{R}$ be a smooth **G**-invariant function on **M** (that is, f is constant on the orbits of **G**). Then a *critical point* (of f) is a point p of **M** where df_p, the differential of f at p vanishes. And a *symmetric point* (of **M**) is an element of the set $\Sigma = \{p\in\mathbf{M} | gp = p \text{ for all } g\in\mathbf{G}\}$ of points fixed under the action of **G**. The Principle states that *in order for a symmetric point p to be a critical point it suffices that it be a critical point of* $f|\Sigma$, *the restriction of f to* Σ; in other words if the directional derivatives $df_p(X)$ vanish for all directions X at p tangent to Σ, then the Principle claims that directional derivatives in directions transverse to Σ also vanish. In particular, for example, an isolated point of Σ (where there are no directions tangent to Σ) should automatically be a critical point of f.

In this generality the Principle unfortunately is *not* valid, as we shall illustrate below by a number of examples. (We shall in particular give examples where Σ is