TSUKUBA J. MATH.
Vol. 40 No. 2 (2017), 125–137

ON DOI-NAGANUMA LIFTING

By

Balesh Kumar and Murugesan Manickam

Abstract. In this paper, we extend the Doi-Naganuma lifting as suggested by Kudla [4], on the lines of Zagier's work [6]. For each fundamental discriminant D associated with a real quadratic field, we prove that there exists a Hecke-equivarient map ι_D which maps the mth Poincare series of weight k, level M and character $\chi_D = \left(\frac{\cdot}{D}\right)$ into a Hilbert cusp form of weight k, level M/D associated with the real quadratic field of discriminant D of class number one. Through this, we get its adjoint ι_D^* with respect to the Petersson inner product.

1. Introduction

In [6], Don Zagier derived the adjoint of the Doi-Naganuma lift by computing its explicit action on Poincare series. More precisely, he considered the space $S_k(D, \chi_D)$ of all cusp forms of weight k, level D, character $\chi_D = \left(\frac{\cdot}{D}\right)$, ($D > 0$ is a fundamental discriminant) and proved that the mth Poincare series in $S_k(D, \chi_D)$ maps into an explicit Hilbert cusp form ω_m in $S_k^{\mathcal{H}}(SL_2(\mathcal{O}))$—the space of Hilbert cusp forms of weight k, level 1 associated with the real quadratic field of discriminant D. He proved that Hecke-eigenforms correspond to each other under the Doi-Naganuma lift.

In this paper, we prove that for each fundamental discriminant D, there exist a Hecke-equivarient map ι_D, which maps the space $S_k(M, \chi_D)$ into $S_k^{\mathcal{H}}(\tilde{\Gamma}_0(M/D))$—the space of Hilbert cusp forms of weight k, level M/D, where M is a square-free positive integer divisible by D. We prove that ι_D takes the mth Poincare series in $S_k(M, \chi_D)$ into a similar kind of Hilbert cusp form ω_m in $S_k^{\mathcal{H}}(\tilde{\Gamma}_0(M/D))$ and then we prove that it is an Hecke equivarient map.

2000 *Mathematics Subject Classification*: 11F41, 11F32.
Key words and phrases: Doi-Naganuma lift, Hilbert modular forms.
Received March 24, 2016.
Revised September 8, 2016.

In his paper [4] Kudla mentioned the possibility of an extension lift of Zagier's type for arbitrary level and character. We treat the case where the level is a square free integer M and for each positive squarefree divisor $D \equiv 1 \pmod 4$ of M, we construct appropriate Hilbert cusp form ω_m and prove our results as done in [6]. The above results can also be seen in terms of Galois representations. For this, we refer to E. Ghate [3].

2. Definition and Properties of ω_m

We will use the following notation:

K a real quadratic number field;
D the discriminant of K;
$\mathcal{O}$ the ring of integers of K;
$\mathfrak{d}$ the different of K (the principal ideal $(\sqrt{D})$);
x' the Galois conjugate over $\mathbf{Q}$ of an element $x \in K$;
$N(x)$ the norm of x, $N(x) = xx'$;
$\mathbf{H}$ the upper half plane $\{z \in \mathbf{C} \mid \operatorname{Im} z > 0\}$;
$\mathbf{Z}$ the set of integers
k a fixed even integer > 2

For an integer $m \geq 0$ and for $z_1, z_2 \in \mathbf{H}$, we define

$$\omega_m(z_1, z_2) = \sideset{}{'}\sum_{\substack{a,\, b \in \mathbf{Z},\ \lambda \in \mathfrak{d}^{-1} \\ N(\lambda) - ab = m/D \\ N \mid a}} (az_1z_2 + \lambda z_1 + \lambda' z_2 + b)^{-k}, \tag{1}$$

where the summation is over all (a, b, λ) satisfying the given conditions, and the notation $\sum'$ indicates that, for $m = 0$, the triple $(0, 0, 0)$ is to be omitted. It can be easily checked that, for $z_1, z_2 \in \mathbf{H}$, the expression $az_1z_2 + \lambda z_1 + \lambda' z_2 + b$ never vanishes. Indeed, $az_1z_2 + \lambda z_1 + \lambda' z_2 + b = 0$ implies $z_1 = (-\lambda' z_2 - b)/(az_2 + \lambda)$, and this is impossible since the determinant of $\begin{pmatrix} -\lambda' & -b \\ a & \lambda \end{pmatrix}$ is ≤ 0 and that the series converges absolutely. Therefore ω_m is a holomorphic function in $\mathbf{H} \times \mathbf{H}$.

Theorem 1.

