EXISTENCE OF THE LEAST AND THE GREATEST ELEMENTS OF A SUBSET OF THE LINDENBAUM ALGEBRA

By

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Introduction.

The purpose of this note is to answer the following problem.

PROBLEM. *Let P be a propositional letter and $G(P)$ a formula. Does the set $\{A \mid \vdash G(A)\}$ always have the least and the greatest elements unless it is empty?*

To be accurate, of course, the above problem must be understood in the Lindenbaum algebra.

The answer is “no” if $\vdash$ denotes the provability in the intuitionistic logic (§ 1), while “yes” if the classical logic is concerned with (§ 2).

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§ 1. The intuitionistic case.

In this section we understand by $\vdash$ the provability in the intuitionistic logic. Then the problem is negative since Proposition 1 below shows an example of a non-empty set which has neither the least nor the greatest element.

PROPOSITION 1. *Let Q and R be distinct propositional letters and $G(P)$ be the formula $[P \sim Q] \vee [P \sim R]$.*

1°) $\vdash G(Q)$ and $\vdash G(R)$.

2°) *There is no formula B such that (a) $\vdash G(B)$ and (b) if $\vdash G(A)$ then $\vdash B \supset A$ for every A .*

3°) *There is no formula B such that (a) $\vdash G(B)$ and (b') if $\vdash G(A)$ then $\vdash A \supset B$ for every A .*

PROOF. 1°) is clear.

2°) We assume, on the contrary to the conclusion, that any formula B

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satisfying (a) and (b) is given. By 1°) and (b) we have $\vdash B \supset Q$ and $\vdash B \supset R$, from which together with (a) follows $\vdash [Q \supset R] \vee [R \supset Q]$ which is absurd. Hence no such B exists.

3°) is proved similarly. □

§ 2. The classical case.

In this section we understand by $\vdash$ the provability in the classical logic. Then we see by Proposition 2 below that the set $\{A \mid \vdash G(A)\}$ has the least element $\neg G(\mathbf{F})$ and the greatest element $G(\mathbf{T})$ unless it is empty, where for brevity's sake we assume that the propositional constants $\mathbf{T}$ (truth) and $\mathbf{F}$ (falsity) are admitted. So the problem stated in the introduction is affirmative.

We remark that Proposition 2, 4°) has, essentially, already been obtained in Maehara [1, Appendix].

PROPOSITION 2. *Let be given a propositional letter P and a formula $G(P)$.*

1°) $\vdash G(A) \supset [\neg G(\mathbf{F}) \supset A]$.

2°) $\vdash G(A) \supset G(\neg G(\mathbf{F}))$.

3°) $\vdash G(A) \supset [A \supset G(\mathbf{T})]$.

4°) $\vdash G(A) \supset G(G(\mathbf{T}))$.

PROOF. We first remark the facts that

$$\vdash A \supset [A \sim \mathbf{T}], \quad \vdash \neg A \supset [A \sim \mathbf{F}], \quad \vdash [A \sim B] \supset [G(A) \sim G(B)],$$

and so

$$\vdash A \supset [G(A) \sim G(\mathbf{T})], \quad \vdash \neg A \supset [G(A) \sim G(\mathbf{F})], \quad \vdash G(\mathbf{T}) \wedge G(\mathbf{F}) \supset G(A).$$

1°) follows from $\vdash \neg A \supset [G(A) \sim G(\mathbf{F})]$.

2°) From $\vdash \neg \neg G(\mathbf{F}) \supset [G(\neg G(\mathbf{F})) \sim G(\mathbf{F})]$ it follows $\vdash A \wedge G(\mathbf{F}) \supset [G(A) \supset G(\neg G(\mathbf{F}))]$, while from 1°) it follows $\vdash \neg A \wedge \neg G(\mathbf{F}) \supset [G(A) \supset G(\neg G(\mathbf{F}))]$; so $\vdash \neg [A \sim \neg G(\mathbf{F})] \supset [G(A) \supset G(\neg G(\mathbf{F}))]$, from which together with $\vdash [A \sim \neg G(\mathbf{F})] \supset [G(A) \sim G(\neg G(\mathbf{F}))]$ follows 2°).

3°) follows from $\vdash A \supset [G(A) \sim G(\mathbf{T})]$.

4°) From $\vdash \neg A \supset [G(A) \sim G(\mathbf{F})]$ and $\vdash G(\mathbf{T}) \wedge G(\mathbf{F}) \supset G(G(\mathbf{T}))$ it follows $\vdash \neg A \wedge G(\mathbf{T}) \supset [G(A) \supset G(G(\mathbf{T}))]$, while from 3°) it follows $\vdash A \wedge \neg G(\mathbf{T}) \supset [G(A) \supset G(G(\mathbf{T}))]$; so $\vdash \neg [A \sim G(\mathbf{T})] \supset [G(A) \supset G(G(\mathbf{T}))]$, from which together with $\vdash [A \sim G(\mathbf{T})] \supset [G(A) \sim G(G(\mathbf{T}))]$ follows 4°. □

Next, consider the following modified problem.

PROBLEM. *Let P be an n -ary predicate letter and $G(P)$ a formula. Does the set*

$$\{\lambda x_1 \cdots x_n A(x_1, \cdots, x_n) \mid \vdash G(\lambda x_1 \cdots x_n A(x_1, \cdots, x_n))\}$$

always have the least and the greatest elements unless it is empty?

The answer is “no” as the following proposition shows. The proof is similar to that of Proposition 1 since it is not the case that $\vdash \forall x[Qx \supset Rx] \vee \forall x[Rx \supset Qx]$.

PROPOSITION 3. *Let Q and R be distinct unary predicate letters and $G(P)$ be the formula $\forall x[Px \sim Qx] \vee \forall x[Px \sim Rx]$.*

1°) $\vdash G(\lambda x Qx)$ and $\vdash G(\lambda x Rx)$.

2°) *There is no $\lambda x B(x)$ such that (a) $\vdash G(\lambda x B(x))$ and (b) if $\vdash G(\lambda x A(x))$ then $\vdash \forall x[B(x) \supset A(x)]$ for every $\lambda x A(x)$.*

3°) *There is no $\lambda x B(x)$ such that (a) $\vdash G(\lambda x B(x))$ and (b') if $\vdash G(\lambda x A(x))$ then $\vdash \forall x[A(x) \supset B(x)]$ for every $\lambda x A(x)$.*

Reference

- [1] Maehara, S., The predicate calculus with ϵ -symbol, J. Math. Soc. Japan 7 (1955), 323-344.

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