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ON CONFORMAL DIFFEOMORPHISMS BETWEEN PRODUCT RIEMANNIAN MANIFOLDS

By

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0. Introduction

Let M and M^* be connected Riemannian manifolds of dimension $n\geqq 3$, and denote the product Riemannian structures by (M, g, F) and (M^*, g^*, G) respectively, where g and g^* are the Riemannian metrics and F and G the product structures of M and M^*. Under a diffeomorphism f of M to M^*, the image of a quntity on M^* to M by the induced map f^* of f will be denoted by the same character as the original. For example, we write g^* for f^*g^* and G for f^*G on M. We say that the product structures F and G are *commutative* with one another at a point P of M under f if $FG=GF$ at P.

In the present paper, a conformal diffeomorphism means a non-homothetic one unless otherwise stated. The purpose is to prove the following

THEOREM 1. *If both M and M^* are complete product Riemannian manifolds, then there is no global conformal diffeomorphism of M onto M^* such that the product structures F and G are not commutative under it in an open subset of M.*

This is an improvement of the main theorem in a previous paper [4] with weaker condition "in a open subset" than "in a dense subset" of the previous. As the contraposition of Theorem 1, we can state the following

THEOREM 2. *If both M and M^* are complete product Riemannian manifolds and there is a global conformal diffeomorphism f of M onto M^*, then the product structures F and G are commutative under f everywhere in M.*

An affirmative example of Theorem 2 was given in [4].

To prove Theorem 1, we first assume that there is an open subset where the product structures F and G are not commutative under a conformal diffeomorphism f of M into M^*. Then we obtain differential equations on the associated scalar field ρ with f. Three considerable cases occur, and we obtain the expression of ρ in each case. Comparison of arc-lengths of some geodesic

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in M and its image in M^* shows the non-existence of global conformal diffeomorphism between complete product manifolds.

1. Preliminaries

We shall recall lemmas and differential equations from [4] as preliminaries. Throughout the present paper we assume that differentiability of manifolds and diffeomorphisms is of class C^∞. For indicating components of tensors, Greek indices $\kappa, \lambda, \mu, \nu, \omega$ run on the range from 1 to n, and other Greek indices run on indicated temporary ranges.

Let M be the product $M_1 \times M_2$ of two Riemannian manifolds M_1 and M_2 of dimension n_1 and n_2 respectively, $n_1+n_2=n$, and (x^h, y^p) a separate coordinate system of M, (x^h) belonging to M_1 and (y^p) to M_2. Here and hereafter Latin indices always run on the following ranges:

$$h, i, j, k=1, 2, \cdots, n_1,$$
$$p, q, r, s=n_1+1, \cdots, n.$$

With respect to a separate coordinate system (x^h, y^p) in M, the metric tensor $g=(g_{\mu\lambda})$ of M has pure components $g_{ji}(x^h)$ and $g_{qp}(y^p)$ only, depending on the coordinates (x^h) and (y^p) respectively, and the product structure $F=(F_\lambda{}^\kappa)$ has pure components $F_i{}^h=\delta_i^h$ and $F_q{}^p=-\delta_q^p$. Covaraint differentiation with respect to g in M will be denoted by ∇, and the parts along M_1 and M_2, expressed by ∇_i and ∇_q respectively, are commutative with one another.

A conformal diffeomorphism f of M to M^* is characterized by a change

$$g^*_{\mu\lambda}=\frac{1}{\rho^2}g_{\mu\lambda} \tag{1.1}$$

of the metric tensors, where ρ is a positive-valued scalar field on M and said to be *associated* with f. We shall put $\rho_\lambda=\nabla_\lambda\rho$ and denote by Y the gradient vector field (ρ^κ) of ρ. The parts (ρ^h) and (ρ^p) of Y belonging to M_1 and M_2 will be denoted by Y_1 and Y_2 respectively, and the squared length of Y by Φ, i.e.,

$$\Phi=|Y|^2=\rho_\kappa\rho^\kappa.$$

Under a conformal diffeomorphism f, the induced tensor G from M^* to M constitutes an almost product Riemannian structure (M, g, G), which is not necessarily integrable. The covariant tensor $G_{\mu\lambda}$ defined by $G_{\mu\lambda}=G_\mu{}^\kappa g_{\kappa\lambda}$ is symmetric in λ and μ. The product structures F and G are commutative if and only if $G_\lambda{}^\kappa$ and $G_{\mu\lambda}$ have pure components only with respect to a separate coordinate system in M.

If the metric g^* of M^* is conformally related to g of M by (1.1), then the integrability of the product structure G with respect to g^* in M^* is equivalent to the differential equation

$$(1.2)\qquad \nabla_\mu G_{\lambda\kappa} = -\frac{1}{\rho}(G_{\mu\lambda}\rho_\kappa + G_{\mu\kappa}\rho_\lambda - g_{\mu\lambda}G_{\kappa\omega}\rho^\omega - g_{\mu\kappa}G_{\lambda\omega}\rho^\omega)$$

on M. Starting from this equation, we proved the following lemmas of local character:

LEMMA 1. *A conformal diffeomorphism f of M into M^* is a homothety if and only if*

$$\nabla_\mu G_{\lambda\kappa} = 0.$$

Then the structures F and G are commutative under f.

LEMMA 2. *If the structures F and G are commutative under a conformal diffeomorphism f, then the associated scalar field ρ is a function on either of the parts M_1 or M_2 only.*

LEMMA 3. *If the associated scalar field ρ depends on one part, say M_1, but is not a constant, then the structure G is commutative with F under f, or the scalar field ρ satisfies the equation*

$$(1.3)\qquad \nabla_j \rho_i = c^2 \rho g_{ji}$$

on M_1, where c is a positive constant, and the squared length Φ of the gradient vector field $Y = Y_1$ is equal to

$$(1.4)\qquad \Phi = \rho_i \rho^i = c^2 \rho^2.$$

We put the subset

$$N_1 = (\mathrm{P} \mid Y_1(\mathrm{P}) = 0\},$$
$$N_2 = \{\mathrm{P} \mid Y_2(\mathrm{P}) = 0\},$$
$$U = \{\mathrm{P} \mid Y_1(\mathrm{P}) \neq 0,\ Y_2(\mathrm{P}) \neq 0\},$$
$$V = \{\mathrm{P} \mid FG \neq FG \text{ at } \mathrm{P}\}.$$

and see the inclusion relations

$$U \subset V \subset M - N_1 \cap N_2$$

by means of Lemmas 1 and 2.