(i) *For each integer $m \geq 0$, $\omega_m(z_1, z_2)$ is a Hilbert modular form of weight k with respect to the congruence subgroup $\tilde{\Gamma}_0(N)$ of the Hilbert modular group $SL_2(\mathcal{O})$.*

(ii) *ω_m is a cusp form for $m > 0$.*

Proof. (i) It is clear from above that ω_m is holomorphic on $\mathbf{H} \times \mathbf{H}$. Also it satisfies the modularity condition as follows; for $z_1, z_2 \in \mathbf{H}$,

$$\omega_m((\alpha z_1+\beta)/(\gamma z_1+\delta),(\alpha' z_2+\beta')/(\gamma' z_2+\delta')) \tag{2}$$
$$=(\gamma z_1+\delta)^k(\gamma' z_2+\delta')^k\omega_m(z_1,z_2)$$

as follows. For any matrix $M=\begin{pmatrix} a & b \\ c & d\end{pmatrix}\in GL_2(\mathbf{R})$, let

$$\phi_M(z_1,z_2)=(\det M)^{-1}\frac{d}{dz_1}(z_2-Mz_1)^{-1}$$
$$=(cz_1z_2-az_1+dz_2-b)^{-2}$$

where $Mz_1=(az_1+b)/(cz_1+d)$;

One easily checks that, for $A_1=\begin{pmatrix} \alpha_1 & \beta_1 \\ \gamma_1 & \delta_1\end{pmatrix}$, $A_2=\begin{pmatrix} \alpha_2 & \beta_2 \\ \gamma_2 & \delta_2\end{pmatrix}\in GL_2(\mathbf{R})$,

$$\phi_M(A_1z_1,A_2z_2)=(\gamma_1z_1+\delta_1)^2(\gamma_2z_2+\delta_2)^2\phi_{A_2^*MA_1}(z_1,z_2), \tag{3}$$

where $A_2^*=\begin{pmatrix} \delta_2 & -\beta_2 \\ -\gamma_2 & \alpha_2\end{pmatrix}=(\det A_2)A_2^{-1}$ is the adjoint of A_2. Let

$$\mathscr{A}=\left\{M=\begin{pmatrix} \alpha & \beta \\ \gamma & \delta\end{pmatrix}\in\mathfrak{M}_2(\mathcal{O})\mid N\mid\gamma, M^*=M'\right\}$$

be the set of matrices whose adjoint equal their conjugates over $\mathbf{Q}$. A typical matrix of $\mathscr{A}$ has the form $\begin{pmatrix} \theta & b\sqrt{D} \\ -a\sqrt{D} & \theta'\end{pmatrix}$ with $a,b\in\mathbf{Z}$ and $N|a,\theta\in\mathcal{O}$. Write $\theta=-\lambda\sqrt{D}$ with $\lambda\in\mathfrak{d}^{-1}$; then $\phi_M(z_1,z_2)=D^{-1}(az_1z_2+\lambda z_1+\lambda' z_2+b)^{-2}$. Hence

$$\omega_m(z_1,z_2)=D^{k/2}\sum\nolimits'_{\substack{M\in\mathscr{A}\\ \det M=-m}}\phi_M(z_1,z_2)^{k/2},$$

where $\sum'$ indicates that, for $m=0$, the zero matrix is to be omitted from the summation. That ω_m satisfies the modularity condition (2) now follows immediately from Eq. (3). Hence ω_m is automatically holomorphic at the cusps of $\tilde{\Gamma}_0(N)$, by the Götzky-Koecher principle. Therefore, ω_m is a Hilbert modular form for the congruence subgroup $\tilde{\Gamma}_0(N)$ of the Hilbert modular group $SL_2\mathcal{O}$.

Since ω_m for $m>0$ is a Hilbert modular form for $\tilde{\Gamma}_0(N)$, we have ω_m is invariant with respect to matrices $\begin{pmatrix} \varepsilon & \mu \\ 0 & \varepsilon^{-1}\end{pmatrix}$ where $\varepsilon\in\mathcal{O}^*$, $\mu\in\mathcal{O}$. That is

$$\omega_m(\varepsilon^2z_1+\varepsilon\mu,\varepsilon'^2z_2+\varepsilon'\mu')=\omega_m(z_1,z_2)$$

Therefore, ω_m has a Fourier expansion at the cusp ∞ of the form

$$\omega_m(z_1, z_2) = c_{m0} + \sum_{\substack{\nu \in \mathfrak{d}^{-1} \\ \nu \gg 0}} c_{m\nu} e^{2\pi i(\nu z_1 + \nu' z_2)} \tag{4}$$

by the Götzky-Koecher principle.

Let be $W = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL_2(K)$ then

$$\begin{aligned}
(\omega_m|W)(z_1, z_2) &= D^{k/2}(\gamma z_1 + \delta)^{-k}(\gamma' z_2 + \delta')^{-k} \sum\nolimits'_{\substack{M \in \mathscr{A} \\ \det M = -m}} \phi_M(Wz_1, W'z_2) \\
&= D^{k/2} \sum\nolimits'_{\substack{M \in \mathscr{A} \\ \det M = -m}} \phi_{W'^* MW}(z_1, z_2)^{k/2} \\
&= D^{k/2} \sum\nolimits'_{\substack{M \in \mathscr{A}_1 \\ \det M = -m}} \phi_M(z_1, z_2)^{k/2}
\end{aligned}$$

where

$$\begin{aligned}
\mathscr{A}_1 &= W'^* \mathscr{A} W \\
&= W'^{-1} \mathscr{A} W \\
&= W'^{-1} \left\{ M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathfrak{M}_2(\mathcal{O}) \,|\, N \,|\, \gamma, M' = M^* \right\} W
\end{aligned}$$

