ON THE DIFFERENCE $f^3(x)-g^2(x)$

By

Saburô UCHIYAMA and Masataka YORINAGA

In 1965 H. Davenport [2] proved that if $f(x)$, $g(x)$ are polynomials in x with arbitrary real or complex coefficients, then we have either

$$f^3(x)-g^2(x)=0 \quad \text{identically,}$$

or

(1) $$\deg\,(f^3(x)-g^2(x))\geqq \frac{1}{2}\deg f(x)+1\,.$$

It is known that for some pairs of polynomials $f(x)$, $g(x)$ the equality holds in (1). Clearly, we have for such pairs of polynomials

$$\deg f(x)=2k\,, \quad \deg g(x)=3k$$

with some integral $k\geqq 1$. Indeed, if

$$f(x)=x^2+2\,, \quad g(x)=x^3+3x\,,$$

then

$$f^3(x)-g^2(x)=3x^2+8\,,$$

and if

$$f(x)=x^4+2x\,, \quad g(x)=x^6+3x^3+\frac{3}{2}\,,$$

then

$$f^3(x)-g^2(x)=-x^3-\frac{9}{4}\,.$$

Some other examples of pairs of polynomials $f(x)$, $g(x)$ of higher degrees satisfying the condition

(2) $$\deg\,(f^3(x)-g^2(x))=\frac{1}{2}\deg f(x)+1$$

are given by B. J. Birch, S. Chowla, Marshall Hall, Jr. and A. Schinzel [1]. They have shown in fact that

$$f(x)=x^6+4x^4+10x^2+6\,,$$

$$g(x)=x^9+6x^7+21x^5+35x^3+\frac{63}{2}x\,,$$

and

$$f(x)=x^{10}+12x^7+60x^4+96x ,$$

$$g(x)=x^{15}+18x^{12}+144x^9+576x^6+1080x^3+432$$

are such pairs. It is interesting to note that, in all of these examples, $f(x)$ and $g(x)$ are polynomials of $Q[x]$, where Q denotes as usual the field of rational numbers. It can be shown that there is also a pair of such polynomials $f(x)$ and $g(x)$ of degrees 8 and 12, respectively, with coefficients in Q. Davenport [2] has found that there exist polynomials $f(x)$ and $g(x)$ satisfying the condition (2), being of degrees 16 and 24, respectively, and having coefficients in the field C of complex numbers. Actually, the polynomials $f(x)$, $g(x)$ in Davenport's example have coefficients in $Q(\sqrt{-3})$. However, the question of whether there exist pairs of polynomials $f(x)$, $g(x)$ with coefficients in C and with $\deg f(x)$, and so $\deg g(x)$ also, arbitrarily large which satisfy (2) remains still open (cf. [1] and [2]).

Our principal aim in this note is to indicate that for $k=7$ and 11 there exist pairs of polynomials $f(x)$, $g(x)$ with real algebraic coefficients such that

$$\deg f(x)=2k , \qquad \deg g(x)=3k ,$$

and the condition (2) is therewith fulfilled. It may be of some interest to note that, in order to produce pairs of polynomials $f(x)$, $g(x)$ of that kind, we have taken full advantage of making machine computations whatever possible, with the aid of programmes in REDUCE-2, a language designed and used for algebraic manipulation of formulas.

1. General considerations.

We begin with describing some general methods, or algorithms, of finding particular pairs of polynomials $f(x)$, $g(x)$ that satisfy the condition (2). The first one is a slight modification of the method proposed in [1].

(I) Let k be a given integer $\geqq 1$ and take $\nu=1, 2,$ or 3. Define

$$h(x)=x^{5k-\nu}+t_1x^{5k-\nu-1}+ \cdots +t_{5k-\nu} ,$$

where the t_j's are parameters whose values are to be determined later. We write

$$h^2(x)=\sum_{i=0}^{10k-2\nu} a_ix^{10k-2\nu-i}=x^{8k-2\nu}f(x)+A(x)$$

with

$$f(x)=\sum_{i=0}^{2k} a_i x^{2k-i}, \qquad \deg A(x)\leqq 8k-2\nu-1,$$

and

$$h^3(x)=\sum_{j=0}^{15k-3\nu} b_j x^{15k-3\nu-j}=x^{12k-3\nu}g(x)+B(x)$$

with

$$g(x)=\sum_{j=0}^{3k} b_j x^{3k-j}, \qquad \deg B(x)\leqq 12k-3\nu-1.$$

We have then

$$f^3(x)-g^2(x)=x^{-24k+6\nu}H(x),$$

where

$$H(x)=(h^2(x)-A(x))^3-(h^3(x)-B(x))^2,$$

so that $\deg (f^3(x)-g^2(x))=k+1$ if and only if $\deg H(x)=25k-6\nu+1$. Therefore, a sufficient condition for $f(x)$, $g(x)$ to satisfy (2) is that

