

ON RATIONAL APPROXIMATIONS TO IRRATIONAL NUMBERS

To Professor Goro AZUMAYA on his sixtieth birthday

By

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A classical theorem of A. Hurwitz concerning rational approximations to irrational numbers states that for any real irrational number ξ there are infinitely many pairs of integers p, q with $q > 0$ satisfying the inequality

$$(1) \quad \left| \xi - \frac{p}{q} \right| < \frac{1}{\sqrt{5} q^2},$$

where the constant $\sqrt{5}$ on the right-hand side of (1) is the best possible number. S. Hartman [1], imposing congruential conditions on the denominator and the numerator of approximating fractions, has proved that if ξ is an irrational number and if a, b and s are fixed integers with $s > 0$, then there exists an infinity of pairs of integers u, v with $v > 0$ satisfying the conditions

$$(2) \quad \left| \xi - \frac{u}{v} \right| < \frac{2s^2}{v^2}$$

and

$$(3) \quad u \equiv a \pmod{s}, \quad v \equiv b \pmod{s}.$$

As is noticed by Hartman, the exponent 2 of s in the right side of (2) cannot be reduced in general; this will readily be seen by putting $a=b=0$.

This result of Hartman has been improved in two ways by J. F. Koksma [2], who proved that if ξ is an irrational number, then for any fixed real number $\varepsilon > 0$ and for any integers $s > 0, a, b$ there exist infinitely many pairs of integers u, v with $v > 0$ such that

$$(4) \quad \left| \xi - \frac{u}{v} \right| < \frac{(1+\varepsilon)s^2}{\sqrt{5} v^2},$$

$$u \equiv a \pmod{s}, \quad v \equiv b \pmod{s},$$

and that if ξ is an irrational number and if $s > 0, a, b$ are integers such that we have not simultaneously $a \equiv 0 \pmod{s}$ and $b \equiv 0 \pmod{s}$, then there are infinitely many pairs of integers u, v with $v \neq 0$ satisfying

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$$(5) \quad \left| \xi - \frac{u}{v} \right| < \frac{s^2}{4|v(v-b)|},$$

$$u \equiv a \pmod{s}, \quad v \equiv b \pmod{s}.$$

Here, in (4) the constant $\sqrt{5}$ is the best possible number in view of the theorem of Hurwitz quoted above, and in (5) the right-hand side is less than

$$\frac{(1+\varepsilon)s^2}{4v^2}$$

for any fixed real number $\varepsilon > 0$ and all sufficiently large values of $|v|$.

Now, the purpose of the present note is to slightly improve this latter result of Koksma's, by establishing the following

THEOREM. *Let ξ be a real irrational number and let a, b and s be fixed integers with $s > 0$. Then there exist infinitely many pairs of integers u, v with $v \neq 0$ satisfying the conditions*

$$\left| \xi - \frac{u}{v} \right| < \frac{s^2}{4v^2}$$

and

$$u \equiv a \pmod{s}, \quad v \equiv b \pmod{s},$$

provided that we have not simultaneously $a \equiv 0 \pmod{s}$ and $b \equiv 0 \pmod{s}$.

It is obvious again by Hurwitz's theorem that our theorem cannot hold in general *without* the proviso that the congruences $a \equiv b \equiv 0 \pmod{s}$ are not satisfied simultaneously, which implies $s \geq 2$.

1. Let ξ be a real irrational number. There is no loss in generality in assuming that $0 < \xi < 1$; in doing so, we replace, if necessary, the value of the integer a by another suitable one. Being thus assumed, the number ξ can be represented by an infinite simple continued fraction

$$\xi = [a_0; a_1, a_2, \dots],$$

where $a_0 = 0$, $a_m > 0$ ($m = 1, 2, \dots$) are integers. We define as usual

$$p_0 = a_0 = 0, \quad p_1 = a_1 a_0 + 1 = 1, \quad p_m = a_m p_{m-1} + p_{m-2} \quad (m \geq 2),$$

