PROXIMITY ON FUNCTION SPACES

By

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Introduction

It has been 30 years since V. A. Efremovič [1] introduced the concept of proximity in 1951. A lot of works have been done and the theory is being perfected. But it is a pity that there is no paper on various proximity on function space.

In this paper we first consider the proximity characterizing the point-wise convergence on function space. Then we investigate the proximity characterizing the uniform convergence on a function space and a condition under which the proximity becomes joint proximally continuous. Last, we study the proximity characterizing the uniform convergence of the function space on a family of subsets and its related properties.

1. Proximity of point-wise convergence

For the definitions and notations used in this section are see to [2]. Let X be a set, $(Y, \mathcal{I})$ be a proximity space, and $F \subset Y^X$. Let

$$A(x) = \{f(x) : f \in A\} \quad \text{for each} \quad A \subset F, \ x \in X.$$

DEFINITION 1.1. The relative proximity $\underset{x \in X}{\times} \mathcal{I}_x | F$ with respect to F is called *proximity of point-wise convergence on F*, denote by ρ-*proximity on F* or $\mathcal{I}^X | F$ for short, where $\underset{x \in X}{\times} \mathcal{I}_x$ is the product proximity of the family of proximity spaces $\{(Y_x, \mathcal{I}_x) : x \in X\}$, and $Y_x = Y$, $\mathcal{I}_x = \mathcal{I}$, for each $x \in X$.

By Definition 1.1, we have the following:

PROPOSITION 1.1. *Let $A, B \subset F$. Then $A \mathcal{I}^X | F B$ iff for any finite decomposition of A and B:*

$$A = \bigcup_{i=1}^{p} A_i, \qquad B = \bigcup_{j=1}^{q} B_j,$$

there are i and j such that $A_i(x) \mathcal{I} B_j(x)$ for each $x \in X$.

PROPOSITION 1.2. *Suppose that the evaluation map $e_x : F \to Y$ is defined by*

$e_x(f)=f(x)(f\in F)$ for each $x\in X$, then proximity $\mathcal{J}^X|F$ on F is the coarsest proximity on F, which makes every evaluation map e_x proximally continuous for each $x\in X$, in other words, $\mathcal{J}^X|F=\mathrm{Sup}\{e_x^{-1}(\mathcal{J}): x\in X\}$.

It is known that the topology of a product proximity space is equal to the product of the topology of each coordinate space, and the product topology is the topology of point-wise convergence. Besides, the convergence with respect to the proximity is the convergence with of the topology induced by the same proximity. Thus we have:

THEOREM 1.1.   The net $\{f_\alpha: \alpha\in D\}\mathcal{J}^X|F$-converges to $f\in F$ in a proximity space $(F, \mathcal{J}^X|F)$ iff the net $\{f_\alpha(x): \alpha\in D\}$ $\mathcal{J}$-converges to $f(x)$ in Y for each $x\in X$.

It is known that a proximity space $(Y, \mathcal{J})$ is compact iff the topological space $(Y, \mathcal{T}_\mathcal{J})$ induced by it is compact. By Tychonoff's Theorem, we have the following immediately.

THEOREM 1.2.   Let $(Y, \mathcal{J})$ be a compact proximity space, then $(F, \mathcal{J}^X|F)$ is a compact proximity space iff

(a) $f\in Y^X\backslash F$ implies $\{f\}\overline{\mathcal{J}^X}F$.

If $(Y, \mathcal{J})$ is separated, then the condition (a) is also a necessary condition under which $(F, \mathcal{J}^X|F)$ is a compact proximity space.

PROOF.   The fact that $(Y, \mathcal{J})$ is a compact proximity space implies that $(Y, \mathcal{T}_\mathcal{J})$ is a compact topological space. By Tychonoff's Theorem $(Y^X, \mathcal{T}_\mathcal{J}^X)$ is a compact topological space. The condition (a) shows that F is a $\mathcal{T}_\mathcal{J}x$-closed set of Y^X, and $\mathcal{T}_\mathcal{J}^X=\mathcal{T}_\mathcal{J}x$, thus $(F, \mathcal{T}_\mathcal{J}^X|F)$ is a compact topological space and so is $(F, \mathcal{T}_\mathcal{J}x_{|F})$, which means that $(F, \mathcal{J}^X|F)$ is a compact proximity space.