(3) $$a_{2k+1}=a_{2k+2}=\cdots=a_{5k-2}=0,$$

(4) $$b_{3k+1}=b_{3k+2}=\cdots=b_{5k-2}=0,$$

and

(5) $$3a_{5k-1}-2b_{5k-1}\neq 0.$$

There are $5k-4$ equations in (3) and (4) with $5k-\nu$ unknowns $t_1, t_2, \cdots, t_{5k-\nu}$ ($\nu=1, 2,$ or 3), and we may find in general the values of the t_j satisfying (3), (4) and (5). With $\nu=1$, for instance, the conditions (3), (4) and (5) can be replaced by

$$a_{2k+1}=a_{2k+2}=\cdots=a_{5k-1}=0$$

and

$$b_{3k+1}=b_{3k+2}=\cdots=b_{5k-2}=0, \qquad b_{5k-1}\neq 0,$$

or by

$$a_{2k+1}=a_{2k+2}=\cdots=a_{5k-2}=0, \qquad a_{5k-1}\neq 0$$

and

$$b_{3k+1}=b_{3k+2}=\cdots=b_{5k-1}=0.$$

Note that we always have

$$a_m=\sum_{\substack{i+j=m\\ i\leqq j}} (2-\delta_{ij})t_i t_j, \qquad b_n=\sum_{i+j=n} a_i t_j,$$

where $a_0=b_0=t_0=1$, and $\delta_{ij}=1$ if $i=j$, and $=0$ if $i\neq j$.

The number of parameters t_j, which is the same as the degree of the basic polynomial $h(x)$, can be reduced as low as to $3k-1$, at the cost of imposing on the coefficients of $h^2(x)$ and of $h^3(x)$ somewhat more complicated conditions to

satisfy than (3), (4) and (5).

A variant of the above method is the following. First, take $k=2l+1$, $l\geq 0$, and put

$$h(x)=x^{5l+1}+t_1x^{5l}+\cdots+t_{5l+1}.$$

We write as before

$$h^2(x)=\sum_{i=0}^{10l+2}a_ix^{10l+2-i}=x^{8l+1}f_0(x)+A(x)$$

with

$$f_0(x)=\sum_{i=0}^{2l+1}a_ix^{2l+1-i},\qquad \deg A(x)\leq 8l,$$

and

$$h^3(x)=\sum_{j=0}^{15l+3}b_jx^{15l+3-j}=x^{12l+2}g_0(x)+B(x)$$

with

$$g_0(x)=\sum_{j=0}^{3l+1}b_jx^{3l+1-j},\qquad \deg B(x)\leq 12l+1.$$

Then, with $f(x)=f_0(x^2)$, $g(x)=xg_0(x^2)$ we have $\deg f(x)=2k$, $\deg g(x)=3k$, and

$$f^3(x)-g^2(x)=x^{-48l-6}H(x),$$

where

$$H(x)=(h^2(x^2)-A(x^2))^3-(h^3(x^2)-B(x^2))^2.$$

We have, therefore, $\deg(f^3(x)-g^2(x))=k+1$ if and only if $\deg H(x)=50l+8$, and sufficient condition for $f(x)$, $g(x)$ to satisfy (2) is that

$$a_{2l+2}=a_{2l+3}=\cdots=a_{5l+1}=0$$

and

$$b_{3l+1}\neq 0,\qquad b_{3l+2}=b_{3l+3}=\cdots=b_{5l+1}=0.$$

Next, taking $k=3l+2$, $l\geq 0$, we define, with

$$h(x)=x^{5l+2}+t_1x^{5l+1}+\cdots+t_{5l+2},$$

$$h^2(x)=\sum_{i=0}^{10l+4}a_ix^{10l+4-i}=x^{8l+3}f_0(x)+A(x),$$

$$f(x)=xf_0(x^3),\qquad f_0(x)=\sum_{i=0}^{2l+1}a_ix^{2l+1-i},\qquad \deg A(x)\leq 8l+2,$$

