TSUKUBA J. MATH.
Vol. 8 No. 1 (1984). 95–100

$\boldsymbol{Z}$-KERNEL GROUPS OF MEASURABLE CARDINALITIES

By

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$\boldsymbol{Z}$-kernel groups are groups obtained by transfinitely iterating use of direct products and sums starting from the group of integers $\boldsymbol{Z}$. In [2] the author defined type for a $\boldsymbol{Z}$-kernel group and proved its uniqueness for a $\boldsymbol{Z}$-kernel group of cardinality less than the least measurable cardinal.

In the present paper we show the uniqueness of type for more general $\boldsymbol{Z}$-kernel groups, e.g., $\prod_{\Lambda_1}\bigoplus_{\Lambda_2}\cdots_{\Lambda_n}\boldsymbol{Z}$ for arbitrary $\Lambda_1, \cdots, \Lambda_n$. In addition we show that the $\boldsymbol{Z}$-dual of such a $\boldsymbol{Z}$-kernel group again becomes a $\boldsymbol{Z}$-kernel group. One of our tools is Zimmermann's trick extended for an arbitrary cardinality from group theory and the other is finitely iterated ultrapowers of the universe from set theory. Our notation and terminology are common with [3] and undefined ones are usual ones in group theory [4] and set theory [5].

DEFINITION 1 [1]. A $\boldsymbol{Z}$-kernel group is a group obtained in the following manner:

(1) The group of integers $\boldsymbol{Z}$ is a $\boldsymbol{Z}$-kernel group;

(2) If G_α is a $\boldsymbol{Z}$-kernel gruop for each $\alpha\in\Lambda$, then $\prod_{\alpha\in\Lambda}G_\alpha$ and $\bigoplus_{\alpha\in\Lambda}G_\alpha$ are $\boldsymbol{Z}$-kernel groups, where Λ is nonempty.

Without loss of generality we may assume that Λ is an ordinal, since we work in ZFC-set theory.

DEFINITION 2 [2]. A type is a pair $(\mu, P), (\mu, S)$ or (μ, M), where μ is an ordinal. For types (μ, X) and $(\nu, Y), (\mu, X)<(\nu, Y)$ holds if $\mu<\nu$, or $\mu=\nu$ and $X\neq M$ and $Y=M$. We say that μ is the ordinal part of a type (μ, X).

Next we define a proper $\boldsymbol{Z}$-kernel ($\boldsymbol{pZk}$) group with type. We denote type of a $\boldsymbol{pZk}$ group G by typ(G) and the ordinal part of it by typ*(G). A rigorous reader should think that $\boldsymbol{Z}$-kernel groups and $\boldsymbol{pZk}$ groups are not just groups but groups with their definitions. Therefore, when we say that two $\boldsymbol{Z}$-kernel groups are isomorphic, it means that group parts of them are isomorphic.

Received June 1983.

DEFINITION 3. A proper $\boldsymbol{Z}$-kernel group ($\boldsymbol{pZk}$) group is a group obtained in the following manner:

(1) For an infinite ordinal Λ, $\prod_{\Lambda}\boldsymbol{Z}$ is a $\boldsymbol{pZk}$ group of type $(1,P)$ and $\bigoplus_{\Lambda}\boldsymbol{Z}$ is a $\boldsymbol{pZk}$ group of type $(1,S)$;

(2) Let Λ be an arbitrary infinite ordinal and G_α a $\boldsymbol{pZk}$ group for each $\alpha<\Lambda$.

(a) $\sup\{\mathrm{typ}^*(G_\alpha):\alpha<\Lambda\}=\mu+1$.

If $\{\alpha:\mathrm{typ}(G_\alpha)=(\mu,S)\}$ is infinite, then $\prod_{\alpha<\Lambda}G_\alpha$ is a $\boldsymbol{pZk}$ group of type $(\mu+1,P)$. If $\{\alpha:\mathrm{typ}(G_\alpha)=(\mu,P)\}$ is infinite, then $\bigoplus_{\alpha<\Lambda}G_\alpha$ is a $\boldsymbol{pZk}$ group of type $(\mu+1,S)$.

(b) $\sup\{\mathrm{typ}^*(G_\alpha):\alpha<\Lambda\}=\mu$: limit.

$\prod_{\alpha<\Lambda}G_\alpha$ is a $\boldsymbol{pZk}$ group of type (μ,P). $\bigoplus_{\alpha<\Lambda}G_\alpha$ is a $\boldsymbol{pZk}$ group of type (μ,S).