Now we suppose that the open subset V is not empty, then by means of Lemma 3 we have to consider the cases where U is empty and $V \subset N_1 \cup N_2$ or where U is not empty.

By pretty long arguments, in every component of U, we obtain the following equations

$$(1.5)\qquad \begin{cases} \nabla_j\rho_i=\dfrac{1}{2\rho}[(\Phi+k\rho^2)g_{ji}+CG_{ji}], \\ \nabla_q\rho_j=\dfrac{C}{2\rho}G_{qi}, \\ \nabla_q\rho_p=\dfrac{1}{2\rho}[(\Phi-k\rho^2)g_{qp}+CG_{qp}], \end{cases}$$

k and C being constants, or

$$(1.6)\qquad \begin{cases} \nabla_j\nabla_i\rho^2=(\Phi+k\rho^2)g_{ji}+CG_{ji}+2\rho_j\rho_i, \\ \nabla_q\nabla_i\rho^2=CG_{qi}+2\rho_q\rho_i, \\ \nabla_q\nabla_p\rho^2=(\Phi-k\rho^2)g_{qp}+CG_{qp}+2\rho_q\rho_p. \end{cases}$$

The squared length Φ of Y is decomposable in U, that is, it is the sum

$$(1.7)\qquad \Phi=\rho_\kappa\rho^\kappa=\Phi_1+\Phi_2$$

of functions Φ_1 of (x^h) and Φ_2 of (y^p), and it satisfies the equations

$$(1.8)\qquad \begin{cases} \nabla_j\nabla_i(\Phi-k\rho^2)=\Omega g_{ji}, \\ \nabla_q\nabla_p(\Phi+k\rho^2)=\Omega g_{qp}, \end{cases}$$

where we have put

$$(1.9)\qquad \Omega=\frac{1}{2\rho^2}(\Phi^2-k^2\rho^4+2CG_{\lambda\kappa}\rho^\lambda\rho^\kappa+C^2).$$

Differentiating the equations (1.6, 1) in y^p and (1.6, 3) in x^i, we have the equations

$$(1.10)\qquad \begin{cases} \nabla_p\nabla_j\nabla_i\rho^2=\nabla_p(\Phi_2+k\rho^2)g_{ji}, \\ \nabla_i\nabla_q\nabla_p\rho^2=\nabla_i(\Phi_1-k\rho^2)g_{qp}. \end{cases}$$

Moreover, comparing these equations (1.10) with the derivatives of (1.8), we see the function Ω equal to

$$(1.11)\qquad \Omega=k(\Phi_1-\Phi_2-k\rho^2)+b,$$

b being a constant. Then the equations (1.8) turn to

$$(1.12)\qquad \begin{cases} \nabla_j\nabla_i(\Phi_1-k\rho^2)=[k(\Phi_1-\Phi_2-k\rho^2)+b]g_{ji}, \\ \nabla_q\nabla_p(\Phi_2+k\rho^2)=[k(\Phi_1-\Phi_2-k\rho^2)+b]g_{qp}, \end{cases}$$

Covariantly differentiating the equations (1.6, 1) in x^k and (1.6, 3) in y^r, we have the equations

$$(1.13)\qquad \begin{cases} \nabla_k\nabla_j\nabla_i\rho^2=(\nabla_k\Phi_1+k\nabla_k\rho^2)g_{ji}+g_{kj}\nabla_i\Phi_1+g_{ki}\nabla_j\Phi_1, \\ \nabla_r\nabla_q\nabla_p\rho^2=(\nabla_r\Phi_2-k\nabla_r\rho^2)g_{qp}+g_{rq}\nabla_p\Phi_2+g_{rp}\nabla_q\Phi_2, \end{cases}$$

and finally the equations

$$(1.14)\qquad \begin{cases} \nabla_k\nabla_j\nabla_i\Phi_1=k(2g_{ji}\nabla_k\Phi_1+g_{kj}\nabla_i\Phi_1+g_{ki}\nabla_j\Phi_1), \\ \nabla_r\nabla_q\nabla_p\Phi_2=-k(2g_{qp}\nabla_r\Phi_2+g_{rq}\nabla_p\Phi_2+g_{rp}\nabla_q\Phi_2), \end{cases}$$

in which the functions Φ_1 and Φ_2 can be replaced with Φ itself. The equations (1.5) to (1.14) are extended on the closure of every component of U, because of the differentiability of ρ. The constants k, C and b might be different in every component, however we shall see that these constants are common all over the manifold M.

On the other hand, a scalar field ρ in a Riemannian manifold M is said to be *special concircular* if it satisfies the equation of the form

$$(1.15)\qquad \nabla_\mu\rho_\lambda=(k\rho+b)g_{\mu\lambda},$$

k and b being constants. The constant k is called the *characteristic* one of ρ. See [2] and [3] as to details on concircular scalar fields.

The trajectories of the gradient vector field $Y=(\rho^\kappa)$ of ρ are geodesics, called *ρ-curves*. In a neighborhood of an ordinary point of ρ, there is a local coordinate system, said to be *adapted*, such that the first coordinate u is the arc-length of ρ-curves and ρ is a function of u. The metric form ds^2 of M is there given in the form

$$(1.16)\qquad ds^2=du^2+\{\rho'(u)\}^2\overline{ds}^2,$$

where prime indicates the derivative in u and $\overline{ds}^2$ is the metric form of an $(n-1)$-dimensional Riemannian manifold $\overline{M}$:

$$(1.17)\qquad \overline{ds}^2=f_{\beta\alpha}du^\beta du^\alpha \qquad (\alpha,\ \beta=2, 3, \cdots, n).$$

The metric tensor $g=(g_{\mu\lambda})$ has components

$$(1.18)\qquad g_{11}=1,\quad g_{1\alpha}=g_{\alpha 1}=0,\quad g_{\beta\alpha}=\rho'^2 f_{\beta\alpha}$$

with respect to an adapted coordinate system, and the Christoffel symbol has components

$$(1.19)\qquad \begin{cases} \left\{ {1 \atop 11} \right\}=\left\{ {\alpha \atop 11} \right\}=\left\{ {1 \atop 1\beta} \right\}=0, \\ \left\{ {1 \atop \gamma\beta} \right\}=-\rho'\rho'' f_{\gamma\beta}, \quad \left\{ {\alpha \atop 1\beta} \right\}=\dfrac{\rho''}{\rho'}\delta^\alpha_\beta, \\ \left\{ {\alpha \atop \gamma\beta} \right\}=\overline{\left\{ {\alpha \atop \gamma\beta} \right\}}, \end{cases}$$

where $\overline{\left\{ {\alpha \atop \gamma\beta} \right\}}$ is the Christoffel symbol composed from the metric (1.17) of $\overline{M}$. Along a ρ-curve, or more generally along any geodesic with arc-length u, the equation (1.15) turns to the ordinary differential equation

$$\rho''(u)=k\rho+b.$$

If there is a stationary point O of ρ, $Y(\mathrm{O})=0$, then a geodesic hypersphere $\overline{M}$ with center O is an $(n-1)$-dimensional sphere.