and

$$h^3(x)=\sum_{j=0}^{15l+6}b_jx^{15l+6-j}=x^{12l+4}g_0(x)+B(x),$$

$$g(x)=g_0(x^3),\qquad g_0(x)=\sum_{j=0}^{3l+2}b_jx^{3l+2-j},\qquad \deg B(x)\leq 12l+3,$$

so that $\deg f(x)=2k$, $\deg g(x)=3k$, and

$$f^3(x)-g^2(x)=x^{-72l-24}H(x),$$

where

$$H(x)=(h^2(x^3)-A(x^3))^3-(h^3(x^3)-B(x^3))^2.$$

It follows from this that $\deg(f^3(x)-g^2(x))=k+1$ if and only if $\deg H(x)=75l+27$, and a sufficient condition for $f(x)$, $g(x)$ to satisfy (2) is that

$$a_{2l+1}\neq 0, \qquad a_{2l+2}=a_{2l+3}=\cdots=a_{5l+2}=0$$

and

$$b_{3l+3}=b_{3l+4}=\cdots=b_{5l+2}=0.$$

(II) Let k be again a given positive integer and consider the polynomials with coefficients in C

$$f(x)=\sum_{i=0}^{2k}a_i x^{2k-i}, \qquad g(x)=\sum_{j=0}^{3k}b_j x^{3k-j},$$

where it is assumed that $a_0=b_0=1$. We wish to show that the pair of polynomials $f(x)$, $g(x)$ satisfies the condition (2), if and only if

(6) $$3f'(x)g(x)-2f(x)g'(x)=c,$$

where c is a non-zero constant.

For any monic polynomial $P=P(x)$ of degree $n\geq 1$ and with coefficients in C we have

$$P(x)=\prod_{i=1}^{n}(x-\xi_i)$$

and therewith define

$$s_0(P)=n \quad \text{and} \quad s_\nu(P)=\sum_{i=1}^{n}\xi_i^\nu \qquad (\nu=1, 2, 3, \cdots).$$

Now, in order to have

$$\deg(f^3(x)-g^2(x))=k+1$$

it is necessary and sufficient that the coefficients of x^m $(k+2\leq m\leq 6k-1)$ in $f^3(x)-g^2(x)$ do vanish and that of x^{k+1} does not, which is obviously equivalent to

(7) $$s_\nu(f^3)-s_\nu(g^2) \begin{cases} =0 & (1\leq\nu\leq 5k-2), \\ \neq 0 & (\nu=5k-1). \end{cases}$$

Since we have $s_\nu(f^3)-s_\nu(g^2)=3s_\nu(f)-2s_\nu(g)$ for all values of ν, it follows from the formal power series expansion

$$\frac{3f'(x)}{f(x)} - \frac{2g'(x)}{g(x)} = \sum_{\nu=0}^{\infty} \frac{s_\nu(f^3) - s_\nu(g^2)}{x^{\nu+1}}$$

that (7) implies (6), and *vice versa.*

It is easily seen that if we write

$$3f'(x)g(x) - 2f(x)g'(x) = \sum_{m=0}^{5k-1} c_m x^{5k-1-m},$$

then

$$c_m = \sum_{i=u}^{v} (2m-5i) a_i b_{m-i} \qquad (0 \leq m \leq 5k-1),$$

where

$$u = \max(0, m-3k), \qquad v = \min(m, 2k).$$

We have

$$c_0 = 0 \qquad \text{automatically,}$$

and

$$c_{5k-1} = 3a_{2k-1}b_{3k} - 2a_{2k}b_{3k-1}.$$

Thus, the condition (6) is equivant to

$$c_1 = c_2 = \cdots = c_{5k-2} = 0, \qquad c_{5k-1} = c \quad (\neq 0).$$

Our equation (6), which in some cases is slightly more convenient to deal with than the original condition (2), may be regarded as an *indefinite* differential equation in polynomials $f(x)$ and $g(x)$. The equation (6) admits a polynomial solution $g(x)$ when the polynomial $f(x)$ is given in such a way that the integral

$$\int (f(x))^{-5/2} dx$$

is a so-called pseudo-hyperelliptic integral; however, though this way of approaching the problem seems to be effective, the situation is in reality not so simple as expected.