A typical matrix $M \in \mathscr{A}_1$ has the form

$$M = \begin{pmatrix} \theta & b\sqrt{D} \\ -a\sqrt{D} & \theta' \end{pmatrix}, \quad \theta \in K,\ a, b \in \mathbf{Q}.$$

Writing $\theta = \lambda\sqrt{D}$, we obtain

$$(\omega_m|W)(z_1, z_2) = \sum\nolimits'_{\substack{(a,b,\lambda) \in \mathbf{L}, \\ N(\lambda) - ab = m/D}} (az_1 z_2 + \lambda z_1 + \lambda' z_2 + b)^{-k}$$

where $\mathbf{L} \subset \mathbf{Q} \times \mathbf{Q} \times K$ is the lattice (i.e. free $\mathbf{Z}$-module of rank 4) of triples (a, b, λ) for which $W' \begin{pmatrix} \lambda\sqrt{D} & b\sqrt{D} \\ -a\sqrt{D} & -\lambda'\sqrt{D} \end{pmatrix} W^{-1} \in \mathfrak{M}_2(\mathcal{O})$. To show that ω_m is a cusp form, enough to show that $c_{m0} = 0$ for the cusp at ∞, because of the similarity between $(\omega_m|W)(z_1, z_2)$ and $\omega_m(z_1, z_2)$, it will be clear that the method used to find the fourier expansion of ω_m can be applied to prove that $(\omega_m|W)(z_1, z_2)$ has a Fourier series whose constant term vanishes. The Fourier coefficients of ω_m for the cusp at ∞ will be computed in the next section. □

3. The Fourier Coefficient of ω_m

THEOREM 2. *For $m > 0$, the Fourier coefficient of $c_{m\nu}$ of $\omega_m(z_1, z_2)$ defined by Eq.* (4) *is given by*

$$c_{m\nu} = (2(2\pi)^k)/((k-1)!)\left\{(-1)^{k/2}\sum_{\substack{r\in\mathbf{N},\, r|\nu\sqrt{D}\\ N(\nu\sqrt{D}/r)=-m}} r^{k-1} + 2\pi D^{(k/2)-1}(N(\nu)/m)^{(k-1)/2}\right.$$
$$\left.\times \sum_{a=1,\, N|a}^{\infty} a^{-1}J_{k-1}((4\pi/a)\sqrt{(mN(\nu))/D})G_a(m,\nu)\right\}$$

if $\nu \gg 0$ and is zero otherwise, where J_{k-1} is the Bessel function of order $k-1$ defined by

$$J_{k-1}(t) = \sum_{r=0}^{\infty}((-1)^r(t/2)^{2r+k-1})/(r!(r+k-1)!)$$

and $G_a(m,\nu)$ is the finite exponential sum

$$G_a(m,\nu) = \sum_{\substack{\lambda\in\mathfrak{d}^{-1}/a\mathcal{O}\\ \lambda\lambda'\equiv m/D (mod\ a\mathbf{Z})}} e^{2\pi i\, Tr(\nu\lambda)/a}$$

PROOF. For $m > 0$, write

$$\omega_m(z_1,z_2) = \sum_{a\in\mathbf{Z},\, N|a}\omega_m^a(z_1,z_2)$$
$$= \omega_m^0(z_1,z_2) + 2\sum_{a=1,\, N|a}^{\infty}\omega_m^a(z_1,z_2),$$

where

$$\omega_m^a(z_1,z_2) = {\sum}'_{\substack{b\in\mathbf{Z},\,\lambda\in\mathfrak{d}^{-1}\\ \lambda\lambda'-ab=m/D}}(az_1z_2+\lambda z_1+\lambda' z_2+b)^{-k}. \quad (5)$$

Observe that ω_m^a satisfies the periodicity property, that is $\omega_m^a(z_1+\theta, z_2+\theta') = \omega_m^a(z_1,z_2)$ $(\theta\in\mathcal{O})$, and hence each ω_m^a has a Fourier expansion

$$\omega_m^a(z_1,z_2) = \sum_{\nu\in\mathfrak{d}^{-1}} c_{m\nu}^a e^{2\pi i(\nu z_1+\nu' z_2)} \quad (6)$$

The Fourier coefficients of ω_m are then given by

$$c_{m\nu} = c_{m\nu}^0 + 2\sum_{a=1}^{\infty} c_{m\nu}^a$$

The rest of the proof follows by the following two propositions as done in [6].