$$q_0 = 1, \quad q_1 = a_1, \quad q_m = a_m q_{m-1} + q_{m-2} \quad (m \geq 2);$$

it is well known and is in fact easily proved by induction on m that

$$(6) \quad p_m q_{m-1} - p_{m-1} q_m = (-1)^{m-1} \quad (m = 1, 2, \dots).$$

Now, following Hartman [1], we consider for $n \geq 1$ the system of congruences

$$p_n x + p_{n-1} y \equiv a \pmod{s}$$

$$q_n x + q_{n-1} y \equiv b \pmod{s}.$$

It will readily be verified in view of (6) that a solution x, y of this system is provided by

$$x = t_{n-1}, \quad y = t_n,$$

where t_m is the integer with

$$(7) \quad -\frac{s}{2} < t_m \leq \frac{s}{2}$$

and determined by the congruence

$$t_m \equiv (-1)^m (a q_m - b p_m) \pmod{s},$$

or equivalently by the conditions

$$t_0 \equiv a \pmod{s}, \quad t_1 \equiv -a a_1 + b \pmod{s}, \quad t_m \equiv -a_m t_{m-1} + t_{m-2} \pmod{s} \quad (m \geq 2).$$

Put

$$u_n = p_n t_{n-1} + p_{n-1} t_n, \quad v_n = q_n t_{n-1} + q_{n-1} t_n.$$

Then $u = u_n, v = v_n$ satisfy the condition (3).

For brevity's sake we set

$$u_n(x, y) = p_n x + p_{n-1} y, \quad v_n(x, y) = q_n x + q_{n-1} y.$$

2. Throughout in what follows we shall suppose that n be even, or more exactly that n increases through the sequence of even integers (alternatively, we may suppose of course that n be odd, without any substantial change in the argument; as a matter of fact, we need only to consider either one of the cases of n even and of n odd).

We begin by observing that one at least of t_n and t_{n-1} is not zero, since otherwise we would have $a \equiv b \equiv 0 \pmod{s}$, which is the case we have excluded by assumption. Thus, if $t_n = 0$ then $t_{n-1} \neq 0$, and so we have, by (7),

$$\left| \xi - \frac{u_n}{v_n} \right| < \frac{1}{q_n^2} = \frac{t_{n-1}^2}{v_n^2} \leq \frac{s^2}{4v_n^2},$$

since there holds the inequality

$$\left| \xi - \frac{p_m}{q_m} \right| < \frac{1}{q_m^2}$$

for all m . Similarly, if $t_{n-1} = 0$ then $t_n \neq 0$ and

$$\left| \xi - \frac{u_n}{v_n} \right| < \frac{1}{q_{n-1}^2} \leq \frac{s^2}{4v_n^2}.$$

Therefore, we may assume in the following that $t_n \neq 0$ and $t_{n-1} \neq 0$.

We now distinguish several cases according to the signs of t_n and t_{n-1} .

Case 1. $t_n t_{n-1} > 0$. We have for all even $n \geq 2$

$$\frac{p_n}{q_n} < \xi < \frac{p_{n-1}}{q_{n-1}}$$

and

$$\frac{p_n}{q_n} < \frac{u_n}{v_n} < \frac{p_{n-1}}{q_{n-1}}.$$

We consider two subcases according as

$$\frac{p_n}{q_n} < \xi < \frac{u_n}{v_n} \quad \text{or} \quad \frac{u_n}{v_n} < \xi < \frac{p_{n-1}}{q_{n-1}}.$$