For a special concircular scalar field ρ, we put

$$(\mathrm{I})\quad k=0,\qquad (\mathrm{II})\quad k=c^2,\qquad (\mathrm{III})\quad k=-c^2\ (c>0)$$

according to the signature of k in (1.15). By a suitable choice of the arc-length u of ρ-curves, ρ is given by

$$(1.20)\qquad \rho(u)=\begin{cases} (\mathrm{I},\mathrm{A}) & au \qquad (b=0), \\ (\mathrm{I},\mathrm{B}) & \frac{1}{2}bu^2+a \qquad (b\neq 0), \\ (\mathrm{II},\mathrm{A}_0) & ae^{cu}-b/c^2, \\ (\mathrm{II},\mathrm{A}_-) & a\sinh cu-b/c^2, \\ (\mathrm{II},\mathrm{B}) & a\cosh cu-b/c^2, \\ (\mathrm{III}) & a\cos cu+b/c^2, \end{cases}$$

where a is an arbitrary constant. The present author [2, 3] proved

THEOREM A. *If a Riemannian manifold M of dimension $n\geqq 2$ is complete and admits a special concircular scalar field ρ, then M is one of the following manifolds corresponding to the expressions* (1.20) *of ρ:*

(I, A) *the product $I\times\overline{M}$ of a straight line I and a complete manifold $\overline{M}$ of dimension $n-1$,*

(I, B) *a Euclidean space,*

(II, A) *a pseudo-hyperbolic space of type* (II, A_0) *or* (II, A_-), *that is, a warped product $I\times\overline{M}$ with metric form* (1.16) *where ρ is given by* (II, A_0) *or* (II, A_-) *of* (1.20).

(II, B) *a hyperbolic space of curvature $-c^2$,*

(III) *a sphere of curvature c^2.*

It is noted that ρ has no stationary point in the cases indicated with A, one in the cases with B and two in the case (III), and that ρ has a zero point in the cases (I, A) and (II, A_-).

2. Case (1) where the subset $U=\emptyset$

Returning to our problem, we first consider the case $U=\emptyset$ but $V\neq\emptyset$. By means of Lemma 3, we may suppose $V\cap(M-N_1)\neq\emptyset$, then we have the equations (1.3) and (1.4) in each connected component of $V\cap(M-N_1)$. By these equations, the associated scalar field ρ is given by

$$\rho=ae^{cu} \qquad (a\neq 0) \tag{2.1}$$

along ρ-curves lying in the part $M_1(\mathrm{P})$ through a point $\mathrm{P}\in V\cap(M-N_1)$. It follows from the differentiability of ρ that the equations (1.3) and (1.4) are extendable first on the part $M_1(\mathrm{P})$, next on the closure of each component of $V\cap(M-N_1)$ and finally all over M. There is no point such that $Y_2\neq 0$, that is, we see $M=N_2$. By virtue of Theorem A, we can state

PROPOSITION 1. *We assume that a product Riemannian manifold $M=M_1\times M_2$ is complete and a conformal diffeomorphism f maps M into a product Riemannian one M^*. In Case* (1), *the associated scalar field ρ is given by* (2.1), *and the part M_1 of dimension $n_1\geqq 2$ is a pseudo-hyperbolic space of type* (II, A_0) *with metric form*

$$ds_1^2=du^2+e^{2cu}\overline{ds}_1^2$$

where $\overline{ds}_1^2$ is the metric form of an (n_1-1)-dimensional manifold $\overline{M}_1$, or M_1 is a 1-*dimensional straight line I.*

Proof of Theorem 1 *in Case* (1). The manifold M^* is supposed to be complete too, and f to be global. The underlying manifold of M_1 is the product $I\times\overline{M}_1$ and copies of I in M_1 are ρ-curves.

Let Γ be a ρ-curve lying on M_1, Γ^* the image $f(\Gamma)$, and s^* the arc-length of Γ^* such that $s^*=0$ corresponding to $u=0$. Then s^* is related to s by the differential equation

$$\frac{ds^*}{du}=\frac{1}{\rho}=\frac{1}{a}e^{-cu}$$

or, by integration, we have the inequality

$$s^*=\frac{1}{ac}(1-e^{-cu})<\frac{1}{ac}.$$

Therefore the length of the image Γ^* is bounded as u tends to the infinity along Γ. This contradicts the globalness of f, see [5]. Thus Theorem 1 is proved in this case.

3. Case (2) where the subset $U\neq\emptyset$ and $k=0$

We shall first consider the case where the subset U is not empty. The copy of the part M_1 passing through a point P will be denoted by $M_1(\mathrm{P})$, the union of $M_1(\mathrm{P})$ for all points P of a subset S by $M_1(S)$, and similar notations will be used as to M_2. Let Γ_1 and Γ_2 be any geodesic curves lying in $M_1(\mathrm{P})$ and $M_2(\mathrm{P})$ with arc-length u and v respectively. Let U_0 be an arbitrary connected component of U and $\bar{U}_0$ the closure of U_0. Along Γ_1 in $M_1(\mathrm{P})\cap\bar{U}_0$ and Γ_2 in $M_2(\mathrm{P})\cap\bar{U}_0$, the equations (1.13) turn to the differential equations

$$
(3.1)\qquad \begin{cases} \dfrac{\partial^3\rho^2}{\partial u^3}=k\dfrac{\partial\rho^2}{\partial u}+3\Phi_1'(u), \\ \dfrac{\partial^3\rho^2}{\partial v^3}=-k\dfrac{\partial\rho^2}{\partial v}+3\Phi_2'(v), \end{cases}
$$

and the equations (1.14) to the ordinary linear differential equations

$$
(3.2)\qquad \Phi_1'''(u)=4k\Phi_1'(u), \qquad \Phi_2'''(v)=-4k\Phi_2'(v),
$$

where primes indicate derivatives in the indicated variables.