If $(Y, \mathcal{J})$ is separated, then $(Y^X, \mathcal{J}^X)$ is separated, and hence the topology $\mathcal{T}_\mathcal{J}x$ is T_2. Thus the condition (a) is the necessary condition under which $(F, \mathcal{J}^X|F)$ is a compact space.

The condition under which $(Y, \mathcal{J})$ is a compact space may be weakened.

THEOREM 1.3.   Let $(Y, \mathcal{J})$ be a proximity space, then the sufficient condition under which $(F, \mathcal{J}^X|F)$ is a compact proximity space is

(a) $f\in Y^X\backslash F$ implies $\{f\}\overline{\mathcal{J}^X}F$.

(b) $F(x)$ has a compact closure for each $x\in X$ in the topology $\mathcal{T}_\mathcal{J}$ in Y.

Furthermore if $(Y, \mathcal{J})$ is separated, then (a) and (b) are necessary conditions

under which $(F, \mathcal{J}^X | F)$ is a compact proximity space.

PROOF. Suppose that $\mathfrak{x}$ is a $\mathcal{J}^X | F$-compressed filter base in F. Then $e_x(\mathfrak{x})$ is a $\mathcal{J}$-compressed filter base in Y for each $x \in X$. By (b), $F(x)^-$ (the closure of the set $F(x)$ in the $\mathcal{T}_\mathcal{J}$ in Y) is $\mathcal{T}_\mathcal{J}$-compact, and hence $\mathcal{J}$-compact. Thus $e_x(\mathfrak{x})$ is a $\mathcal{J}$-compressed filter base in $F(x)^-$, so there is a $y \in F(x)^-$ such that $e_x(\mathfrak{x})$ $\mathcal{J}$-converges to y denoted by $f(x)$. Thus we get an $f \in Y^X$, and $\mathfrak{x}$ point-wisely converges to f, and hence $\mathfrak{x}$ $\mathcal{J}^X$-converges to f. By condition (a), $f \in F$, and hence $\mathfrak{x}$ $\mathcal{J}^X | F$-converges to $f \in \mathfrak{x}$, that is $(F, \mathcal{J}^X | F)$ is a compact proximity space.

For each $x \in X$, since $(F, \mathcal{J}^X | F)$ is a compact proximity space, $F(x) = e_x(F)$ is $\mathcal{T}_\mathcal{J}$-compact. Furthermore, since $(Y, \mathcal{J})$ is separated, $(Y, \mathcal{T}_\mathcal{J})$ is a T_2-space. Thus $F(x)$ is $\mathcal{T}_\mathcal{J}$-closed, that is $F(x)^- = F(x)$ is $\mathcal{T}_\mathcal{J}$-compact.

2. Joint proximal continuity

DEFINITION 2.1. Let $(X, \mathcal{P})$ and $(Y, \mathcal{J})$ be proximity spaces, $F \subset Y^X$, and $\mathcal{R}$ be a proximity on F. If $\mathcal{R}$ makes a function $P: F \times X \to Y$, defined by $P(f, x) = f(x)$, $(\mathcal{R} \times \mathcal{P}, \mathcal{J})$-proximally continuous, then we call $\mathcal{R}$ a joint proximally continuous proximity on F, denoted by J. P. C.-proximity for short. And we denote the family of all $(\mathcal{P}, \mathcal{J})$-proximally continuous functions by $C(X, Y)$.

PROPOSITION 2.1. *If there is a J. P. C.-proximity on F, then $F \subset C(X, Y)$.*

PROOF. Suppose that $\mathcal{R}$ is a J. P. C.-proximity on F. For each $f \in F$, $A \subset X$, $B \subset X$ if $A \mathcal{P} B$, then

$$(f \times A) \mathcal{R} \times \mathcal{P} (f \times B) \tag{2.1}$$

In fact, for any finite decomposition of $(f \times A)$ and $(f \times B)$

$$f \times A = \bigcup_{i=1}^{n} f \times A_i \qquad f \times B = \bigcup_{j=1}^{m} f \times B_j,$$

accordingly, we have a finite decomposition of A, B

$$A = \bigcup_{i=1}^{n} A_i, \qquad B = \bigcup_{j=1}^{m} B_j.$$

Thus there are i and j such that $A_i \mathcal{P} B_j$. Furthermore we have

$$p_1(f \times A_i) = \{f\}, \qquad p_1(f \times B_j) = \{f\},$$

and hence $p_1(f \times A_i) \mathcal{R} p_1(f \times B_j)$. Note that

$$p_2(f \times A_i) = A_i, \qquad p_2(f \times B_j) = B_j,$$

thus $p_2(f \times A_i) \mathcal{P} p_2(f \times B_j)$. ($p_1$ and p_2 represent the projections on F and on X respectively). And by the definition of product proximity (2.1) holds.