PROPOSITION 1. *For* $m > 0$, $\nu \in \mathfrak{d}^{-1}$, *the Fourier coefficient* $c^0_{m\nu}$ *defined by* (6) *is zero unless* $\nu \gg 0$ *and* $\nu = r\lambda$ *with* $r \in \mathbf{N}$, $\lambda \in \mathfrak{d}^{-1}$, $\lambda\lambda' = m/D$, *in which case*

$$c^0_{m\nu} = 2c_k r^{k-1}, \quad c_k = (2\pi i)^k/(k-1)!.$$

PROPOSITION 2. *For* $m > 0$, $\nu \in \mathfrak{d}^{-1}$, $a > 0$, *the fourier coefficient* $c^a_{m\nu}$ *defined by* (6) *is zero unless* $\nu \gg 0$ *and is then given by*

$$c^a_{m\nu} = (2\pi)^{k+1}/((k-1)!)(D^{(k/2)-1}/a)(N(\nu)/m)^{(k-1)/2}$$
$$\times G_a(m,\nu)J_{k-1}((4\pi/a)\sqrt{(mN(\nu))/D}). \qquad \square$$

4. Poincare Series for $\Gamma_0(M)$

Let M be a square-free positive integer. Let D be the fundamental discriminant of a real quadratic field K such that $D \equiv 1 \pmod 4$ and dividing M.

Let $\chi : \Gamma_0(M) \to \{\pm 1\}$ be such that

$$\chi\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \epsilon(a) = \epsilon(d) \quad \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(M)\right).$$

where $\epsilon = \epsilon_D$ is the fundamental character of K with $\epsilon(p) = \left(\frac{p}{D}\right)$ for $p \nmid 2D$. The space $S_k(\Gamma_0(M), \chi)$ is usually denoted by $S(M, k, \epsilon)$. For $x/y, x'/y' \in \mathbf{Q} \cup \{\infty\}$ with $(x', y') = (x, y) = 1$, the equation

$$x'/y' = (ax + by)/(cx + dy), \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(M)$$

can be solved if and only if $(y, M) = (y', M)$. The equivalence classes of $\mathbf{Q} \cup \{\infty\}$ modulo $\Gamma_0(M)$ are thus described by the positive divisions of M. Let D_1 be a divisior of D. Let the cusp P be given by $D_1 N$, $(N = M/D)$ and write $D_2 = D/D_1$; then $(D_1 N, D_2) = 1$, since M is square-free, and we can find $p, q \in \mathbf{Z}$ such that $pD_1 N + qD_2 = 1$; choose

$$A_P = \begin{pmatrix} D_2 & -p \\ D_1 N & q \end{pmatrix} \in SL_2(\mathbf{Z}) \tag{7}$$

The cusp P is easily checked to have width D_2. we will denote the cusp simply by $D_1 N$; thus for $f \in S(M, k, \epsilon)$ and $D_1 | D$ we have the Fourier expansion

$$(f|A^{-1}_{D_1 N})(z) = \sum\nolimits_{n=1}^{\infty} a^{D_1 N}_n(f) e^{2\pi i n z/D_2}$$

The coefficients $a_n^{D_1N}(f)$ being independent of the choice of p, q in (7) and given by

(8) $$a_n^{D_1N}(f) = ((4\pi n)^{k-1}/(k-2)!)D_2^{-k}(f, G_n^{D_1N})$$

where $(f, G_n^{D_1N})$ denotes the Petersson inner product of f with $G_n^{D_1N}$ where $G_n^{D_1N}$ is the nth Poincare series at the cusp D_1N defined by

(9) $$G_n^{D_1N}(z) = 2^{-1}\sum\nolimits_{A\in\Gamma_{D_1N}\backslash A_{D_1N}\Gamma}\chi(A_{D_1N}^{-1}A)j(A,z)^{-k}e^{2\pi inAz/D_2}$$
$$= \sum\nolimits_{m=1}^{\infty} g_{nm}^{D_1N}e^{2\pi imz}$$

where $\Gamma = \Gamma_0(M)$, $\Gamma_{D_1N} = A_{D_1N}\Gamma A_{D_1N}^{-1}\cap\begin{pmatrix}1 & \mathbf{Z}\\ 0 & 1\end{pmatrix}$, $j(A,z) = (cz+d)$, where $A = \begin{pmatrix}a & b\\ c & d\end{pmatrix}\in SL_2(\mathbf{Z})$.

$$g_{nm}^{D_1N} = \delta_{D_1N,M}\delta_{n,m} + 2\pi(-1)^{k/2}((mD_2)/n)^{(k-1)/2}\sum\nolimits_{\substack{c=1\\(c,M)=D_1N}}^{\infty} H_c^{D_1N}(n,m)$$
$$\times J_{k-1}((4\pi/c)\sqrt{(mn)/D_2}),$$

(10) $$H_c^{D_1N}(n,m) = c^{-1}\sum\nolimits_{\substack{d(\mathrm{mod}\ c)\\(d,c)=1}}\chi\left(A_{D_1N}^{-1}\begin{pmatrix}a & b\\ c & d\end{pmatrix}\right)e^{2\pi ic^{-1}(na/D_2+md)}.$$

Let $M\geq 1$ be a square-free integer. Let D be a fundamental discriminant dividing M, that is, $D|M$. Let χ be the fundamental character which is a quadratic character of the associated quadratic field K. Write $N := M/D$.