Let us prove the following

LEMMA 4. *The subsets N_1 and N_2 are border sets in M the constants k, C and b are common to all connected components of U and the equations* (1.5) *to* (1.14) *all valid in the whole manifold M.*

PROOF. Let N_1^0 and N_2^0 be the open kernels of N_1 and N_2 respectively. Suppose that N_2^0 is not empty and Q its point. Then the equation (1.10. 1) means that, for each p, $\nabla_p\rho^2$ is a special concircular scalar field in $M_1(\mathrm{Q})\cap\bar{U}_0$ and identically vanishes in $M_1(\mathrm{Q})\cap\bar{N}_2^0$. The equation (1.10, 1) holds with vanishing right hand side in $M_1(\mathrm{Q})\cap\bar{N}_1^0$. Since a special concircular scalar field has at most two stationary points unless it is constant, $\nabla_p\rho^2$ should be constant and consequently $\nabla_p\rho^2=0$ on $M_1(\mathrm{Q})$. Hence we have $Y_2=(\rho^p)=0$ in $M_1(\bar{N}_2^0)$, that is, $M_1(\bar{N}_2^0)$ is contained in N_2 and there vanish all successive derivatives of ρ and Φ in y^p. Simlarly, $M_2(\bar{N}_1^0)\subset N_1$ if N_1^0 is not empty.

Take an arbitrary point $\mathrm{R}\in M-M_2(\bar{N}_1^0)$. The intersection $M_2(\mathrm{R})\cap M_1(\bar{N}_2^0)$ is not emply. In order for ρ to be differentiably continuable beyond the border of $M_2(\mathrm{R})\cap M_1(\bar{N}_2^0)$, it follows from (3.2, 2) that Φ_2 should be constant along any geodesic curve lying in $M_2(\mathrm{R})$, then from (3.1) that so be ρ and $Y_2=0$ in $M_2(\mathrm{R})$. Hence $M-M_2(\bar{N}_1^0)$ is contained in N_2 and we have $M=N_1\cap N_2$. This contradicts the assumption $U\neq\emptyset$. Thus N_2 is a border set in M, and similarly so is N_1.

By the similar arguments to the above, neither the border sets N_1 nor N_2 contain points where all the successive derivatives of ρ vanish. By comparison of the equations (1.14) on the border of adjoining connected components of U, the constant k is common to all connected components, and so are the constants C and b by means of (1.5) and (1.12). As a consequence the equations (1.5) to (1.14) are valid over the manifold M. Q. E. D.

In the remaining of this paragraph, we consider the case of $k=0$. The equations (1.10) and (1.13) together make a tensor equation

$$\nabla_\nu\nabla_\mu\nabla_\lambda\rho^2=g_{\nu\mu}\nabla_\lambda\Phi+g_{\nu\lambda}\nabla_\mu\Phi+g_{\mu\lambda}\nabla_\nu\Phi\,. \tag{3.3}$$

It follows from (1.11) that $\Omega=b$, and by account of the decomposability of Φ, the equations (1.12) turn to the tensor equation

$$\nabla_\mu\nabla_\lambda\Phi=bg_{\mu\lambda}\,. \tag{3.4}$$

Furthermore this case splits to the three following cases.

(a) If Φ is constant in M, then the equation (3.3) is reduced to the equation

$$\nabla_\nu\nabla_\mu\nabla_\lambda\rho^2=0\,. \tag{3.5}$$

Now we shall prove the following

LEMMA 5. *In Case* (2, a) *where* $k=0$ *and* Φ *is constant, we have the tensor equation*

$$\nabla_\mu\nabla_\lambda\rho^2=2\Phi g_{\mu\lambda}\,. \tag{3.6}$$

PROOF. We suppose that M is a product of irreducible parts, regarding a 1-dimensional part to be irreducible. One of the parts may be M_1. Then, by the irreducibility, the equation (3.5) implies

$$\nabla_q\nabla_i\rho^2=0\,, \qquad \nabla_j\nabla_i\rho^2=2a_1g_{ji}$$

in M_1, α_1 being a non-zero constant. Hence ρ^2 is decomposable, and there is an adapted coordinate system (u^h) in M_1 such that ρ^2 is expressed as

$$\rho^2=a_1(u^1)^2+2\beta\,, \tag{3.7}$$

where β is a function independent of u^h. Substituting the derivatives of the expression (3.7) into $\rho^2\Phi=\rho^2\rho_\lambda\rho^\lambda$, we obtain the relation

$$\Phi\{a_1(u^1)^2+2\beta\}=a_1^2(u^1)^2+\beta_q\beta^q\,,$$

putting $\beta_q=\nabla_q\beta$. Comparing the coefficients of $(u^1)^2$ in the two sides, we see $a_1=\Phi$. Applying the same argument to the irreducible parts, we obtain the equation (3.6). Q. E. D.

The scalar field ρ^2 is of type (I, B) in (1.20). There is an adapted coordinate system (u, u^α), $\alpha=2, \cdots, n$, in M such that ρ^2 is given by

$$\rho^2=\Phi u^2 \tag{3.8}$$

and the metric form of M by

$$ds^2=du^2+u^2\overline{ds}^2\,. \tag{3.9}$$

By virtue of Theorem A, we have

PROPOSITION 2, (a). *In Case* (2, a), *under the same assumption as that of Proposition* 1, *the manifold M is a Euclidean space and the associated scalar field ρ is given by* (3.8).

Proof of Theorem 1 *in Case* (2, a). The associated scalar field ρ vanishes at the origin O corresponding to $u=0$, and there is no conformal diffeomorphism of M onto M^*. The theorem is proved in this case.

(b) If Φ is not constant and $\Omega=b=0$, then we have the equation

$$\nabla_\mu \nabla_\lambda \Phi=0$$

by means of (3.4). The manifold M is the Riemannian product of a 1-dimensional manifold M_1 and an $(n-1)$-dimensional one M_2, and there is a separate coordinate system (u, y^p) such that Φ is expressed as

$$\Phi=2au, \tag{3.10}$$

a being a constant.