Case 1.1. $p_n/q_n < \xi < u_n/v_n$. In this case we define for integral $j \geq 0$

$$u_{n,j} = u_n(t_{n-1} \pm js, t_n), \quad v_{n,j} = v_n(t_{n-1} \pm js, t_n),$$

where the sign of j in the parentheses is fixed by $\pm = \text{sgn } t_n (= \text{sgn } t_{n-1})$. Since the fraction $u_{n,j}/v_{n,j}$ (n being fixed) monotonically decreases as j increases and since

$$\lim_{j \rightarrow +\infty} \frac{u_{n,j}}{v_{n,j}} = \frac{p_n}{q_n},$$

there is a unique integer $k \geq 1$ such that

$$\frac{u_{n,k}}{v_{n,k}} < \xi < \frac{u_{n,k-1}}{v_{n,k-1}}.$$

Here, it will readily be verified that

$$\begin{aligned} \left| \xi - \frac{u_{n,k}}{v_{n,k}} \right| + \left| \xi - \frac{u_{n,k-1}}{v_{n,k-1}} \right| &= \frac{u_{n,k-1}}{v_{n,k-1}} - \frac{u_{n,k}}{v_{n,k}} \\ &= \frac{|st_n|}{|v_{n,k-1}v_{n,k}|} \\ &\leq \frac{1}{2} \left(\frac{|st_n|}{v_{n,k}^2} + \frac{|st_n|}{v_{n,k-1}^2} \right). \end{aligned}$$

It follows that one at least of the two fractions $u_{n,k}/v_{n,k}$, $u_{n,k-1}/v_{n,k-1}$, say u/v , satisfies the inequality

$$(8) \quad \left| \xi - \frac{u}{v} \right| < \frac{|st_n|}{2v^2} \leq \frac{s^2}{4v^2},$$

where

$$u \equiv u_n \equiv a \pmod{s}, \quad v \equiv v_n \equiv b \pmod{s}$$

and the condition (3) is fulfilled.

Case 1.2. $u_n/v_n < \xi < p_{n-1}/q_{n-1}$. In this case we define for integral $j \geq 0$

$$u_{n,j} = u_n(t_{n-1}, t_n \pm js), \quad v_{n,j} = v_n(t_{n-1}, t_n \pm js),$$

where we take as before $\pm = \text{sgn } t_n$ ($= \text{sgn } t_{n-1}$). Since the fraction $u_{n,j}/v_{n,j}$ (n being fixed) monotonously increases with j and since

$$\lim_{j \rightarrow +\infty} \frac{u_{n,j}}{v_{n,j}} = \frac{p_{n-1}}{q_{n-1}},$$

there is a unique integer $k \geq 1$ such that

$$\frac{u_{n,k-1}}{v_{n,k-1}} < \xi < \frac{u_{n,k}}{v_{n,k}}.$$

where

$$\frac{u_{n,k}}{v_{n,k}} - \frac{u_{n,k-1}}{v_{n,k-1}} = \frac{|st_{n-1}|}{|v_{n,k}v_{n,k-1}|}.$$

Hence, just as in Case 1.1, we may conclude that at least one (call this u/v) of the two fractions $u_{n,k-1}/v_{n,k-1}$, $u_{n,k}/v_{n,k}$ satisfies the inequality

$$(9) \quad \left| \xi - \frac{u}{v} \right| < \frac{|st_{n-1}|}{2v^2} \leq \frac{s^2}{4v^2},$$

where

$$u \equiv u_n \equiv a \pmod{s}, \quad v \equiv v_n \equiv b \pmod{s}.$$

Case 2. $t_n > 0, t_{n-1} < 0$. It is obvious that $v_n \neq 0$ for all sufficiently large n , since g. c. d. $(q_n, q_{n-1}) = 1$ and since q_m indefinitely increases with m . We distinguish two subcases according as

$$v_n > 0 \quad \text{or} \quad v_n < 0,$$

assuming that n be sufficiently large.