Now, suppose that $k = 2l+1$, $l \geq 0$, and put

$$f_0(x) = \sum_{i=0}^{2l+1} a_i x^{2l+1-i}, \qquad g_0(x) = \sum_{j=0}^{3l+1} b_j x^{3l+1-j}.$$

Then, the polynomials $f(x)$, $g(x)$, defined by

$$f(x) = f_0(x^2), \qquad g(x) = x g_0(x^2),$$

so that $\deg f(x) = 2k$, $\deg g(x) = 3k$, satisfy the equation (6), if and only if

$$c_1 = c_2 = \cdots = c_{5l+1} = 0, \qquad c_{5l+2} = \frac{1}{2} c \quad (\neq 0),$$

where

(8)
$$c_m = \sum_{i=u}^{v} (2m - 5i) a_i b_{m-i} \qquad (0 \le m \le 5l+2)$$

with

$$u = \max(0, m - 3l - 1), \qquad v = \min(m, 2l+1).$$

If $k = 3l+2$, $l \ge 0$, and if

$$f_0(x) = \sum_{i=0}^{2l+1} a_i x^{2l+1-i}, \qquad g_0(x) = \sum_{j=0}^{3l+2} b_j x^{3l+2-j},$$

then the polynomials $f(x)$, $g(x)$ defined by

$$f(x) = x f_0(x^3), \qquad g(x) = g_0(x^3),$$

so that $\deg f(x) = 2k$, $\deg g(x) = 3k$, satisfy the condition (6), if and only if

$$c_1 = c_2 = \cdots = c_{5l+2} = 0, \qquad c_{5l+3} = \frac{1}{3} c \quad (\ne 0),$$

where

(9)
$$c_m = \sum_{i=u}^{v} (2m - 5i) a_i b_{m-i} \qquad (0 \le m \le 5l+3)$$

with

$$u = \max(0, m - 3l - 2), \qquad v = \min(m, 2l+1).$$

2. Some specific examples.

First we shall give examples of pairs of polynomials $f(x)$, $g(x)$, satisfying the condition (2) with $\deg f(x) = 2k$, $\deg g(x) = 3k$, for some small values of k. Our method of determining such pairs of polynomials $f(x)$, $g(x)$ will chiefly be the method (II) which we have just described above.

(i) $k = 1$. Here, we have $k = 2l+1$, $l = 0$, and put

$$f_0(x) = a_0 x + a_1, \qquad g_0(x) = b_0 x + b_1,$$

where $a_0 = b_0 = 1$. The coefficients c_m given by (8) with $l = 0$ are:

$$c_0 = 0 \quad \text{automatically},$$

$$c_1 = 2 a_0 b_1 - 3 a_1 b_0 = 2 b_1 - 3 a_1,$$

$$c_2 = - a_1 b_1.$$

Accordingly, if we take

$$a_1 = 2z, \qquad b_1 = 3z \quad \text{with } z \in \mathbf{Z}, \ z \ne 0,$$

where $\mathbf{Z}$ denotes the set of rationnal integers, then we have

$$c_1=0, \qquad c_2=-6z^2\neq0.$$

Thus we have

$$f_0(x)=x+2z, \qquad g_0(x)=x+3z,$$

$$f(x)=f_0(x^2)=x^2+2z, \qquad g(x)=xg_0(x^2)=x^3+3zx,$$

and

$$f^3(x)-g^2(x)=3z^2x^2+8z^3,$$

$$3f'(x)g(x)-2f(x)g'(x)=2c_2=-12z^2.$$

We note that the method (I) will furnish as a general solution of (6), i.e. of (2),

$$f(x)=x^2+2t_1x+t_1^2+2t_2,$$

$$g(x)=x^3+3t_1x^2+3(t_1^2+t_2)x+t_1(t_1^2+3t_2),$$

where t_1, t_2 are free parameters with $t_2\neq0$.