The equation (3.3) splits into the following equations:

$$\begin{cases} \nabla_1\nabla_1\nabla_1\rho^2=6a, & \nabla_q\nabla_1\nabla_1\rho^2=0, \\ \nabla_q\nabla_p\nabla_1\rho^2=2ag_{qp}, & \nabla_r\nabla_q\nabla_p\rho^2=0 \end{cases} \tag{3.11}$$

with respect to the system. Since the equation (3.11, 2) implies that $\nabla_1\rho^2$ is decomposable, we put

$$\nabla_1\rho^2=\alpha(u)+\beta(y^p),$$

where α and β are functions of the indicated variables respectively. Substituting this expression into (3.11, 1), and integrating, we may put

$$\alpha=3au^2+2bu$$

b being a constant. Then it follows from (3.11) that ρ has the expression

$$\rho^2=au^3+bu^2+u\beta+\gamma, \tag{3.12}$$

where γ is a function of y^p.

Since the squared length Φ of $Y=(\rho^\kappa)$ is equal to

$$\Phi=\rho_1{}^2+g^{qp}\rho_q\rho_p$$

in the system (u, y^p), we substitute (3.10) and derivatives of (3.12) into this equation, and obtain the identity

$$8au(au^3+bu^2+u\beta+\gamma)=(3au^2+2bu+\beta)^2+g^{qp}(u\beta_q+\gamma_q)(u\beta_p+\gamma_p)$$

putting $\gamma_q=\partial_q\gamma$. Comparing the coefficients of u^4, we see $a=0$. This is a contradiction and the case (b) does not occur.

(c) If $b\neq 0$, then Φ is of type (I, B) in (1.20) and there is the same adapted coordinate system (u, u^α) as that in the case (a), and Φ is expressed as

$$\Phi=\frac{1}{2}bu^2+c, \tag{3.13}$$

c being a constant, and the metric form of M is given by (3.9). If we denote the metric tensor of $\overline{M}$ by $(f_{\gamma\beta})$, then non-vanishing components of the Christoffel symbol of M with respect to the adapted coordinate system (u, u^α) are

$$\left\{ {1 \atop \gamma\beta} \right\}=-uf_{\gamma\beta},\quad \left\{ {\alpha \atop 1\beta} \right\}=\frac{1}{u}\delta^\alpha_\beta,\quad \left\{ {\alpha \atop \gamma\beta} \right\}=\overline{\left\{ {\alpha \atop \gamma\beta} \right\}},$$

where $\overline{\left\{ {\alpha \atop \gamma\beta} \right\}}$ is the Christoffel symbol composed from the metric tensor $f_{\gamma\beta}$ of $\overline{M}$. The covariant differentiation with respect to $\overline{\left\{ {\alpha \atop \gamma\beta} \right\}}$ will be denoted by $\overline{\nabla}$.

The components of the second covariant derivative $\nabla_\mu\nabla_\lambda\rho^2$ are expressed as

$$\begin{cases} \nabla_1\nabla_1\rho^2=\partial_1\partial_1\rho^2, \\ \nabla_\beta\nabla_1\rho^2=\partial_\beta\partial_1\rho^2-\dfrac{1}{u}\partial_\beta\rho^2. \\ \nabla_\beta\nabla_\alpha\rho^2=\overline{\nabla}_\beta\overline{\nabla}_\alpha\rho^2+uf_{\beta\alpha}\partial_1\rho^2 \end{cases} \tag{3.14}$$

with respect to (u, u^α), and essentials of the equation (3.3) are

$$\begin{cases} \nabla_1\nabla_1\nabla_1\rho^2=\partial_1\partial_1\partial_1\rho^2=3bu, \\ \nabla_\beta\nabla_1\nabla_1\rho^2=\partial_\beta\partial_1\partial_1\rho^2-\dfrac{2}{u}\nabla_\beta\nabla_1\rho^2=0, \\ \nabla_1\nabla_\beta\nabla_\alpha\rho^2=\partial_1\nabla_\beta\nabla_\alpha\rho^2-\dfrac{2}{u}\nabla_\beta\nabla_\alpha\rho^2=bu^3f_{\beta\alpha}, \\ \nabla_\gamma\nabla_\beta\nabla_\alpha\rho^2=\overline{\nabla}_\gamma\nabla_\beta\nabla_\alpha\rho^2+uf_{\gamma\beta}\nabla_\alpha\nabla_1\rho^2+uf_{\gamma\alpha}\nabla_\beta\nabla_1\rho^2=0. \end{cases} \tag{3.15}$$

Substituting (3.14, 2) into (3.15, 2), we see that $\partial_1\rho^2-(2/u)\rho^2$ is decomposable and put in the form

$$\partial_1\rho^2-\frac{2}{u}\rho^2=\alpha(u)+\beta(u^\alpha), \tag{3.16}$$

α being a function of u and β a function of (u^α) belonging to $\overline{M}$. Substituting the third derivative of ρ^2 in u into (3.15, 1), we have the equation

$$u\alpha''+2\alpha'=3bu^2. \tag{3.17}$$

The solution of this equation is given by

$$\alpha=\frac{1}{4}bu^3+\frac{B}{u}, \tag{3.18}$$

where B is a constant and a constant term has been transferred into β of (3.16)

Subsituting (3.18) into (3.16) and dividing by u^2, we have

$$\partial_1\left(\frac{1}{u^2}\rho^2\right)=\frac{1}{4}bu+\frac{B}{u^3}+\frac{\beta}{u^2}$$

and consequently ρ^2 is expressed as

$$\rho^2=\frac{1}{8}bu^4-\frac{B}{2}-uB+u^2\gamma, \tag{3.19}$$

γ being a function of u^α belonging to $\bar{M}$. Taking account of the expressions (3.14, 3), (3.19) and (3.17), we have

$$\nabla_1\nabla_\beta\nabla_\alpha\rho^2=\nabla_\beta\nabla_\alpha\beta+(bu^3+\beta)f_{\beta\alpha}$$

and, comparing this equation with (3.15, 3),

$$\nabla_\beta\nabla_\alpha\beta=-\beta f_{\beta\alpha}. \tag{3.20}$$

Subsituting (3.14, 2), (3.14, 3) and (3.19) into (3.15, 3) and taking account of (3.20), we see that the function γ satisfies the equation

$$\nabla_\gamma\nabla_\beta\nabla_\alpha\gamma=-(2f_{\beta\alpha}\nabla_\gamma\gamma+f_{\gamma\beta}\nabla_\alpha\gamma+f_{\gamma\alpha}\nabla_\beta\gamma) \tag{3.21}$$

on $\bar{M}$.