Furthermore we have $P(f \times A) = f(A)$, $P(f \times B) = f(B)$. With the fact that the map P is $(\mathcal{R} \times \mathcal{P}, \mathcal{J})$-proximally continuous, it follows that $f(A) \mathcal{J} f(B)$, which means, f is a $(\mathcal{P}, \mathcal{J})$-proximally continuous map, $f \in C(X, Y)$.

For further discussion of joint proximal continuity, we first obtain a sufficient and necessary condition of proximal continuity, then give the concept of the family of equiproximally continuous functions (The terms and notations used in the following can be seen in [3]).

THEOREM 2.1. *Suppose that* $(X, \mathcal{P})$ *and* $(Y, \mathcal{J})$ *are proximity spaces.* $f : X \to Y$. *Then* f *is* $(\mathcal{P}, \mathcal{J})$-*proximally continuous iff for each* $\mathcal{J}$-*uniform cover* $\mathcal{A} = \{A_i : i = 1, 2, \cdots, n\}$ *on* Y, *there is a* $\mathcal{P}$-*uniform cover* $\mathcal{B} = \{B_j : j = 1, 2, \cdots, m\}$ *on* X *such that for each* B_j, *there is an* A_i *satisfying* $f(B_j) \subset A_i$.

PROOF. Suppose that $\mathcal{V}_{\mathcal{P}}$ on X and $\mathcal{V}_{\mathcal{J}}$ on Y stand for totally bounded uniformities which are compatible with $\mathcal{P}$ and $\mathcal{J}$, respectively.

Necessity: Suppose that $\mathcal{A}' = \{A'_k : k = 1, 2, \cdots, l\}$ is a star-refinement $\mathcal{J}$-uniform cover of $\mathcal{A}$, then $V' = \bigcup_{k=1}^{l} A'_k \times A'_k \in \mathcal{V}_{\mathcal{J}}$. The fact that f is $(\mathcal{P}, \mathcal{J})$-proximally continuous implies that f is $(\mathcal{V}_{\mathcal{P}}, \mathcal{V}_{\mathcal{J}})$-uniformly continuous. Then there is a $\mathcal{P}$-uniform cover $\mathcal{B} = \{B_j : j = 1, 2, \cdots, m\}$ on X. Suppose that $f(x_0) \in A'_{k_0}$ for a fixed $x_0 \in B_j$, then $(f(x_0), f(x)) \in V'$ for each $x \in B_j$. Thus there is an index k such that $(f(x_0), f(x)) \in A'_k \times A'_k$, which means $f(x_0) \in A'_{k_0} \times A'_k$, and hence $f(x) \in (A'_{k0})^*$ and $\mathcal{A}'$ is a star-refienment of $\mathcal{A}$, and there is an $A_i \in \mathcal{A}$ such that $(A'_{k_0})^* \subset A_i$, thus $f(B_j) \subset A_i$.

Sufficiency: Suppose C, $D \subset X$ and $C \mathcal{P} D$. By the hypothesis, for any $\mathcal{J}$-uniform cover $\mathcal{A} = \{A_i : i = 1, 2, \cdots, n\}$, on Y, there is a $\mathcal{P}$-uniform cover $\mathcal{B} = \{B_j : j = 1, 2, \cdots, m\}$ on X and there is an A_i such that $f(B_j) \subset A_i$ for each B_j. Since $C \mathcal{P} D$, there is a B_j such that $C \cap B_j \neq \emptyset$ and $D \cap B_j \neq \emptyset$. Therefore, $f(C) \cap A_i \neq \emptyset$ and $f(D) \cap A_i \neq \emptyset$. This means $f(C) \mathcal{J} f(D)$, and hence f is a $(\mathcal{P}, \mathcal{J})$-proximally continuous map.