In this paper M stands for a transitive model of set theory of class type. For a definable operation or notion Ψ, Ψ^M is the one relativized to M as usual. The Boolean algebra of all subsets of X is denoted by $\boldsymbol{P}(X)$.

DEFINITION 4. By a finitely obtainable universe (*UFO*), we mean a transitive model M obtained in the following manner:

(1) The universe V is a *UFO*.

(2) Let M be a *UFO* and F a countable complete maximal filter ($c.c.$ max-filter) of $\boldsymbol{P}^M(\Lambda)$ for some ordinal Λ. Then, the transitive collapse M_F of an ultrapower of M($i.e.$, $(M^\Lambda)^M/F$) is a *UFO*.

Since the super class of *UFO*'s is definable in *ZFC*, our proofs in the following can be performed in *ZFC*.

LEMMA 1. For a *UFO* M, every function from the least measurable cardinal M_c to M belongs to M($i.e.$, $M^{M_c}\subseteq M$). Especially, $j(\alpha)=\alpha$ for $\alpha<M_c$ where $j: V\to M$ is the elementary embedding.

PROOF. By induction on the definition of a *UFO*. Since $M^{M_c}\subseteq M$, the countably completeness of F implies the M_c-completeness. Let $j_F: V\to M_F(\simeq(M^\Lambda)^M/F)$ be the elementary embedding, then $j_F(\alpha)=\alpha$ for $\alpha<M_c$ and $(M_F)^{M_c}\subseteq M_F$ holds by the same argument as Proposition 1.7 of [6].

LEMMA 2. Let M be a *UFO* and G a ($\boldsymbol{Z}$-kernel)^M group. Then, G is a reduced torsion free group.

The proof is clear.

LEMMA 3. Let $M_k(1\leq k\leq n)$ be *UFO*'s and Λ_k ordinals. Let a sequence of groups

$(A_\alpha^k : \alpha < \Lambda_k)$ be in M_k for each $1 \leq k \leq n$ and $G_j (j \in J)$ reduced torsion free groups. If $h : \prod_{\alpha<\Lambda}^{M_1} A_\alpha^1 \oplus \cdots \oplus \prod_{\alpha<\Lambda}^{M_n} A_\alpha^n \to \bigoplus_{j\in J} G_j$ is a homomorphism, then there exist $c.c.$ max-filters $F_1, \cdots, F_m$ of $\boldsymbol{P}^{M_1}(\Lambda_1) \times \cdots \times \boldsymbol{P}^{M_n}(\Lambda_n)$ and a finite subset $\bar{J}$ of J such that $h(K_{F_1} \cdots {}_{F_m}) \subseteq \bigoplus_{j\in\bar{J}} G_j$, where $x \in K_{F_1} \cdots {}_{F_m}$ iff $\{\alpha : x(\alpha)=0\} \in F_i$ for every $1 \leq i \leq m$. In addition we may regard that F_i is a $c.c.$ max-filter of $\boldsymbol{P}^{M_k}(\Lambda_k)$ for some but unique k.

Proof. To apply Theorem 1 (3) of [3], we define a quasi sheaf (S, ρ) as follows: For $(X_1, \cdots, X_n) \in \boldsymbol{P}^{M_1}(\Lambda_1) \times \cdots \times \boldsymbol{P}^{M_n}(\Lambda_n)$ $S(X_1, \cdots, X_n) = \prod_{\alpha\in X_1}^{M_1} A_\alpha^n \oplus \cdots \oplus \prod_{\alpha\in X_n}^{M_n} A^n$ and $\rho_{(Y_1,\cdots,Y_n)}^{(X_1,\cdots,X_n)}$ the restriction for $Y_k \subseteq X_k (1 \leq k \leq n)$. Since $\boldsymbol{P}^{M_1}(\Lambda_1) \times \cdots \times \boldsymbol{P}^{M_n}(\Lambda_n)$ is countably complete and (S, ρ) is a quasi sheaf over it, the lemma is clear by Theorem 1 (3) of [3]. An additional part holds, because F_i is a maximal filter for each i.

Lemma 4. Let $M, M_1, \cdots, M_n$ be *UFO*'s. Let G be a $(p\boldsymbol{Z}k)^M$ group with $\mathrm{typ}^{*M}(G) = \mu < M_c$. If $\mathrm{typ}^M(G) = (\mu, P)$, then G is not isomorphic to a summand of $G_1 \oplus \cdots \oplus G_n$ for any $(p\boldsymbol{Z}k)^{M_k}$ group G_k such that $\mathrm{typ}^{M_k}(G_k) \leq (\mu, S)$ for $1 \leq k \leq n$. In case $\mathrm{typ}^M(G) = (\mu, S)$, the dual statement holds.