(ii) $k=3$. In this case we have $k=2l+1$, $l=1$, and take

$$f_0(x)=a_0x^3+a_1x^2+a_2x+a_3,$$

$$g_0(x)=b_0x^4+b_1x^3+b_2x^2+b_3x+b_4,$$

where $a_0=b_0=1$. We have, by (8),

$$c_m=\sum_{i=u}^{v}(2m-5i)a_ib_{m-i} \qquad (0\leqq m\leqq7)$$

with $u=\max(0, m-4)$, $v=\min(m, 3)$, where $c_0=0$ and the c_m $(1\leqq m\leqq7)$ should satisfy

$$c_1=2b_1-3a_1=0,$$

$$c_2=4b_2-a_1b_1-6a_2=0,$$

$$c_3=6b_3+a_1b_2-4a_2b_1-9a_3=0,$$

$$c_4=8b_4+3a_1b_3-2a_2b_2-7a_3b_1=0,$$

$$c_5=5a_1b_4-5a_3b_2=0,$$

$$c_6=2a_2b_4-3a_3b_3=0,$$

$$c_7=-a_3b_4\neq0.$$

We introduce a new set of parameters t_j $(1\leqq j\leqq3)$ and write

$$a_1=2t_1,$$

$$a_2=t_1^2+2t_2,$$

$$a_3=2t_1t_2+2t_3.$$

Then, the equations $c_m = 0$ $(1 \leqq m \leqq 4)$ will give

$$b_1 = 3t_1 ,$$

$$b_2 = 3t_1^2 + 3t_2 ,$$

$$b_3 = t_1^3 + 6t_1t_2 + 3t_3 ,$$

$$b_4 = 3t_1^2 t_2 + 3t_1 t_3 + \frac{3}{2} t_2^2 ,$$

and the equation $c_5 = 0$ becomes, when reduced

$$t_2(t_1 t_2 + 2t_3) = 0 .$$

Now, if $t_2 = 0$ then $c_6 = 0$ implies $t_3 = 0$, giving $b_4 = 0$. So, we must have $t_2 \neq 0$, and

$$t_1 t_2 + 2t_3 = 0 .$$

On the other hand, it follows from $c_5 = c_6 = 0$ that

$$3t_1^2 t_2 + 3t_1 t_3 - 2t_2^2 = 0 .$$

Substituting in here $t_3 = (-1/2)t_1 t_2$, we get $t_2(3t_1^2 - 4t_2) = 0$, or

$$3t_1^2 - 4t_2 = 0 ;$$

a general integer solution of this last equation is

$$t_1 = 2z, \qquad t_2 = 3z^2 \qquad \text{with } z \in \mathbf{Z}, \ z \neq 0 ,$$

and so

$$t_3 = -3z^3 .$$

Hence

$$a_1 = 4z, \qquad a_2 = 10z^2, \qquad a^3 = 6z^3 :$$

$$b_1 = 6z, \qquad b_2 = 21z^2, \qquad b_3 = 35z^3, \qquad b_4 = \frac{63}{2} z^4 .$$

We thus have

$$f(x) = f_0(x^2) = x^6 + 4zx^4 + 10z^2 x^2 + 6z^3 ,$$

$$g(x) = x g_0(x^2) = x^9 + 6zx^7 + 21z^2 x^5 + 35z^3 x^3 + \frac{63}{2} z^4 x ,$$

where

$$3f'(x)g(x) - 2f(x)g'(x) = -378z^7 .$$

This with $z = 1$ is one of the examples given in [1].

(iii) $k = 2$. For this value of k we have $k = 3l + 2$, $l = 0$, and we easily find, referring to (9), a pair

$$f(x) = x^4 + 2zx,$$

$$g(x) = x^6 + 3zx^3 + \frac{3}{2}z^2$$

with $z \in Z$, $z \neq 0$. Here

$$3f'(x)g(x) - 2f(x)g'(x) = 9z^3.$$

(iv) $k=5$. In this case $k=3l+2$, $l=1$, and one may proceed in a manner similar to that for the case of $k=3$, with (9) in place of (8), to obtain a pair

$$f(x) = x^{10} + 6zx^7 + 15z^2x^4 + 12z^3x,$$

$$g(x) = x^{15} + 9zx^{12} + 36z^2x^9 + 72z^3x^6 + \frac{135}{2}z^4x^3 + \frac{27}{2}z^5$$

with $z \in Z$, $z \neq 0$, where

$$3f'(x)g(x) - 2f(x)g'(x) = 486z^8.$$

For $z=2$ this reduces to another example found in [1].