DEFINITION 2.2. Suppose that $(X, \mathcal{P})$ and $(Y, \mathcal{J})$ are proximity spaces, $F \subset Y^X$. If for each $\mathcal{J}$-uniform cover $\mathcal{A} = \{A_i : i = 1, 2, \cdots, n\}$ on Y, there is a $\mathcal{P}$-uniform cover $\mathcal{B} = \{B_j : j = 1, 2, \cdots, m\}$ on X such that for each $f \in F$, $B_j \in \mathcal{B}$ there is an $A_i \in \mathcal{A}$ such that $f(B_j) \subset A_i$, then F is called a *family of* $(\mathcal{P}, \mathcal{J})$-*equiproximally continuous functions*.

We obtain the following from Theorem 2.1 and Definition 2.2.

PROPOSITION 2.2. *If* F *is a family of* $(\mathcal{P}, \mathcal{J})$-*equiproximally continuous functions, then* $F \subset C(X, Y)$.

3. Uniformly convergent proximity

Let X be a set, $(Y, \mathcal{V})$ be a uniform space, and $F \subset Y^X$. For $V \in \mathcal{V}$, let

$$W(V) = \{(f, g) \in F \times F : (f(x), g(x)) \in V, \text{ for each } x \in X\}$$

We have known that the uniformity with $\{W(V) : V \in \mathcal{V}\}$ as a base is called the uniformity of uniform convergence, denoted by u. c.-uniformity on F.

DEFINITION 3.1. The proximity on F generated by a u. c.-uniformity on F is called the *uniformly convergent proximity*, denoted by u. c.-proximity, or is called u. c.-proximity induced by $\mathcal{V}$.

PROPOSITION 3.1. *Suppose that $\mathcal{R}$ is a u. c.-proximity on F, $A \subset F$ and $B \subset F$, then $A\mathcal{R}B$ iff for each $V \in \mathcal{V}$, there are $f \in A$ and $g \in B$ such that $(f(x), g(x)) \in V$ for each $x \in X$.*

THEOREM 3.1. *Let $\mathcal{R}$ be a u. c.-proximity on F, then the net $\{f_\alpha : \alpha \in D\}$ in $F\mathcal{R}$-converges to $f \in F$ iff the net $\mathcal{V}$-uniformly converges to f on X.*

PROOF. Suppose $G \subset F$ and $\{f\}\overline{\mathcal{R}}F\backslash G$. Thus there is a $V \in \mathcal{V}$ such that for each $g \in F\backslash G$ there is an $x \in X$ such that

$$(f(x), g(x)) \notin V \tag{3.1}$$

We choose a symmetric element $U \in \mathcal{V}$ such that $U \circ U \subset V$. Since the net $\{f_\alpha : \alpha \in D\}$ uniformly converges to f on X, there is an $\alpha_0 \in D$ such that

$$(f(x), f_\alpha(x)) \in U \quad \text{for each} \quad \alpha \geq \alpha_0, \ x \in X. \tag{3.2}$$

If $\{f_\alpha : \alpha \geq \alpha_0, \alpha \in D\}\mathcal{R}F\backslash G$, then there is an α, $\alpha \geq \alpha_0$ and $\alpha \in D$, and a $g \in F\backslash G$, such that

$$(f_\alpha(x), g(x)) \in U \quad \text{for each} \quad x \in X. \tag{3.3}$$

By (3.2) and (3.3), $(f(x), g(x)) \in U \circ U \subset V$ for any $x \in X$, which contradicts (3.1), hence $\{f_\alpha : \alpha \geq \alpha_0, \alpha \in D\}\overline{\mathcal{R}}F\backslash G$. Which means the net $\{f_\alpha : \alpha \in D\}\mathcal{R}$-converges to f.

Let $G = W(V)[f]$ for each symmetric element $V \in \mathcal{V}$, then $\{f\}\overline{\mathcal{R}}F\backslash G$. Otherwise, there must be a $g \notin G$ such that $(f(x), g(x)) \in V$ for each $x \in X$, that means $g \in W(V)[f]$, a desired contradiction.

Since the net $\{f_\alpha : \alpha \in D\}$ $\mathcal{R}$-converges to f, there is an $\alpha_0 \in D$ such that

$$\{f_\alpha : \alpha \geq \alpha_0, \alpha \in D\}\overline{\mathcal{R}}F\backslash G.$$

Hence $f_\alpha \in G$ for each $\alpha \geq \alpha_0$, which implies $(f(x), f_\alpha(x)) \in V$ for each $x \in X$. This means $\{f_\alpha : \alpha \in D\}$ $\mathcal{V}$-uniformly converges to an f on X.

