TSUKUBA J. MATH.
Vol. 4 (1980). 77—81

DIRECT SUM OF τ-INJECTIVE MODULES

By

Kanzo MASAIKE and Takanori HORIGOME

Throughout this paper R is a ring with identity and every R-module is unital. Let τ be a hereditary torsion theory with respect to R (see Golan [3]). A submodule N of a right R-module M is said to be τ-dense in M, if M/N is a τ-torsion module. We shall denote the set of all τ-dense right ideals of R by $\mathcal{L}_\tau$. A right R-module M is called τ-injective, if for every right R-module L and its τ-dense submodule N every R-homomorphism $N\rightarrow M$ is extended to $L\rightarrow M$. Let us denote by $E(L)$ the injective hull of the right R-module L. Then, $E_\tau(L)=\{x\in E(L) \mid \text{there exists } I\in\mathcal{L}_\tau \text{ such that } xI\subset L\}$ is said to be τ-injective hull of L. If N is a submodule of L, $E_\tau(N)$ is contained in $E_\tau(L)$.

By a result of Matlis [5] and Papp, R is right Noetherian, if and only if every injective right R-module is a direct sum of (injective) indecomposable submodules. It is to be noted that this result was generalized to injective τ-torsion free right R-modules by Teply [7]. Let τ_G be the Goldie torsion theory with respect to R. Clearly, injective indecomposable right R-modules coincide with those modules each of which is a τ_G-injective hull of its every non-zero submodule. Furthermore, if R is right Noetherian, the ring of quotient of R with respect to τ_G is semi-simple Artinian (cf. Kutami and Oshiro [4]) and hence τ_G is a perfect torsion theory (see [3], [6]). Now, concerning the above result of Matlis and Papp we shall study in this paper a right R-module M such that $M=\bigoplus_{i\in I} M_i$, where each M_i is a τ-injective hull of its every non-zero submodule. In the following such a module M will be said to be τ-completely decomposable.

Now, at first we shall prove the next

THEOREM 1. *Let τ be a hereditary torsion theory with respect to R. Then, $\mathcal{L}_\tau$ satisfies the ascending chain condition, if and only if, every τ-injective τ-torsion R-module is τ-completely decomposable.*

A ring R is called right semi-Artinian, if every non-zero right R-module has a non-zero socle. Then, we shall prove the following

Received June 25, 1979. Revised October 10, 1979.
The authors wish to express their gratitude to the referee for his helpful advice.

THEOREM 2. *The following are equivalent, if τ is a perfect torsion theory.*

(i) *$\mathcal{L}_\tau$ satisfies the ascending chain condition and the ring of quotient R_τ of R with respect to τ is right semi-Artinian.*

(ii) *Every τ-injective right R-module is an essential extension of a τ-injective τ-completely decomposable R-module.*

Let τ_G be a Goldie torsion theory. Then every τ_G-torsion free direct summand of a τ_G-completely decomposable module is quasi-injective (see [4]). In Theorem 3, we shall see that this result remains true, even if τ is an arbitrary hereditary torsion theory.

LEMMA 1. *Let M be a right R-module such that M is a τ-injective hull of its every non-zero submodule. Then* $\mathrm{End}(M_R)$ *is a local ring.*

PROOF. Let $f \in \mathrm{End}(M_R)$ be a non-zero element. Since M is uniform and $\mathrm{Ker}\, f \cap \mathrm{Ker}(1-f)=0$, f or $1-f$ is a monomorphism and hence a unit.

LEMMA 2. *Let M be a submodule of a right R-module K such that $M=\bigoplus_{a\in A} M_a$ and $K=\bigoplus_{b\in B} K_b$, which are τ-complete decompositions. If the cardinal number $|A|$ is at most countable, there exists a subset C of B such that $M\cong \bigoplus_{c\in C} K_c$.*

PROOF. Let $a\in A$. Put $A_a=\{d\in A \mid M_d\cong M_a\}$ and $B_a=\{b\in B \mid K_b\cong M_a\}$. It sufficies to show $|A_a|\leqq|B_a|$. Let $d_1, d_2, \cdots, d_n$ be distinct elements of A_a. Assume x_i be a non-zero element of M_{d_i}, $i=1, 2, \cdots, n$. Then, $\bigoplus_{i=1}^{n} M_{d_i}=E_\tau(\bigoplus_{i=1}^{n} x_iR)$, which is contained in a direct sum of finite number of K_b, $b\in B$. Let $b_1, b_2, \cdots, b_t$ be elements of B such that there exists a monomorphism $f: \bigoplus_{i=1}^{n} M_{d_i} \to \bigoplus_{j=J}^{t} K_{b_j}$ and for every p ($p=1, 2, \cdots, t$) there is no monomorphism from $\bigoplus_{i=1}^{n} M_{d_i}$ to $K_b \oplus K_{b_2}\oplus \cdots \oplus K_{b_{p-1}}\oplus K_{b_{p+1}}\oplus \cdots \oplus K_{b_t}$. Let π_{b_p} be the projection $K\to K_{b_p}$. Then, for every p ($p=1, \cdots, t$) we can pick up $y_p\in \bigoplus_{i=1}^{n} M_{d_i}$ so that $\pi_{b_p}f(y_p)\neq 0$ and $\pi_{b_s}f(y_p)=0$, $s\neq p$. Put $N=y_1R\oplus \cdots \oplus y_tR$. Since $f(N)=f(y_1)R\oplus \cdots \oplus f(y_t)R\subset K_{b_1}\oplus \cdots \oplus K_{b_t}$, we have $E_\tau(\bigoplus_{i=1}^{t} f(y_i)R)=K_{b_1}\oplus \cdots \oplus K_{b_t}$. Therefore, $\bigoplus_{i=1}^{n} M_{d_i}=\bigoplus_{j=1}^{t} K_{b_j}$. Then, by Lemma 1 and a theorem of Azumaya [1] every K_{b_j} is isomorphic to M_a. Since $t=n$, the consequence is immediate when $|A|$ is finite. Assume A_a is infinite. Then, the above argument implies B_a is an infinite set. Hence $|A_a|\leqq|B_a|$.