So far everything is quite simple and no machine computations are needed.

3. Further examples.

We are now going to describe the examples newly found of pairs of polynomials $f(x)$, $g(x)$ satisfying the condition (2), or equivalently the equation (6), such that

$$\deg f(x) = 2k, \qquad \deg g(x) = 3k,$$

where $k=4, 7, 8$ and 11. Our computations were done on an electronic computer HITAC M-200H (VOS 3) in the Information Processing Center, Hiroshima University, Hiroshima.

(v) $k=4$. We have

$$f(x) = \sum_{i=0}^{8} a_i x^{8-i}, \qquad g(x) = \sum_{j=0}^{12} b_j x^{12-j},$$

where $a_0 = b_0 = 1$ and, with free parameters s, t, $t \neq 0$,

$$a_1 = 8s + 2t,$$

$$a_2 = 28s^2 + 14st + 7t^2,$$

$$a_3 = 56s^3 + 42s^2t + 42st^2 + 6t^3,$$

$$a_4 = 70s^4 + 70s^3t + 105s^2t^2 + 30st^3 + 11t^4,$$

$$a_5 = 56s^5 + 70s^4t + 140s^3t^2 + 60s^2t^3 + 44st^4 - 4t^5,$$

$$a_6=28s^6+42s^5t+105s^4t^2+60s^3t^3+66s^2t^4-12st^5\,,$$

$$a_7=8s^7+14s^6t+42s^5t^2+30s^4t^3+44s^3t^4-12s^2t^5-12t^7\,,$$

$$a_8=s^8+2s^7t+7s^6t^2+6s^5t^3+11s^4t^4-4s^3t^5-12st^7+t^8\,;$$

$$b_1=12s+3t\,,$$

$$b_2=66s^2+33st+12t^2\,,$$

$$b_3=220s^3+165s^2t+120st^2+19t^3\,,$$

$$b_4=495s^4+495s^3t+540s^2t^2+171st^3+39t^4\,,$$

$$b_5=792s^5+990s^4t+1440s^3t^2+684s^2t^3+312st^4+24t^5\,,$$

$$b_6=924s^6+1386s^5t+2520s^4t^2+1596s^3t^3+1092s^2t^4+168st^5+30t^6\,,$$

$$b_7=792s^7+1386s^6t+3024s^5t^2+2394s^4t^3+2184s^3t^4+504s^2t^5+180st^6-36t^7\,,$$

$$b_8=495s^8+990s^7t+2520s^6t^2+2394s^5t^3+2730s^4t^4+840s^3t^5$$
$$+450s^2t^6-180st^7-15t^8\,,$$

$$b_9=220s^9+495s^8t+1440s^7t^2+1596s^6t^3+2184s^5t^4+840s^4t^5$$
$$+600s^3t^6-360s^2t^7-60st^8-60t^9\,,$$

$$b_{10}=66s^{10}+165s^9t+540s^8t^2+684s^7t^3+1092s^6t^4+504s^5t^5$$
$$+450s^4t^6-360s^3t^7-90s^2t^8-180st^9+\frac{27}{2}t^{10}\,,$$

$$b_{11}=12s^{11}+33s^{10}t+120s^9t^2+171s^8t^3+312s^7t^4+168s^6t^5$$
$$+180s^5t^6-180s^4t^7-60s^3t^8-180s^2t^9+27st^{10}-\frac{9}{2}t^{11}\,,$$

$$b_{12}=s^{12}+3s^{11}t+12s^{10}t^2+19s^9t^3+39s^8t^4+24s^7t^5+30s^6t^6$$
$$-36s^5t^7-15s^4t^8-60s^3t^9+\frac{27}{2}s^2t^{10}-\frac{9}{2}st^{11}+\frac{29}{2}t^{12}\,.$$

The polynomials $f(x)$, $g(x)$ satisfy

$$3f'(x)g(x)-2f(x)g'(x)=-513t^{19}\,.$$

(vi) $k=7$. For this value of k we have $k=2l+1$, $l=3$, and put

$$f(x)=\sum_{i=0}^{7}a_ix^{14-2i}\,,\qquad g(x)=\sum_{j=0}^{10}b_jx^{21-2j}$$

with $a_0=b_0=1$. The coefficients a_i $(1\leqq i\leqq 7)$ and b_j $(1\leqq j\leqq 10)$ are given in the following way.