Proposition 3.
(i)

$$H_{aD_1N}^{D_1N}(n,m) = (aD_1N)^{-1}\sum\nolimits_{\substack{d(mod\ aD_1N)\\(d,aD_1N)=1}}\chi\left(A_{D_1N}^{-1}\begin{pmatrix}a & b\\ aD_1N & d\end{pmatrix}\right)e^{2\pi i(aD_1N)^{-1}(naD_2^{-1}+md)}$$
$$= (aD_1N)^{-1}\left(\frac{aD_1N}{D_2}\right)\left(\frac{N}{D_2}\right)\sum\nolimits_{\substack{d(mod\ aD_1N)\\(d,aD_1N)=1}}\left(\frac{-d}{D_1}\right)$$
$$\times e^{2\pi i(aD_1N)^{-1}(nD_2^{-1}d^{-1}+md)}$$

(ii) *Write*

(11) $$H_b(n,m) = \sum\nolimits_{\substack{D=D_1D_2\\D_2|n,(b,D_2)=1}}(\psi(D_2)/D_2)H_{bD_1}^{D_1}(n/D_2,m),$$ *as defined in* [6],

then

$$H_{Na}(n,m) = \sum_{\substack{D=D_1D_2 \\ D_2|n,\,(a,D_2)=1}} \left(\frac{N}{D_2}\right)(\psi(D_2)/D_2) H_{aD_1N}^{D_1N}(n/D_2, m).$$

where $\psi(D_2)$ *is the Gauss sum defined by*

$$\psi(D_2) = \sum_{x(mod\ D_2)} \left(\frac{x}{D_2}\right) e^{-2\pi i D_1 x/D_2} = \begin{cases} \left(\frac{D_1}{D_2}\right)\sqrt{D_2} & if\ \ D_1 \equiv D_2 \equiv 1\ \ (mod\ 4) \\ -i\left(\frac{D_1}{D_2}\right)\sqrt{D_2} & if\ \ D_1 \equiv D_2 \equiv 3\ \ (mod\ 4) \end{cases}$$

$$= \left(\frac{-4}{D_2}\right)^{-1/2} \left(\frac{D_1}{D_2}\right)\sqrt{D_2}$$

Proof. (i) Since $A_{D_1N} = \begin{pmatrix} D_2 & -p \\ D_1N & q \end{pmatrix} \in SL_2(\mathbf{Z})$, $pD_1N + qD_2 = 1$. Therefore

$$A_{D_1N}^{-1}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} aq+pc & bq+dp \\ -aD_1N + cD_2 & -bD_1N + dD_2 \end{pmatrix} \in \Gamma_0(M)$$

only if $D_2|a$ (since $c = D_1N$.)

So a is determined by (mod cD_2) by $ad \equiv 1$ (mod c), $D_2|a$. So

$$\chi\left(A_{D_1N}^{-1}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \epsilon(aq+pc) = \left(\frac{aq+pc}{D}\right)$$

$$= \left(\frac{aq+pc}{D_1}\right)\left(\frac{aq+pc}{D_2}\right)$$

$$= \left(\frac{aq}{D_1}\right)\left(\frac{pc}{D_2}\right)$$

$$= \left(\frac{aD_2^{-1}}{D_1}\right)\left(\frac{c(D_1N)^{-1}}{D_2}\right)$$

$$= \left(\frac{(dD_2)^{-1}}{D_1}\right)\left(\frac{c(D_1N)^{-1}}{D_2}\right)$$

$$= \left(\frac{-d}{D_1}\right)\left(\frac{c}{D_2}\right)\left(\frac{N}{D_2}\right)$$

where in the last line we have used the quadratic reciprocity and $D_1D_2 \equiv 1$ (mod 4) to set $\left(\frac{D_1}{D_2}\right)\left(\frac{D_2}{D_1}\right) = \left(\frac{-1}{D_1}\right)$. Therefore

$$H^{D_1N}_{aD_1N}((Dvv')/(r^2D_2), m) = (aD_1N)^{-1}\left(\frac{aD_1N}{D_2}\right)\left(\frac{N}{D_2}\right)\sum\nolimits_{\substack{d(\text{mod } aD_1N)\\(d, aD_1N)=1}}\left(\frac{-d}{D_1}\right)$$
$$\times\, e^{2\pi i(aD_1N)^{-1}((Dvv')/(r^2D_2)D_2^{-1}d^{-1}+md)}.$$

(ii) Follows by the definition of $H_b(n,m)$. □

Proposition 4. *See* [6]. *For* $a, m \in \mathbf{Z}$, $v \in \mathfrak{d}^{-1}$, $a > 0$,

$$(a\sqrt{D})^{-1}G_a(m, v) = \sum\nolimits_{r|v,\, r|a} H_{a/r}((Dvv')/r^2, m).$$

where $G_a(m, v)$ *is the sum defined as in theorem* (2), *and* $H_b(n, m)$ *is the sum as defined in eqn* (11).