Since the squared length Φ of $Y=(\rho^\kappa)$ is given by

$$\Phi=\rho_1^2+\frac{1}{u^2}f^{\beta\alpha}\rho_\beta\rho_\alpha$$

with respect to the adapted coordinate system (u, u^α), we substitute (3.13) and derivatives of (3.19) into this equation, and obtain the identity

$$4\left(\frac{1}{8}bu^4-\frac{B}{2}-u\beta+u^2\gamma\right)\left(\frac{1}{2}bu^2+c\right)$$
$$=\left(\frac{1}{2}bu^3-\beta+2u\gamma\right)^2+f^{\beta\alpha}(u\gamma_\beta-\beta_\beta)(u\gamma_\alpha-\beta_\alpha),$$

where $\beta_\alpha=\partial_\alpha\beta$ and $\gamma_\alpha=\partial_\alpha\gamma$. Comparing the coefficients of u^4, u^3 and u^2 in the both sides, we have $c=0$, $\beta=0$ and

$$-bB=4\gamma^2+f^{\beta\alpha}\gamma_\beta\gamma_\alpha.$$

Thus the assoiated scalar field ρ is expressed as

$$\rho^2=\frac{1}{8b}(b^2u^4+8bu^2\gamma+16\gamma^2+4f^{\beta\alpha}\gamma_\beta\gamma_\alpha) \tag{3.22}$$
$$=\frac{1}{8b}[(bu^2+4\gamma)^2+4f^{\beta\alpha}\gamma_\beta\gamma_\alpha]$$

by use of a solution γ of (3.21) on $\bar{M}$.

Since Φ satisfies the equation (3.4) with $b\neq 0$, by virtue of Theorem A, we have the following

PROPOSITION 2, (c). *In Case* (2, c) *under the same assumption as that of Proposition* 1, *the manifold M is a Euclidean space and ρ is given by* (3.22).

Proof of Theorem 1 *in Case* (2, c). The squared length Φ given by (3.13) has a staionary point O corresponding to $u=0$ in (u, u^{α}) and the hypersurface $\bar{M}$ is the unit hypersphere with point O as center. The rays issuing from O are ρ-curves of Φ, and the function γ is constant on each of the rays.

Let Γ be one of the rays and Γ^* the image $f(\Gamma)$ in M^*. Since the constant b in (3.22) should be positive, we put $b=16a^2$, $a>0$. We can take a value u_0 so large that

$$\rho(u)>au^2 \qquad (u>u_0) \tag{3.23}$$

holds. The arc-length s^* of Γ^* is related to u of Γ by the equation

$$\frac{ds^*}{du}=\frac{1}{\rho}.$$

Denoting by s_0^* the value of s^* corresponding to u_0 and taking account of the inequality (3.23), we obtain the inequality

$$s^*-s_0^*<\frac{1}{a}\left(\frac{1}{u_0}-\frac{1}{u}\right)<\frac{1}{au_0} \qquad (u>u_0).$$

Hence the length of the image Γ^* is bounded as u tends to the infinity. This contradicts the globalness of f. Theorem 1 is thus proved in this case.

4. Case (3) where $U\neq\emptyset$ and $k\neq 0$.

In this paragraph we shall consider the case where the subset U is not empty and the constant k is not equal to 0. We may suppose k is positive without loss of generality and put $k=1$ for simplicity. Moreover this case splits into the two following cases.

(a) If the function Ω is constant, then Ω is equal to zero as easily seen from (1.8) and (1.11), and we have

$$\rho^2=\Phi_1-\Phi_2+b. \tag{4.1}$$

The square ρ^2 itself is decomposable and satisfies the equations

$$\nabla_k\nabla_j\nabla_i\rho^2=2g_{ji}\nabla_k\rho^2+g_{kj}\nabla_i\rho^2+g_{ki}\nabla_j\rho^2,$$

$$\nabla_r\nabla_q\nabla_p\rho^2=-(2g_{qp}\nabla_r\rho^2+g_{rq}\nabla_p\rho^2+g_{rp}\nabla_q\rho^2)$$

of the same type as the equations (1.14). The field ρ is given by the expression (4.1) in which Φ_1 and Φ_2 are solutions of the respective equations of (1.14).

Proof of Theorem 1 *in Case* (3, a). Let Γ be a geodesic curve with arc-length u lying in the part $M_1(\mathrm{P})$, $\mathrm{P}\in U$, and Γ^* the image $f(\Gamma)$ in M^*. Then, along Γ, ρ^2 is given by

$$\rho^2=Ae^{2u}+Be^{-2u}+C,$$

A, B and C being constants. Since the arc-length u of Γ is extendable to the infinities of two sides, A and B should be non-negative and at least one of them be positive. Putting $A=2a^2$, $a>0$, we can take a value u_0 so large that

$$\rho(u)>ae^u \qquad (u>u_0)$$

holds. Using the same notations and arguments at the end of § 3, the arc-length s^* of Γ^* is bounded as

$$s^*-s_0^*<e^{-u_0}/a \qquad (u>u_0).$$

This contradicts the globalness of f, and Theorem 1 is proved in this case.

(b) If the function Ω is not constant, then the equations (1.12) show that $\Phi_1-\rho^2$ and $\Phi_2+\rho^2$ are special concircular scalar fields with characteristic constant 1 and -1 in $M_1(\mathrm{P})$ and $M_2(\mathrm{P})$ passing through every point P of U, respectively. By virtue of Theorem A, we have the following

PROPOSITION 3. *In Case* (3, b) *under the same assumption as that of Proposition* 1, *the manifold M is the product Riemannian manifold $M_1\times M_2$, in which M_1 is a pseudo-hyperbolic space or a hyperbolic sapce and M_2 a sphere. Either the part M_1 or M_2 may be a straight line.*

The part M_1 admits a special concircular scalar field σ and an adapted coordinate system (u, u^α), $\alpha, \beta, \gamma=2, \cdots, n_1$, such that u is the arc-length of ρ-curves of σ, σ satisfies the equation

$$\sigma''(u)=\sigma, \tag{4.2}$$

prime indicating derivatives in u, and the metric form ds_1^2 of M_1 is given by

$$ds_1^2=du^2+\sigma'^2\overline{ds_1}^2,$$

where $\overline{ds_1}^2=f_{\beta\alpha}du^\beta du^\alpha$ is the metric form of an (n_1-1)-dimensional Riemannian manifold $\overline{M}_1$. By suitable choice of the arc-length u of ρ-curves of σ, we take the function

$$\sigma=e^u-Be^{-u} \tag{4.3}$$

as representative of σ in M_1. For the later use, we notice that σ satisfies the equations

$$\sigma'^2-\sigma^2=4B \tag{4.4}$$

and

$$(\sigma''/\sigma')'+(\sigma''/\sigma')^2=1. \tag{4.5}$$

Components of the Christoffel symbol of M_1 are given by (1.19) with σ for ρ. Then the equation (1.12, 1) splits into

$$\begin{cases} \partial_1\partial_1(\Phi_1-\rho^2)=\Phi_1-\Phi_2-\rho^2+b, \\ \partial^1\partial_\beta(\Phi_1-\rho^2)-\dfrac{\sigma''}{\sigma'}\partial_\beta(\Phi_1-\rho^2)=0, \\ \nabla_\gamma\nabla_\beta(\Phi_1-\rho^2)+\sigma'\sigma'' f_{\gamma\beta}\partial_1(\Phi_1-\rho^2)=(\Phi_1-\Phi_2-\rho^2+b)\sigma'^2 f_{\gamma\beta}. \end{cases} \tag{4.6}$$