THEOREM 3.2. *Let* $(X, \mathscr{P})$ *and* $(Y, \mathscr{J})$ *be proximity spaces and* F *be a family of* $(\mathscr{P}, \mathscr{J})$-*equiproximally continuous functions. Then the u. c.-proximity* $\mathscr{R}$ *on* F *induced by* $\mathcal{V}_{\mathscr{J}}$, *a totally bounded uniformity and is compatible with* $\mathscr{J}$, *is J. P. C.-proximity.*

PROOF. Take arbitrary $C \subset F \times X$, $D \subset F \times X$, such that $C \mathscr{R} \times \mathscr{P} D$, and a $\mathscr{J}$-uniform cover $\mathcal{A} = \{A_i : i=1, 2, \cdots, n\}$ on Y, we have a $\mathscr{J}$-uniform cover $\mathcal{A}' = \{A'_i : i=1, 2, \cdots, n'\}$ on Y, which is a star-refinement of $\mathcal{A}$. By equicontinuity of F, there is a $\mathscr{P}$-uniform cover $\mathscr{B} = \{B_j : j=1, 2, \cdots, m\}$ on X such that for each $f \in F$, $B_j \in \mathscr{B}$, there is an $A'_i \in \mathcal{A}'$ such that $f(B_j) \subset A'_i$. We take another $\mathscr{P}$-uniform cover $\mathscr{B}'$ on X, as a star-refinement of $\mathscr{B}$.

Let $F_{ij} = \{f \in F : f(B'_j) \subset A'_i\}$,

$C_{ij} = C \cap (F_{ij} \times B'_j)$,

$D_{ij} = D \cap (F_{ij} \times B'_j)$.

We obtain finite decompositions of C and D: $C = \cup C_{ij}$, $D = \cup D_{ij}$. There are (i, j) and (i', j'), such that $p_1(C_{ij}) \mathscr{R} p_1(D_{i'j'})$ and $p_2(C_{ij}) \mathscr{P} p_2(D_{i'j'})$. Hence there are $f \in p_1(C_{ij})$ and $g \in p_1(D_{i'j'})$. For each $x \in X$ there is an e such that $(f(x), g(x)) \in A'_e \times A'_e$, and there is a B'_{j_0} such that $p_2(C_{ij}) \times p_2(D_{i'j'}) \cap B'_{j_0} \times B'_{j_0} \neq \varnothing$. Also, $p_2(C_{ij}) \subset B'_j$, $p_2(D_{i'j'}) \subset B'_{j'}$, and hence $B'_j \times B'_{j'} \cap B'_{j_0} \times B'_{j_0} \neq \varnothing$.

And becaus of $f \in p_1(C_{ij})$, there is an $x \in B'_j$ such that $(f, x) \in C_{ij}$. Similarly $g \in p_1(D_{i'j'})$, there is an $x' \in B'_{j'}$ such that $(g, x) \in D_{i'j'}$, and hence $x, x' \in (B'_{j_0})^* \subset B_j$. Thus $(f(x), f(x')) \in A'_i \times A'_i$, $(g(x), g(x')) \in A'_k \times A'_k$ and $(f(x), g(x)) \in A'_e \times A'_e$, and hence $(f(x), g(x')) \in (A'_e)^* \times (A'_e)^*$. Note that $(A'_e)^* \subset A_i$. Which means both $f(x)$ and $g(x')$ belong to A_i. We have known that $(f, x) \in C$, $(g, x') \in D$, so $P(C) \cap A_i \neq \varnothing$ and $P(D) \cap A_i \neq \varnothing$, which implies that $P(C) \mathscr{J} P(D)$.