5. The Doi-Naganuma Lifting

Let $M \geq 1$ be a square-free integer. Let D be a fundamental discriminant dividing M. Let χ_D be the fundamental quadratic character associated with the quadratic field $K = \mathbf{Q}\sqrt{D}$ of class number one. Write $N := M/D$ and $S_k^{\mathcal{H}}(\tilde{\Gamma}_0(N))$ denote the space of Hilbert cusp forms of weight k for the congruence subgroup $\tilde{\Gamma}_0(N)$ for the Hilbert modular group $SL_2(\mathcal{O})$. Let $f \in S_k(M, \chi_D)$, then the Doi-Naganuma lifting of f is defined by

$$\iota_D(f) = \sum\nolimits_{\substack{v\in\mathfrak{d}^{-1}\\ v\gg 0}} c((v)\mathfrak{d})e^{2\pi i(vz_1+v'z_2)},$$

where

$$c((v)\mathfrak{d}) = \sum\nolimits_{r|(v)\mathfrak{d}} r^{k-1}\sum\nolimits_{D_2|(D,\, N(v)N(\mathfrak{d})/r^2)}\left(\frac{N}{D_2}\right)\psi(D_2)D_2^{k-1}a^{D_1N}_{(N(v)N(\mathfrak{d}))/(r^2D_2)}(f),$$
$$(D_1|D)$$

and where the first sum is over all positive integers r dividing $(v)\mathfrak{d}$, the second sum over all positive integers dividing D and $N((v)\mathfrak{d})/r^2$, $D_1 = D/D_2$ and $\psi(D_2)$ is the Gauss sum defined as in proposition 3.

Theorem 3. *Let* $m \geq 1$ *be an integer. Let* G_m *be the mth Poincare series for the cusp at* ∞ *of* $\Gamma_0(M)$ *with character* χ_D, *then* $\iota_D(G_m) = (k-1)!/(2(2\pi)^k)(-1)^{k/2}\omega_m \in S_k^{\mathcal{H}}(\tilde{\Gamma}_0(N))$ *where* ω_m *is as defined by* (1).

PROOF. It suffices to show that $c((v)\mathfrak{d}) = (k-1)!/(2(2\pi)^k)(-1)^{k/2}c_{mv}$. Now

$$c((v)\mathfrak{d}) = \sum_{r|(v)\mathfrak{d}} r^{k-1} \sum_{D_2|(D,(N(v)N(\mathfrak{d}))/r^2)} \left(\frac{N}{D_2}\right) \psi(D_2) D_2^{k-1} a^{D_1N}_{(N(v)N(\mathfrak{d}))/(r^2D_2)}(G_m)$$

$$= \sum_{r|(v)\sqrt{D}} r^{k-1} \sum_{D_2|(D,(Dvv')/r^2)} \left(\frac{N}{D_2}\right) \psi(D_2) D_2^{k-1} (4\pi(Dvv')/(r^2D_2))^{k-1}$$

$$\times ((k-2)!)^{-1} D_2^{-k}(G_m, G^{D_1N}_{(Dvv')/(r^2D_2)}) \text{ by Eqn (8)}$$

$$= \sum_{r|(v)\sqrt{D}} r^{k-1} \sum_{D_2|(D,(Dvv')/r^2)} \left(\frac{N}{D_2}\right) \psi(D_2) D_2^{-1} (4\pi(Dvv')/(r^2D_2))^{k-1}$$

$$\times ((k-2)!)^{-1}(k-2)!/(4\pi m)^{k-1}(m\text{-th coefficient of } G^{D_1N}_{(Dvv')/(r^2D_2)})$$

$$= \sum_{r|(v)\sqrt{D}} r^{k-1} \sum_{D_2|(D,(Dvv')/r^2)} \left(\frac{N}{D_2}\right) \psi(D_2) D_2^{-1} ((Dvv')/(r^2D_2m))^{k-1}$$

$$\times \Bigg\{ \delta_{D_1N,M}\delta_{(Dvv')/(r^2D_2),m} + 2\pi(-1)^{k/2}(mD_2)^{(k-1)/2}((Dvv')/(r^2D_2))^{-(k-1)/2}$$

$$\times \sum_{\substack{a=1\\(a,M)=D_1N}}^{\infty} H_a^{D_1N}((Dvv')/(r^2D_2), m) J_{k-1}((4\pi/a)\sqrt{(mDvv')/(r^2D_2^2)}) \Bigg\}$$

$$= \sum_{r|(v)\sqrt{D}} r^{k-1}((Dvv')/(r^2m))^{k-1}\delta_{(Dvv')/r^2,m}$$

$$+ \sum_{r|(v)\sqrt{D}} r^{k-1} \sum_{D_2|(D,(Dvv')/r^2)} \left(\frac{N}{D_2}\right) \psi(D_2) D_2^{-1} ((Dvv')/(r^2D_2m))^{k-1}$$