From (4.6, 2) we see $(\Phi_1-\rho^2)/\sigma'$ decomposable in M_1 and put

$$\Phi_1-\rho^2=\sigma'[\alpha_1(u, y^p)+\gamma(u^\alpha, y^p)], \tag{4.7}$$

where α_1 and γ are functions dependent on the indicated variables respectively. Substituting (4.7) into (4.6, 1) and taking account of (4.2), we have the equation

$$\sigma'\partial_1\partial_1\alpha_1+2\sigma''\partial_1\alpha_1=-\Phi_2+b,$$

by means of which we may put

$$\sigma'^2\partial_1\alpha_1=(-\Phi_2+b)\sigma+4\lambda_2(y^p), \tag{4.8}$$

λ_2 being a function of y^p. Substituting (4.7) into (4.6, 3) and using (4.4) and (4.8), we have the equation

$$\sigma'(\nabla_\gamma\nabla_\beta\gamma-4B\gamma f_{\gamma\beta})=4[B(\sigma'\alpha_1-\Phi_2+b)-\sigma\lambda_2]f_{\gamma\beta}. \tag{4.9}$$

The expression in the parentheses in the left hand side is independent of u and the expression in the brackets in the right hand side independent of u^α. Hence, if $B\neq 0$, then we may put

$$B(\sigma'\alpha_1-\Phi_2+b)-\sigma\lambda_2=B\sigma'\mu_2(y^p),$$

$$\nabla_\gamma\nabla_\beta\gamma=4B(\gamma+\mu_2)f_{\gamma\beta},$$

where μ_2 is a function of y^p. Substituting the expression of $\sigma'\alpha_1$ obtained from (4.7) into the first equation and putting

$$\beta_2=\lambda_2/B, \qquad \gamma_1=\gamma+\mu_2,$$

we have the relation

$$\Phi_1-\rho^2=\Phi_2-b+\beta_2\sigma+\gamma_1\sigma', \tag{4.10}$$

where β_2 is a function of y^p, γ_1 is of u^α and y^p, and satisfies the equation

$$\nabla_\gamma\nabla_\beta\gamma_1=4B\gamma_1 f_{\gamma\beta}. \tag{4.11}$$

By similar arguments, the part M_2 admits a special concircular scalar field τ satisfying the equation

$$\tau''(v)=-\tau \tag{4.12}$$

in the arc-length v of ρ-curves of τ, and an adapted coordinate system (v, v^ξ), ξ, η, $\zeta=n_1+2, \cdots, n$, where the metric form $ds_2{}^2$ of M_2 is of the form

$$ds_2{}^2=dv^2+\tau'^2\overline{ds_2}^2$$

and $\overline{ds_2}^2=f_{\eta\xi}dv^\eta dv^\xi$ is the metric form of an (n_2-1)-dimensional Riemannian manifold $\overline{M}_2$. We take the function

$$\tau=-\cos v \tag{4.13}$$

as representative of τ in M_2. We can obtain the relation

$$\Phi_2+\rho^2=\Phi_1+b+\beta_1\tau+\gamma_2\tau', \tag{4.14}$$

where β_1 is a function of x^h, γ_2 is of x^h and v^ξ, and satisfies the eqaution

$$\nabla_\eta\nabla_\xi\gamma_2=-\gamma_2 f_{\eta\xi}. \tag{4.15}$$

Comparing the relations (4.10) and (4.14) and taking account of linear independence among σ, σ', τ and τ', we can see that ρ^2 is expressed in the form

$$\rho^2=\Phi_1-\Phi_2+b+(A\tau+\beta\tau')\sigma+(\alpha\tau+\gamma\tau')\sigma', \tag{4.16}$$

where A is a constant, α a function of u^α, β one of v^ξ and γ one of u^α and v^ξ, and they satisfy the equations

$$\nabla_\gamma\nabla_\beta\alpha=4B\alpha f_{\gamma\beta}, \qquad \nabla_\gamma\nabla_\beta\gamma=4B\gamma f_{\gamma\beta},$$
$$\nabla_\eta\nabla_\xi\beta=-\beta f_{\eta\xi}, \qquad \nabla_\eta\nabla_\xi\gamma=-\gamma f_{\eta\xi}$$

respectively.

If $B=0$ in (4.3), or $\sigma=e^u$, then the equation (4.8) turns to

$$\partial_1\alpha_1=-(\Phi_2-b)e^{-u}+4\lambda_2 e^{-2u}$$

and the solution is given by

$$\alpha_1=(\Phi_2-b)e^{-u}-2\lambda_2 e^{-2u}+\mu_2(y^p),$$

μ_2 being a function of y^p. Substituting this expression into (4.7), we have the relation

$$\Phi_1-\rho^2=\Phi_2-b-2\lambda_2 e^{-u}+\gamma_1 e^u, \tag{4.17}$$

where γ_1 is defined by $\gamma_1=\gamma+\mu_2$ and satisfies the equation

$$\overline{\nabla}_\gamma\overline{\nabla}_\beta\gamma_1=-4\lambda_2 f_{\gamma\beta} \tag{4.18}$$

as seen from (4.9). On the other hand, we have again the relations (4.14), and, by comparing this relation with (4.17), see that ρ^2 is expressed in the form

$$\rho^2=\Phi_1-\Phi_2+b+(A\tau+\beta\tau')e^{-u}+(\alpha\tau+\gamma\tau')e^u. \tag{4.19}$$

The functions A, α, β and γ are of the same properties as those in (4.16) but α and γ satisfy the equations of type (4.18) in $\overline{M}_1$.