THEOREM 3.3. *Let* $(X, \mathscr{P})$ *be a proximity space,* $(Y, \mathcal{V})$ *be a uniform space,* Y^X *possess u. c.-proximity* $\mathscr{R}$ *induced by* $\mathcal{V}$ *and* $\mathscr{J}_{\mathcal{V}}$ *be a proximity on* Y *generated by* $\mathcal{V}$. *If a net of* $(\mathscr{P}, \mathscr{J}_{\mathcal{V}})$-*proximally continuous maps* $\{f_\alpha : \alpha \in D\}$ $\mathscr{R}$-*converges to* f *in* Y^X, *then* f *is a* $(\mathscr{T}, \mathscr{J}_{\mathcal{V}})$-*proximally continuous map.*

PROOF. If f is not a $(\mathscr{P}, \mathscr{J}_{\mathcal{V}})$-proximally continuous map, there is $A \subset X$, $B \subset X$, $A \mathscr{P} B$, but $f(A) \overline{\mathscr{J}_{\mathcal{V}}} f(B)$. Thus there is a symmetric element $V \in \mathcal{V}$ such that

$$(f(x),\ f(y)) \notin V \qquad \text{for each} \quad x \in A,\ y \in B \tag{3.4}$$

Choose a symmetric element $U \in \mathcal{V}$ such that $U \circ U \circ U \subset V$, since the net $\{f_\alpha : \alpha \in D\}$ $\mathcal{R}$-converges to f there is an $\alpha_0 \in D$ such that

$$(f(x),\ f_\alpha(x)) \in U \qquad \text{for each} \quad \alpha \geq \alpha_0,\ x \in X. \tag{3.5}$$

From the $(\mathcal{P},\ \mathcal{S}_\mathcal{V})$-proximal continuity of f_α, we know $f_\alpha(A)\mathcal{S}_\mathcal{V}f_\alpha(B)$, implying there are $a \in A$ and $b \in B$ such that

$$(f_\alpha(a),\ f_\alpha(b)) \in U. \tag{3.6}$$

By (3.5), we obtain

$$(f(a),\ f_\alpha(a)) \in U. \tag{3.7}$$

$$(f(b),\ f_\alpha(b)) \in U. \tag{3.8}$$

By (3.6), (3.7) and (3.8), we have

$$(f(a),\ f(b)) \in U \circ U \circ U \subset V. \tag{3.9}$$

where $a \in A$, $b \in B$, which contradicts (3.4).

COROLLARY. *Under the hypothesis of the theorem, the family $C(X,\ Y)$ of all $(\mathcal{P},\ \mathcal{S}_\mathcal{V})$-proximally continuous functions is a $\mathcal{T}_\mathcal{R}$-closed set in Y^X.*

4. Uniformly convergent proximity on a family of subsets

Let X be a set, $(Y,\ \mathcal{V})$ be a uniform space, $F \subset Y^X$ and $\mathcal{K}$ be a family of subsets of X closed for finite union, in other words, if $K_1 \in \mathcal{K}$, $K_2 \in \mathcal{K}$, then $K_1 \cup K_2 \in \mathcal{K}$.

THEOREM 4.1. *Suppose that S is a binary relation on the family of all subsets of F. If for each $A \subset F$, $B \subset F$, ASB iff for each $V \in \mathcal{V}$, $K \in \mathcal{K}$, there are $f \in A$ and $g \in B$ such that $(f(x),\ g(x)) \in V$ for each $x \in K$, then S is a proximity on F.*

PROOF. It is obvious that S satisfies (P_1)-(P_4) (cf. [2]). If $A\tilde{S}C$, and $B\tilde{S}C$, then there are $V_1 \in \mathcal{V}$ and $K_1 \in \mathcal{K}$ such that for each $f \in A$, $g \in C$, there is an $x_1 \in K_1$ such that $(f(x_1),\ g(x_1)) \notin V$, and there are $V_2 \in \mathcal{V}$ and $K_2 \in \mathcal{K}$ such that for each $\varphi \in B$, $g \in C$, there is an $x_2 \in K_2$ such that $(\varphi(x_2),\ g(x_2)) \notin V_2$.

Thus there are $V = V_1 \cap V_2 \in \mathcal{V}$ and $K = K_1 \cup K_2 \in \mathcal{K}$ such that there is an $x \in K_1 \cup K_2$ satisfying $(\psi(x),\ g(x)) \notin V$ for each $\psi \in A \cup B$, $g \in C$. So it satisfies (P_5).