Proof.

Case (1): $\mu = 1$. Since a finite sequence is absolute among transive models of set theory, a direct sum is absolute. Hence, $\mathrm{typ}^M(G) = (1, S)$ implies $G = \bigoplus_\Lambda \boldsymbol{Z}$ for some ordinal Λ. Let M' be a *UFO* and $h : \prod_\Lambda^{M'} \boldsymbol{Z} \to \bigoplus_{\Lambda'} \boldsymbol{Z}$ a homomorphism. Let $S(X) = \prod_X^{M'} \boldsymbol{Z}$ and ρ_Y^X the restriction for $X, Y \in \boldsymbol{P}^{M'}(\Lambda)$. Then, (S, ρ) is a quasi sheaf over $ccBa\, \boldsymbol{P}^{M'}(\Lambda)$ and $\hat{S} \simeq \prod_\Lambda^{M'} \boldsymbol{Z}$. Since a free group is slender, the range of h is of finite rank by of Theorem 1 (1) of [3]. Hence the theorem holds for both cases $\mathrm{typ}(G) = (1, P)$ and $\mathrm{typ}(G) = (1, S)$.

Case (2): μ is limit. Since the proof for this case is simple, we omit it.

Case (3): $\mu = \nu + 1$ and $\nu \neq 0$. Let $\mathrm{typ}^M(G) = (\mu, P)$ and $G = \prod_{\alpha<\Lambda}^M A_\alpha$ and $\mathrm{typ}^{*M}(A_\alpha) < \mathrm{typ}^{*M}(G)$ for $\alpha < \Lambda$. If $\mathrm{typ}^{M_k}(G_k) \leq (\nu, P)$ for every k, we can simply reduce the case to that of ν. Hence a critical case is that $\mathrm{typ}^{M_k}(G_k) = (\mu, S)$or (ν, S). In both cases we may assume that $G_k = \bigoplus_{j\in J_k} B_j^k$ and $\mathrm{typ}^{M_k}(B_j^k) \leq (\nu, P)$. Suppose that $G_1 \oplus \cdots \oplus G_n$ contains G as a summand. Then, by Lemma 3 there exist finite $J \subseteq \bigcup_{k=1}^n J_k$ and $c.c.$ max-filters $F_1, \cdots, F_m$ of $\boldsymbol{P}^M(\Lambda)$ such that $K_{F_1} \cdots {}_{F_m} \subseteq \bigoplus_{j\in J} B_j$, where for each $j \in J$ there are unique k and $i \in J_k$ such that $B_j = B_i^k$. Since there are infinitely many α such that $\mathrm{typ}^M(A_\alpha) = (\nu, S)$, there exists α_0 such that $\mathrm{typ}^M(A_{\alpha_0}) = (\nu, S)$ and $\{\alpha_0\} \notin F_k$ for every k, *i.e.*, $A_{\alpha_0} \subseteq K_{F_1} \cdots {}_{F_n}$. Since A_{α_0} is a summand of G, it is a summand of $\bigoplus_{j\in J} B_j$, which is a contradiction.

On the other hand, let $\mathrm{typ}^M(G) = (\mu, S)$ and $G = \bigoplus_{\alpha<\Lambda} B_\alpha$ and $\mathrm{typ}^{*M}(B_\alpha) < \mathrm{typ}^{*M}(G)$