Case 2.1 $v_n > 0$. In this case we have

$$\frac{p_n}{q_n} < \xi < \frac{p_{n-1}}{q_{n-1}} < \frac{u_n}{v_n}.$$

We define for integral $j \geq 0$

$$u_{n,j} = u_n(t_{n-1} + js, t_n), \quad v_{n,j} = v_n(t_{n-1} + js, t_{n-1}).$$

Since the fraction $u_{n,j}/v_{n,j}$ (n being fixed large enough) monotonically decreases as j increases and since

$$\lim_{j \rightarrow +\infty} \frac{u_{n,j}}{v_{n,j}} = \frac{p_n}{q_n},$$

there exists a unique integer $k \geq 1$ such that

$$\frac{u_{n,k}}{v_{n,k}} < \xi < \frac{u_{n,k-1}}{v_{n,k-1}},$$

where

$$\frac{u_{n,k-1}}{v_{n,k-1}} - \frac{u_{n,k}}{v_{n,k}} = \frac{st_n}{v_{n,k-1}v_{n,k}}.$$

It follows that one at least of the two fractions $u_{n,k}/v_{n,k}$, $u_{n,k-1}/v_{n,k-1}$, say u/v , satisfies the inequality (8), where we have

$$u \equiv u_n \equiv a \pmod{s}, \quad v \equiv v_n \equiv b \pmod{s}.$$

Case 2.2. $v_n < 0$. In this case we have

$$\frac{u_n}{v_n} < \frac{p_n}{q_n} < \xi < \frac{p_{n-1}}{q_{n-1}}.$$

We define for integral $j \geq 0$

$$u_{n,j} = u_n(t_{n-1}, t_n - js), \quad v_{n,j} = v_n(t_{n-1}, t_n - js).$$

Since the fraction $u_{n,j}/v_{n,j}$ (n being fixed large enough) monotonously increases with j and since

$$\lim_{j \rightarrow +\infty} \frac{u_{n,j}}{v_{n,j}} = \frac{p_{n-1}}{q_{n-1}},$$

there is a unique integer $k \geq 1$ such that

$$\frac{u_{n,k-1}}{v_{n,k-1}} < \xi < \frac{u_{n,k}}{v_{n,k}},$$

where

$$\frac{u_{n,k}}{v_{n,k}} - \frac{u_{n,k-1}}{v_{n,k-1}} = \frac{|st_{n-1}|}{|v_{n,k}v_{n,k-1}|}.$$

Hence, we have again the inequality (9) with $u/v = u_{n,k}/v_{n,k}$ or $u_{n,k-1}/v_{n,k-1}$, where

$$u \equiv u_n \equiv a \pmod{s}, \quad v \equiv v_n \equiv b \pmod{s}.$$

Case 3. $t_n < 0, t_{n-1} > 0$. We can deal with this case just as in Case 2, by considering two subcases according as $v_n > 0$ or $v_n < 0$ and suitably defining $u_{n,j}$ and $v_{n,j}$. Namely:

Case 3.1. $v_n > 0$. In this case we have

$$\frac{u_n}{v_n} < \frac{p_n}{q_n} < \xi < \frac{p_{n-1}}{q_{n-1}}$$

and define for $j \geq 0$

$$u_{n,j} = u_n(t_{n-1}, t_n + js), \quad v_{n,j} = v_n(t_{n-1}, t_n + js).$$

Case 3.2. $v_n < 0$. In this case we have

$$\frac{p_n}{q_n} < \xi < \frac{p_{n-1}}{q_{n-1}} < \frac{u_n}{v_n}$$

and define for $j \geq 0$

$$u_{n,j} = u_n(t_{n-1} - js, t_n), \quad v_{n,j} = v_n(t_{n-1} - js, t_n).$$

We omit the details which can be easily filled out.

This concludes the proof of our theorem.

3. We note that it can also be proved by the present method that if ξ is a real irrational number and if a, b and s are arbitrary but fixed integers with $s > 0$, then there are infinitely many pairs of integers u, v with $v > 0$ satisfying the conditions

$$(10) \quad \left| \xi - \frac{u}{v} \right| < \frac{s^2}{2v^2}$$

and

$$u \equiv a \pmod{s}, \quad v \equiv b \pmod{s}.$$

However, the inequality (10) is apparently weaker than Koksma's (4).

References

- [1] Hartman, S., Sur une condition supplémentaire dans les approximations diophantiques. *Colloq. Math.* 2 (1951), 48-51.
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