If $A\tilde{S}B$, then there are $V \in \mathcal{V}$ and $K \in \mathcal{K}$, for each $f \in A$, $g \in B$, there is an $x \in K$ such that

$$(f(x),\ g(x))\overline{\in}V.\qquad\qquad\qquad\qquad (4.1)$$

Take a symmetric element $U\in \mathcal{CV}$, such that $U\circ U\subset V$. Let

$$P=\{f\in F:\ \text{there is a } g\in A,\ (f(x),\ g(x))\in U \text{ for each } x\in K\},$$

$$Q=\{f\in F:\ \text{there is a } g\in B,\ (f(x),\ g(x))\in U \text{ for each } x\in K\},$$

If $P\cap Q\neq\emptyset$, then there is $f\in P\cap Q$, $g\in A$, $g'\in B$ such that $(f(x),\ g(x))\in U$ and $(f(x),\ g'(x))\in U$ for each $x\in K$, thus $(g(x),\ g'(x))\in V$ for each $x\in K$. It is contrary to (4.1). So $P\cap Q=\emptyset$.

In addition, there are $U\in\mathcal{CV}$ and $K\in\mathcal{K}$ such that there must be an $x\in X$ such that $(g(x),f(x))\overline{\in}U$ for each $g\in A,\ f\in F\backslash P$. Hence $A\overline{S}F\backslash P$. Similarly, $B\overline{S}F\backslash Q$. Which means that (P_6) is satisfied and S is a proximity on F.

DEFINITION 4.1. The proximity on F defined by Theorem 4.1 is called the *uniformly convergent proximity on* $\mathcal{K}$ *induced by* $\mathcal{CV}$, denoted by $(\mathcal{K})$ u.c.-proximity.

In view of Definition 4.1, we obtain the following proposition immediately:

PROPOSITION 4.1. *If the* $(\mathcal{K}_1)$ *u. c.-proximity on* F *induced by* $\mathcal{CV}$ *is* S_1 *and* $(\mathcal{K}_2)$ *u. c.-proximity is* S_2, *then:*

(a) *If* $\mathcal{K}_1\subset\mathcal{K}_2$, *then* $S_1<S_2$,

(b) *If there is a* $K_0\in\mathcal{K}_2$ *such that* $K_0\supset\bigcup\limits_{K\in\mathcal{K}_1} K$, *then* $S_1<S_2$,

(c) *u. c.-proximity is more refined than any other* $(\mathcal{K})$ *u. c.-proximity,*

(d) *If* $\mathcal{K}_0$ *consists of all finite subsets of* X, *then* $(\mathcal{K}_0)$ *u. c.-proximity is the coarsest one of all* $(\mathcal{K})$ *u. c.-proximity satisfying* $\bigcup\limits_{K\in\mathcal{K}} K=X$. $((\mathcal{K}_0)$ *u. c.-proximity is called a point-wise convergent u. c.-proximity*).

THEOREM 4.2. *Suppose that the* $(\mathcal{K})$ *u. c.-proximity on* F *induced by* $\mathcal{CV}$ *is* S, *then a net* $\{f_\alpha:\ \alpha\in D\}$ *in* F *S-converges to* $f\in F$ *iff the net* $\mathcal{CV}$-*uniformly converges to* f *on* K *for each* $K\in\mathcal{K}$.

The concept of joint proximally continuous proximity may be generalized to the family $\mathcal{K}$ of subsets of X.

DEFINITION 4.2. Let $(X,\ \mathcal{P})$ and $(Y,\ \mathcal{J})$ be proximity spaces, $F\subset Y^X$ and $\mathcal{K}$ be a family of subsets of X which is closed for finite join operation. If there is a proximity $\mathcal{R}$ such that $P:\ F\times K\to Y$ $(P(f,\ x)=f(x))$ is a $(\mathcal{R}\times\mathcal{P}|K,\ \mathcal{J})$-proximally continuous map for each $K\in\mathcal{K}$, then $\mathcal{R}$ is called a *joint proximally continuous proximity on* $\mathcal{K}$ *on* F, denoted by $(\mathcal{K})$ J. P. C.-proximity for short.

Theorem 3.2 can be generalized as follows:

THEOREM 4.3.   Let $(X, \mathcal{P})$ and $(Y, \mathcal{I})$ be proximity spaces, F be a family of $(\mathcal{P}, \mathcal{I})$-equiproximally continous functions and $\mathcal{K}$ be a family of subsets which is closed for finite join operation.   Then the $(\mathcal{K})$ u. c.-proximity $\mathcal{R}$ on F induced by $\mathcal{V}_\mathcal{I}$ is a $(\mathcal{K})$ J. P. C.-proximity.

The proofs of above two theorems are omitted.

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