Let r be any one of the roots of the irreducible quintic equation

$$x^5 - \frac{8}{3}x^3 + \frac{15}{4}x^2 - \frac{10}{3}x + \frac{4}{3} = 0,$$

which admits a uniqe real root in the interval $(-3, -2)$, and set

$$s = \left(\frac{24r^2 - 24r}{3r^2 + 2r - 4}\right)^{1/5}.$$

Then, with a free parameter $t \neq 0$,

$$a_1 = \frac{-144r^4 - 108r^3 + 288r^2 - 348r + 243}{16} s^4 t,$$

$$a_2 = \frac{46212r^4 + 36852r^3 - 88576r^2 + 108647r - 78957}{1248} s^3 t^2,$$

$$a_3 = \frac{-24072r^4 - 20172r^3 + 44764r^2 - 54790r + 40731}{312} s^2 t^3,$$

$$a_4 = \frac{103068r^4 + 88872r^3 - 187572r^2 + 230573r - 170597}{1248} s t^4,$$

$$a_5 = \frac{-23520r^4 - 20244r^3 + 41216r^2 - 54880r + 37653}{624} t^5,$$

$$a_6 = \frac{42576r^4 + 35100r^3 - 78496r^2 + 98508r - 77623}{9216} s^4 t^6,$$

$$a_7 = \frac{-60r^4 - 48r^3 + 112r^2 - 145r + 100}{96} s^3 t^7;$$

$$b_1 = \frac{-432r^4 - 324r^3 + 864r^2 - 1044r + 729}{32} s^4 t,$$

$$b_2 = \frac{9408r^4 + 7536r^3 - 17984r^2 + 22069r - 16083}{104} s^3 t^2,$$

$$b_3 = \frac{-469512r^4 - 397068r^3 + 862412r^2 - 1051214r + 788907}{1352} s^2 t^3,$$

$$b_4 = \frac{2210508r^4 + 1962390r^3 - 3987258r^2 + 4789471r - 3641527}{2704} s t^4,$$

$$b_5 = \frac{-250644r^4 - 229116r^3 + 443516r^2 - 530080r + 401607}{208} t^5,$$

$$b_6 = \frac{158004r^4 + 162756r^3 - 271392r^2 + 233244r - 314765}{2304} s^4 t^6,$$

$$b_7 = \frac{-97404r^4 - 91020r^3 + 174280r^2 - 187081r + 168035}{768} s^3 t^7,$$

$$b_8 = \frac{294840r^4+271572r^3-522144r^2+601874r-488901}{3072}s^2t^8,$$

$$b_9 = \frac{-257748r^4-236268r^3+437440r^2-557307r+420451}{9216}st^9,$$

$$b_{10} = \frac{60r^4+48r^3-40r^2+193r-100}{64}t^{10}.$$

Here we have

$$3f'(x)g(x)-2f(x)g'(x) = \frac{102r^4+102r^3-170r^2+187r-153}{48}s^3t^{17}.$$

(vii) $k=8$. Here, we have $k=3l+2$, $l=2$, and

$$f(x) = \sum_{i=0}^{5} a_i x^{16-3i}, \qquad g(x) = \sum_{j=0}^{8} b_j x^{24-3j},$$

where $a_0=b_0=1$ and, with $t \neq 0$,

$$a_1 = \frac{-26+6\sqrt{-3}}{3}t,$$

$$a_2 = \frac{241-105\sqrt{-3}}{9}t^2,$$

$$a_3 = \frac{-140+72\sqrt{-3}}{3}t^3,$$

$$a_4 = \frac{301-147\sqrt{-3}}{6}t^4,$$

$$a_5 = \frac{-64+24\sqrt{-3}}{3}t^5;$$

$$b_1 = (-13+3\sqrt{-3})t,$$

$$b_2 = \frac{383-183\sqrt{-3}}{6}t^2,$$

$$b_3 = \frac{-4600+3384\sqrt{-3}}{27}t^3,$$

$$b_4 = \frac{1835-1725\sqrt{-3}}{6}t^4,$$

$$b_5 = \frac{-1177+1221\sqrt{-3}}{3}t^5,$$

$$b_6 = \frac{1043-1008\sqrt{-3}}{3}t^6,$$

$$b_7 = (-178 + 138\sqrt{-3})t^7 ,$$

$$b_8 = (20 - 12\sqrt{-3})t^8 .$$

With these polynomials we find

$$3f'(x)g(x) - 2f(x)g'(x) = (-416 + 1248\sqrt{-3})t^{13} .$$

For $t=1$ our polynomials $f(x)$, $g(x)$ will reduce to the ones found by Davenport [2].