$$\times \Bigg\{ 2\pi(-1)^{k/2}(mD_2)^{(k-1)/2}((Dvv')/(r^2D_2))^{-(k-1)/2}$$

$$\times \sum_{\substack{a=1\\(a,M)=D_1N}}^{\infty} H_a^{D_1N}((Dvv')/(r^2D_2), m) J_{k-1}((4\pi/a)\sqrt{(mDvv')/(r^2D_2^2)}) \Bigg\}$$

$$= \sum_{\substack{r|(v)\sqrt{D}\\ N(v\sqrt{D}/r)=-m}} r^{k-1} + \sum_{r|(v)\sqrt{D}} ((Dvv')/m)^{(k-1)/2} 2\pi(-1)^{k/2} \sum_{D_2|(D,Dvv'/r^2)}$$

$$\times \left(\frac{N}{D_2}\right) \psi(D_2) D_2^{-1} \sum_{\substack{a=1\\(a,M)=D_1N}}^{\infty} H_a^{D_1N}((Dvv')/(r^2D_2), m)$$

$$\times J_{k-1}((4\pi/(arD_2))\sqrt{mDvv'})$$

$$= \sum_{\substack{r|(v)\sqrt{D} \\ N(v\sqrt{D}/r)=-m}} r^{k-1} + 2\pi(-1)^{k/2}((Dvv')/m)^{(k-1)/2} \sum_{r|(v)\sqrt{D}} \sum_{a=1}^{\infty}$$

$$\times \sum_{\substack{D_2|(D, Dvv'/r^2) \\ (a, D_2)=1}} \left(\frac{N}{D_2}\right) \psi(D_2) D_2^{-1} H_{aD_1N}^{D_1N}((Dvv')/(r^2D_2), m)$$

$$\times J_{k-1}((4\pi/(aD_1NrD_2))\sqrt{mDvv'})$$

$$= \sum_{\substack{r|(v)\sqrt{D} \\ N(v\sqrt{D}/r)=-m}} r^{k-1} + 2\pi(-1)^{k/2}((Dvv')/m)^{(k-1)/2} \sum_{r|(v)\sqrt{D}} \sum_{a=1}^{\infty}$$

$$\times H_{aN}((Dvv')/r^2, m) J_{k-1}((4\pi/(aNr))\sqrt{(mvv')/D}) \text{ by proposition 3(ii).}$$

$$= \sum_{\substack{r|(v)\sqrt{D} \\ N(v\sqrt{D}/r)=-m}} r^{k-1} + 2\pi(-1)^{k/2}((Dvv')/m)^{(k-1)/2} \sum_{\substack{a=1 \\ N|a}}^{\infty} \sqrt{D}^{-1}$$

$$\times \sum_{\substack{r|(v)\sqrt{D} \\ r|a}} \sqrt{D} H_{a/r}((Dvv')/r^2, m) J_{k-1}((4\pi/a)\sqrt{(mvv')/D})$$

$$= \sum_{\substack{r|(v)\sqrt{D} \\ N(v\sqrt{D}/r)=-m}} r^{k-1} + 2\pi(-1)^{k/2} D^{(k/2)-1} (N(v)/m)^{(k-1)/2}$$

$$\times \sum_{\substack{a=1 \\ N|a}}^{\infty} a^{-1} G_a(m, v) J_{k-1}((4\pi/a)\sqrt{(mN(v))/D}), \text{ by Proposition 4}$$

$$= ((k-1)!)/(2(2\pi)^k)(-1)^{k/2} c_{mv}$$

Hence the theorem. □

THEOREM 4. *ι_D sends Hecke eigenforms to Hecke eigenforms.*

PROOF. Follows similar to that given in [2]. □

We now describe the relationship of Theorem 3 to the analogous construction of K. Doi and H. Naganuma [1] [5]. Let $f(z) = \sum_{n=1}^{\infty} a_n e^{2\pi i n z} \in S_k(\Gamma_0(M), (\frac{\cdot}{D}))$ be a cusp form of weight k, where $D(D|M)$ is the discriminant of real quadratic field $K = \mathbf{Q}(\sqrt{D})$ of class number one. We assume that f is an eigen function of all the Hecke operators T_n, normalized with $a_1 = 1$ then the associated Dirichlet's series

$$L(f, s) = \sum_{n=1}^{\infty} a_n n^{-s} \quad (\mathrm{Re}\, s \gg 1)$$

has an Euler product expansion of the form

$$L(f,s) = \prod_q \left(1 - a_q q^{-s} + \left(\frac{q}{D}\right)\left(\frac{N}{q}\right)^2 q^{k-1-2s}\right)^{-1}$$