for $\alpha<\Lambda$. As in the case of (μ, P), we may assume that $G_k=\prod_{\beta<\Lambda_k}{}^{M_k}A_\beta^k$ and $\mathrm{typ}^{M_k}(A_\beta^k)\leq(\nu, S)$ for $\beta<\Lambda_k$. Suppose that $G_1\oplus\cdots\oplus G_n$ contains G as a summand. Let $\sigma: G_1\oplus\cdots\oplus G_n\rightarrow G$ be the projection. By Lemma 3 there exist finite $J\subseteq\Lambda$ and $c.c.$ max-filters $F_1,\cdots F_m$ of a $ccBa$ $\boldsymbol{P}^{M_1}(\Lambda_1)\times\cdots\times\boldsymbol{P}^{M_n}(\Lambda_n)$ such that $\sigma(K_{F_1}\cdots{}_{F_m})\subseteq\oplus B_\alpha$. Since we assume that F_i is a $c.c.$ max-filter of $\boldsymbol{P}^{M_k}(\Lambda_k)$ for some k, $\hat{S}/F_i\simeq\prod_{\beta<\Lambda_k}{}^{M_k}A_\beta^k/F_i$, where S is a quasi sheaf as defined in the proof of Lemma 3. Let $p_J:\bigoplus_{\alpha<\Lambda}B_\alpha\rightarrow\bigoplus_{\alpha\in\Lambda-J}B_\alpha$ be the canonical projection. Since $F_1,\cdots,F_m$ are distinct, any element of $\hat{S}/K_{F_1}\cdots{}_{F_m}$ can be written of form $[x]_{F_1}+\cdots+[x]_{F_m}$ for some $x\in\hat{S}$, where $[x]_{F_i}$ is the coset relative to F_i. We now define $\tilde{\sigma}:\hat{S}/K_{F_1}\cdots{}_{F_m}(\simeq\hat{S}/F_1\oplus\cdots\oplus\hat{S}/F_m)\rightarrow\bigoplus_{\alpha\in\Lambda-J}B_\alpha$ as to be $\tilde{\sigma}([x]_{F_1}+\cdots+[x]_{F_m})=p_J\cdot\sigma(x)$. Then $\tilde{\sigma}$ is well-defined and a homomorphism. For the canonical homomorphism h: $\hat{S}\rightarrow\hat{S}/K_{F_1}\cdots{}_{F_m}$, the restriction of h to $\bigoplus_{\alpha\in\Lambda-J}B_\alpha$ is injective and $\tilde{\sigma}\cdot h$ is the identity on $\bigoplus_{\alpha\in\Lambda-J}B_\alpha$. Hence, $\bigoplus_{\alpha\in\Lambda-J}B_\alpha$ is isomorphic to a summand of $\hat{S}/F_1\oplus\cdots\oplus\hat{S}/F_m$. There exsts $\alpha_0\in\Lambda-J$ such that $\mathrm{typ}^M(B_{\alpha_0})=(\nu, P)$. Let $j_i:(M_k{}^{\Lambda_k})^{M_k}/F_i\rightarrow M_i'$ be the transitive collapsing isomorphism. Then, $j_i(\prod_{\beta<\Lambda_k}{}^{M}A_\beta^k/F_i)$ is a $\boldsymbol{pZk}$ group in M_i' and its type in M_i' is equal to or less than (ν, S) by Lemma 2. Now a contradiction occurs.

If $\mathrm{typ}^*(G)<M_c$ for a $\boldsymbol{pZk}$ group G, then G has only one type by virtue of Lemma 4. We restate Lemma 3 and Definition 3 of [2].

LEMMA 5 [2, Lemma 3]. Every $\boldsymbol{Z}$-kernel group of infinite rank is a $\boldsymbol{pZk}$ group or a direct sum of two $\boldsymbol{pZk}$ groups G and G' such that $\mathrm{typ}^*(G)=\mathrm{typ}^*(G')=\mu$, $\mathrm{typ}(G)=(\mu, P)$ and $\mathrm{typ}(G')=(\mu, S)$.

DEFINITION 5 [2]. Let G be a $\boldsymbol{Z}$-kernel group. If G is a $\boldsymbol{pZk}$ group, $\mathrm{typ}(G)$ has been already defined. Let $\mathrm{typ}(\boldsymbol{Z})=(0, P)=(0, S)^{(4)}$ and $\mathrm{typ}(G)=(0, M)$ for a $\boldsymbol{Z}$-kernel group $G(\neq\boldsymbol{Z})$ of finite rank. If G is of infinite rank and not a $\boldsymbol{pZk}$ group, then it is a direct sum of two $\boldsymbol{pZk}$ groups, the existence of which is assured by Lemma 5. Let $\mathrm{typ}(G)=(\mu, M)$, where μ is the same as in Lemma 5.

LEMMA 6. Let G be a $\boldsymbol{Z}$-kernel group. If $\mu\leq\mathrm{typ}^*(G)$, then there exists a summand of G which is isomorphic to some $\boldsymbol{pZk}$ group G' of type (μ, P) or (μ, S).

A straight proof by induction on the definition goes well.

Since Lemmas 5 and 6 hold in every *UFO*, we obtain the following theorem by Lemma 4.