(viii) $k=11$. In this case we have $k=3l+2$, $l=3$, and

$$f(x) = \sum_{i=0}^{7} a_i x^{22-3i} , \qquad g(x) = \sum_{j=0}^{11} b_j x^{33-3j} ,$$

where $a_0 = b_0 = 1$ as before and the coefficients a_i ($1 \leqq i \leqq 7$) and b_j ($1 \leqq j \leqq 11$) are given as follows.

Let r be any one of the roots of the irreducible cubic equation

$$x^3 + 3x^2 + 7x + \frac{31}{3} = 0 ,$$

which has a unique real root in the interval $(-3, -2)$, and set

$$s = \left(\frac{24r^2 + 36r + 156}{7} \right)^{1/5} .$$

Then, with a free parameter $t \neq 0$,

$$a_1 = \frac{15r^2 + 18r - 49}{64} s^4 t ,$$

$$a_2 = \frac{-267r^2 + 1878r + 6661}{896} s^3 t^2 ,$$

$$a_3 = \frac{-369r^2 - 2566r - 5097}{224} s^2 t^3 ,$$

$$a_4 = \frac{4797r^2 + 18798r + 31093}{896} s t^4 ,$$

$$a_5 = \frac{-3063r^2 - 9162r - 15391}{448} t^5 ,$$

$$a_6 = \frac{693r^2 + 5502r + 11501}{4096} s^4 t^6 ,$$

$$a_7 = \frac{-3r^2 - 18r - 43}{64} s^3 t^7 ;$$

$$b_1 = \frac{45r^2+54r-147}{128} s^4t \,,$$

$$b_2 = \frac{-153r^2+1188r+3993}{224} s^3t^2 \,,$$

$$b_3 = \frac{-12141r^2-79248r-143775}{1568} s^2t^3 \,,$$

$$b_4 = \frac{146799r^2+536490r+720999}{3136} st^4 \,,$$

$$b_5 = \frac{-7641r^2-19494r-21249}{64} t^5 \,,$$

$$b_6 = \frac{20163r^2+77202r+90699}{1024} s^4t^6 \,,$$

$$b_7 = \frac{-23949r^2-36423r-75717}{1024} s^3t^7 \,,$$

$$b_8 = \frac{75141r^2+191070r+199485}{4096} s^2t^8 \,,$$

$$b_9 = \frac{-38467r^2-85074r-106315}{4096} st^9 \,,$$

$$b_{10} = \frac{2493r^2+4926r+8277}{896} t^{10} \,,$$

$$b_{11} = \frac{-3r^2-12r-21}{256} s^4t^{11} \,.$$

Here, we could replace, of course, t by st (or by $s^{-4}t$), thus eliminating s in the above expressions for the a_i and the b_j; this replacement, however, would not seem to bring any particular improvement on these expressions.[*]

We have

$$3f'(x)g(x)-2f(x)g'(x)= \frac{27r^2+18r+27}{128} s^2t^{18} \,.$$

References

[1] Birch, B. J., S. Chowla, Marshall Hall, Jr., and A. Schinzel, On the difference x^3 $-y^2$. Norske Vid. Selsk. Forh. (Trondheim), **38** (1965), 65-69.

[2] Davenport, H., On $f^3(t)-g^2(t)$. Norske Vid. Selsk. Forh. (Trondheim), **38** (1965), 86-87.

[*] The same remark applies also in expressing the coefficients of polynomials of our example for $k=7$ given in (vi) above.

Cf. also: The Collected Works of Harold Davenport, edited by B. J. Birch, H. Halberstam, C. A. Rogers, Vol. IV. Academic Press, London et al., 1977. Especially, pp. 1743-1744, 1757.

Institute of Mathematics
University of Tsukuba
Sakura-mura, Niihari-gun
Ibaraki Pref., 305 Japan

Deparment of Mathematics
Faculty of Education—Fukuyama Branch
Hiroshima University
Fukuyama, 720 Japan