(product over all rational primes q, $N = M/D$). Consider the series

$$L(\bar{f},s) = \sum_{n=1}^{\infty} \bar{a}_n n^{-s}$$

whose coefficients are the complex conjugates of those of $L(f,s)$. More precisely,

$$\bar{a}_n = \left(\frac{n}{D}\right) a_n, \quad (n, M) = 1 \tag{12}$$

Consider

$$\begin{aligned} \Phi(s) &= L(f,s)L(\bar{f},s) \\ &= \prod\nolimits_{\mathfrak{q} \nmid N} (1 - b(\mathfrak{q})N(\mathfrak{q})^{-s} + N(\mathfrak{q})^{k-1-2s})^{-1} \prod\nolimits_{\mathfrak{q} \mid N} (1 - b(\mathfrak{q})N(\mathfrak{q})^{-s})^{-1} \end{aligned} \tag{13}$$

where the product is extended over all prime ideals $\mathfrak{q}$ of $\mathbf{Q}(\sqrt{D})$ and the coefficients are defined by

$$b(\mathfrak{q}) = a_q \qquad \text{if } q \text{ splits and } (q, M) = 1 \tag{14a}$$

$$b(\mathfrak{q}) = a_q^2 + 2q^{k-1} \qquad \text{if } q \text{ inert and } (q, M) = 1 \tag{14b}$$

Indeed, for splits primes q, we know by (12) that $a_q = \bar{a}_q$, so the factor $(1 - a_q q^{-s} + q^{k-1-2s})^{-1}$ occurs twice in $L(f,s)L(\bar{f},s)$, and since there are two prime ideals with norm q, it also occurs twice in the product (13). For inert primes q, (12) tells us that $a_q = -\bar{a}_q$, so the corresponding local factor in $L(f,s)L(\bar{f},s)$ is

$$\begin{aligned} &(1 - a_q q^{-s} + q^{k-1-2s})^{-1}(1 + a_q q^{-s} + q^{k-1-2s})^{-1} \\ &= (1 - a_q^2 q^{-2s} - 2q^{k-1-2s} + q^{2k-2-4s})^{-1} \\ &= (1 - b(\mathfrak{q})N(\mathfrak{q})^{-s} + N(\mathfrak{q})^{k-1-2s})^{-1} \end{aligned}$$

with $\mathfrak{q} = (q)$, $b(\mathfrak{q})$ as in (14b), $N(\mathfrak{q}) = q^2$.

THEOREM 5. *Let $f \in S_k\left(\Gamma_0(M), \left(\frac{\cdot}{D}\right)\right)$ be a normalized Hecke eigen function, where M is a square-free integer and $D|M$, $D \equiv 1 \pmod 4$ is a fundamental discriminant of $K = \mathbf{Q}(\sqrt{D})$ of class number one. Let*

$$L(i_D(f),s) = \sum\nolimits_{\mathfrak{m}} c(\mathfrak{m})N(\mathfrak{m})^{-s}, \quad Re\, s > \frac{k}{2} + 1$$

be the associated Dirichlet's series to $i_D(f)$, where $i_D(f)$ is as defined in theorem (3) *and the summation is over all integral ideals* $\mathfrak{m}$ *of* K. *Then*

$$L(i_D(f), s) = L(f, s)L(\bar{f}, s).$$

Proof. As we know, by the definition of $c(\mathfrak{a})$, for primes $\mathfrak{p} \nmid N$,

$$c(\mathfrak{p}) = \begin{cases} a_p & \text{if } p \text{ splits.} \\ a_p^2 + 2p^{k-1} & \text{if } p \text{ is inert.} \end{cases}$$

Also, using the Euler product of $L(f, s)L(\bar{f}, s)$, we find that $L(i_D(f), s)$ and $L(f, s)L(\bar{f}, s)$ agree up to finitely many Euler factors, but they satisfies the same functional equation. Hence they are equal. □

Acknowledgements

The authors are thankful to the referee. The first author is grateful to Prof. E. Ghate for the helpful discussions. Major parts of this work was done when the first named author was visiting Kerala School of Mathematics, India. The first named author thanks Kerala School of Mathematics for their hospitality during his visits.

References

[1] Doi, K. and Naganuma, H., On the functional equation of certain Dirichlet series, Invent. Math. **9** (1969), 1–14.
[2] Geer Van der, G., Hilbert modular surfaces, Springer-Verlag Berlin, 1988.
[3] Ghate, E., Congruences between base-change and non-base-change Hilbert modular forms, In Cohomology of arithmetic groups, L-functions and automorphic forms, (Mumbai, 1998/1999), Tata Inst. Fund. Res. Stud. Math. **15** (2001), 35–62.
[4] Kudla, S. S., Theta-functions and Hilbert modular forms, Nagoya Math. J. **69** (1978), 97–106.
[5] Naganuma, H., On the coincidence of two Dirichlet series associated with cusp forms of Hecke's Neben-type and Hilbert modular forms over real quadratic field, J. Math. Soc. Japan. **25** (1973), 547–555.
[6] Zagier, D., Modular forms associated with real quadratic fields, Invent. Math. **30** (1975), 1–46.

Balesh Kumar
Harish-Chandra Research Institute
HBNI, Chhatnag Road, Jhunsi, Allahabad-211019, India
E-mail: baleshkumar@hri.res.in

Murugesan Manickam
Kerala School of Mathematics
Kunnamangalam, Kozhikode-673571, Kerala, India
E-mail: murugumanick@gmail.com