THEOREM 1. Le G and G' be $\boldsymbol{Z}$-kernel groups in *UFO*'s M and M' respectively and $\mathrm{typ}^{*M}(G)<M_c$ or $\mathrm{typ}^{*M'}(G')<M_c$. If there exists a summand of G' which is isomorphic to G, then $\mathrm{typ}^M(G)\leq\mathrm{typ}^{M'}(G')$.

COROLLARY 1. Let G and G' be $\boldsymbol{Z}$-kernel groups and $\mathrm{typ}^*(G)<M_c$ or $\mathrm{typ}^*(G')<M_c$. If there exists a summand of G' which is isomorphic to G, then $\mathrm{typ}(G)\leq\mathrm{typ}(G')$. Consequently typ $(G)=\mathrm{typ}(G')$, if G' is isomorphic to G.

Next we study $\boldsymbol{Z}$-dual groups of $\boldsymbol{Z}$-kernel groups.

LEMMA 7. Let G be a $\boldsymbol{pZk}$ group in a *UFO* M and $\mathrm{typ}^{*M}(G)<M_c$. Then, $\mathrm{Hom}(G,\boldsymbol{Z})$ is a $\boldsymbol{pZk}$ group and $\mathrm{typ}(\mathrm{Hom}(G,\boldsymbol{Z}))=(\mu,P)$ or (μ,S), according to $\mathrm{typ}^M(G)=(\mu,S)$ or (μ,P).

PROOF. By induction on $\mathrm{typ}^{*M}(G)$. It is clear for $\mu=1$ and a routine in case that μ is limit. Let $\mu=\nu+1$. Let $G=\prod^M_{\alpha<\Lambda}G_\alpha$ and $\mathrm{typ}^M(G)=(\mu,P)$. We may assume that $\mathrm{typ}^M(G_\alpha)\leq(\nu,S)$ for each $\alpha<\Lambda$. Let $j_F:(M^\Lambda)^M/F\to M_F$ be the transitive collapsing isomorphism. Then, $\mathrm{typ}^{M_F}(j_F(\prod^M_{\alpha<\Lambda}G_\alpha/F))\leq(\nu,S)$ for every $c.c.$ max-filter F of $\boldsymbol{P}^M(\Lambda)$. Since there are infinitely many α such that $\mathrm{typ}^M(G_\alpha)=(\nu,S)$, there are infinitely many $c.c.$ max-filters F such that $\mathrm{typ}^{M_F}(j_F(\prod^M_{\alpha<\Lambda}G_\alpha/F))=(\nu,S)$. Now, $\mathrm{typ}(\mathrm{Hom}(G,\boldsymbol{Z}))=\mathrm{typ}(\bigoplus_{F\in\mathcal{F}}\mathrm{Hom}(\prod^M_{\alpha<\Lambda}G_\alpha/F,\boldsymbol{Z}))=(\mu,S)$, where $\mathcal{F}$ is the set of all $c.c.$ max-filters of $\boldsymbol{P}^M(\Lambda)$.

In case $\mathrm{typ}^M(G)=(\mu,S)$ it is similar and simpler.

By Lemmas 5 and 7 we obtain the following,

THEOREM 2. Let G be a $\boldsymbol{Z}$-kernel group in a *UFO* M and $\mathrm{typ}^{*M}(G)<M_c$. Then, $\mathrm{Hom}(G,\boldsymbol{Z})$ is a $\boldsymbol{Z}$-kernel group and typ $(\mathrm{Hom}(G,\boldsymbol{Z}))=(\mu,P),(\mu,S)$ or (μ,M) according to $\mathrm{typ}^M(G)=(\mu,S),(\mu,P)$ or (μ,M) respectively.

At the end we indicate a limitation of our method. Let G_α be a $\boldsymbol{pZk}$ group of type (μ,S) for each $\mu<M_c$ and $G=\prod_{\alpha<M_c}G_\alpha$. Let F be a non-normal, non-principal $c.c.$ max-filter of $\boldsymbol{P}(M_c)$, then the transitive collapse of $\prod_{\alpha<M_c}G_\alpha/F$ has type (μ,S) in M_F, where μ is greater than M_c. Hence, if $\mathrm{Hom}(G,\boldsymbol{Z})$ is a $\boldsymbol{Z}$-kernel group, it must have a higher type than (M_c,S). This implies the necessity of the condition $\mathrm{typ}^*(G)<M_c$ in Theorem 2. However, this does not refuse the possibility that Corollary 1 would hold beyond M_c. We do not know the answer, but think it possible.

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(A foot note)

p. 7 (i) Here we identify (0, P) and (0, S) as a special case.