Next we shall seek for the expressions of Φ_1 and Φ_2. The equation (1.14, 1) splits into the four essential equations

$$\begin{cases} \nabla_1\nabla_1\nabla_1\Phi_1=\partial_1\partial_1\partial_1\Phi_1=4\partial_1\Phi_1, \\ \nabla_\beta\nabla_1\nabla_1\Phi_1=\partial_\beta\partial_1\partial_1\Phi_1-2\dfrac{\sigma''}{\sigma'}\nabla_\beta\nabla_1\Phi_1=2\partial_\beta\Phi_1, \\ \nabla_1\nabla_\beta\nabla_\alpha\Phi_1=\partial_1\nabla_\beta\nabla_\alpha\Phi_1-2\dfrac{\sigma''}{\sigma'}\nabla_\beta\nabla_\alpha\Phi_1=2\sigma'^2 f_{\beta\alpha}\partial_1\Phi_1, \\ \nabla_\gamma\nabla_\beta\nabla_\alpha\Phi_1=\overline{\nabla}_\gamma\nabla_\beta\nabla_\alpha\Phi_1+\sigma'\sigma'' f_{\gamma\beta}\nabla_\alpha\nabla_1\Phi_1+\sigma'\sigma'' f_{\gamma\alpha}\nabla_\alpha\nabla_1\Phi_1 \\ \qquad =\sigma'^2(2f_{\beta\alpha}\nabla_\gamma\Phi_1+f_{\gamma\beta}\nabla_\alpha\Phi_1+f_{\gamma\alpha}\nabla_\beta\Phi_1) \end{cases} \tag{4.20}$$

with respect to the adapted system (u, u^α) in M_1. Substituting the expression

$$\nabla_\beta\nabla_1\Phi_1=\partial_\beta\partial_1\Phi_1-\frac{\sigma''}{\sigma'}\partial_\beta\Phi_1$$

into (4.20, 2) and using the relation (4.5), we may put

$$\partial_1\Phi_1-2\frac{\sigma''}{\sigma'}\Phi_1=\varphi_1+2\psi_1, \tag{4.21}$$

where φ_1 is a function of u and ψ_1 of the other u^α in M_1. By means of (4.4), the third derivative of Φ_1 in u is equal to

$$\partial_1\partial_1\partial_1\Phi_1=4\partial_1\Phi_1+2\frac{\sigma''}{\sigma'}\varphi_1'+\varphi_1''.$$

Therefore it follows from (4.20, 1) that φ_1' is expressed as

$$\varphi_1'=\frac{8C_1}{\sigma'^2}$$

C_1 being a constant. Substituting the expression (4.3), the function φ_1 is given by

$$\varphi_1=\frac{-4C_1}{e^{2u}+B} \tag{4.22}$$

where the integral constant has been transferred into ψ_1. By integration of (4.21) substituted with (4.22), the function Φ_1 is given by

$$(4.23)\qquad \Phi_1=\omega_1(e^u+Be^{-u})^2-\psi_1+(C_1-B\psi_1)e^{-2u},$$

where ω_1 is a function of u^α belonging to $\overline{M}_1$. Substituting this expression into (4.20, 3) and (4.20, 4), we see the functions ψ_1 and ω_1 satisfying the equations

$$\nabla_\beta\nabla_\alpha\psi_1=4(B\psi_1-C_1)f_{\beta\alpha}$$

and

$$\nabla_\gamma\nabla_{\beta\alpha}\nabla\omega_1=4B(2f_{\beta\alpha}\partial_\gamma\omega_1+f_{\gamma\beta}\partial_\alpha\omega_1+f_{\gamma\alpha}\partial_\beta\omega_1)$$
$$-2(f_{\beta\alpha}\partial_\gamma\psi_1+f_{\gamma\beta}\partial_\alpha\psi_1+f_{\gamma\alpha}\partial_\beta\psi_1)$$

on $\overline{M}_1$ respectively.

By similar arguments to those on Φ_1, the part Φ_2 of Φ is expressed as

$$(4.24)\qquad \Phi_2=C_2-2\psi_2\sin v\cos v+\omega_2\sin^2 v$$

with respect to the adapted coordinate system (v, v^ξ) in M_2, where C_2 is a constant, ψ_2 and ω_2 are functions of v^ξ of $\overline{M}_2$ and satisfy the equations

$$\nabla_\eta\nabla_\xi\psi_2=-\psi_2 f_{\eta\xi}$$

and

$$\nabla_\zeta\nabla_\eta\nabla_\xi\omega_2=-(2f_{\eta\xi}\partial_\zeta\omega_2+f_{\zeta\eta}\partial_\xi\omega_2+f_{\zeta\xi}\partial_\eta\omega_2)$$

on $\overline{M}_2$ respectively.

Thus the associated scalar field ρ has the expression (4.16) or (4.19) with σ, τ, Φ_1 and Φ_2 given by (4.3), (4.13), (4.23) and (4.24) respectively.

Proof of Theorem 1 *in Case* (3, b). Let P be a point of U, Γ an arbitrary ρ-curve of σ which lies in M_1(P), and Γ^* the image $f(\Gamma)$ of Γ in M^*. The coefficients A, α, β, γ in (4.16) or (4.19), ψ_1, ω_1 in (4.23) and ψ_2, ω_2 in (4.24) are all constant on Γ. The function Φ_1 is of the second order in e^u and e^{-u}, and the other terms in (4.16) or (4.19) are of less order. The value of ω_1 on Γ should not be negative for $\rho^2>0$ for any value of u.

If $\omega_1\neq 0$ on Γ and we put $\omega_1=2a^2$, then we can take a value u_0 so large that

$$\rho>ae^u$$

for $u>u_0$. By the same argument as that in Case (3, a), we can see that the length of the image Γ^* is bounded as u tends to the infinity. This contradicts the globalness of f.

If ω_1 identically vanishes, then the coefficient $C_1-B\psi_1$ of e^{-2u} should not be negative. If $C_1-B\psi_1>0$, we can apply the similar argument to this case and yield a contradiction.

If $C_1-B\psi_1=0$ everywhere, then ψ_1 is constant and so is Φ_1 in M. Then, in order for $\rho^2>0$ everywhere, the coefficients

$$-A\cos v+\beta\sin v \quad\text{and}\quad -\alpha\cos v+\gamma\sin v$$

in (4.16) or (4.19) should be always positive, but it cannot occur.

As consequence of discussions in three Cases (1), (2) and (3), we have completed a proof of Theorem 1.

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