PROOF OF THEOREM 1.

Assume $\mathcal{L}_\tau$ satisfies the ascending chain condition. Let M be a τ-injective

τ-torsion R-module. If $0 \neq x \in M$, then $xR \cong R/I$ for some $I \in \mathcal{L}_\tau$. Therefore, R/I is a Noetherian right R-module and hence contains a uniform submodule U. Since $E_\tau(U)$ is contained in M, M have a τ-injective submodule which is a τ-injective hull of its every non-zero submodule. Let $\{M_j\}_{j \in J}$ be a maximal independent set of submodules of M such that each M_j is a τ-injective hull of its every non-zero submodule, where J is an index set. Let $y \in M$ be a non-zero element. As is shown above $E_\tau(yR)$ contains a submodule which is a τ-injective hull of its every non-zero submodule and hence $\bigoplus_{j \in J} M_j \cap yR \neq 0$. It follows $\bigoplus_{j \in J} M_j$ is an essential submodule of M. On the other hand, since $\mathcal{L}_\tau$ satisfies the ascending chain condition, by [3, p. 128, Proposition 14.2] $\bigoplus_{j \in J} M_j$ is τ-injective. Since M is a τ-injective τ-torsion module, $\bigoplus_{j \in J} M_j$ is a direct summand of M. Thus we have $M = \bigoplus_{j \in J} M_j$.

Conversely, assume every τ-injective τ-torsion R-module is τ-completely decomposable. Let $\{P_i\}_{i \in I}$ be a class of (non-isomorphic) representatives of all τ-injective τ-torsion uniform R-modules. Then, each P_i is the τ-injective hull of its every non-zero submodule. Let us denote $P_i^{(N)}$ the direct sum of countably copies of P_i. Since $E_\tau(\bigoplus_{i \in I} P_i^{(N)})$ is a τ-torsion module, it has a τ-complete decomposition such that $E_\tau(\bigoplus_{i \in I} P_i^{(N)}) = \bigoplus_{j \in J} Q_j$, where J is an index set. From Lemma 2 $\bigoplus_{i \in I} P_i^{(N)}$ is isomorphic to a direct summand of $\bigoplus_{j \in J} Q_j$. Hence we have $(\bigoplus_{i \in I} P_i)^{(N)}$ is τ-injective. Let $K_1 \subset K_2 \subset K_3 \subset \cdots$ be a strictly ascending chain of right ideals in $\mathcal{L}_\tau$. Then, R/K_j $(j=1, 2, \cdots)$ is a submodule of a τ-torsion τ-completely decompossable R-module $E_\tau(R/K_j)$. It follows K_j is an annihilator right ideal of a subset of $\bigoplus_{i \in I} P_i$. Therefore, we can choose $a_1, a_2, a_3, \cdots \in \bigoplus_{i \in I} P_i$ such that $a_j K_j = 0$ and $a_j K_{j+1} \neq 0$ $(j=1, 2, 3, \cdots)$. Put $K = \bigcup_{j=1} K_j$. Clearly, the map $f: K \to (\bigoplus_{i \in I} P_i)^{(N)}$ by $f(x) = (a_1 x, a_2 x, \cdots)$, $x \in K$, is an R-homomorphism. Since $K \in \mathcal{L}_\tau$, f is extended to $R \to (\bigoplus_{i \in I} P_i)^{(N)}$. However this is a contradiction, since for every integer $n > 0$ there exists $x \in K$ such that $a_n x \neq 0$. This complete the proof.

In the following let us denote $T_\tau(M)$ the τ-torsion submodule of M.

Proof of Theorem 2.

(i) $\Rightarrow$ (ii). Let M be a τ-injective R-module. Since $E_\tau(T_\tau(M))$ is contained in M, it is equal to $T_\tau(M)$. Then, $T_\tau(M)$ is τ-completely decomposable by Theorem 1. We may assume $T_\tau(M)$ is not an essential submodule of M. Let N be a closed submodule of M such that $N \cap T_\tau(M) = 0$ and $N \oplus T_\tau(M)$ is essential in M. Since N has no essential extension in M, N is τ-injective. Then, N becomes a right R_τ-module, which has an essential socle $S=\bigoplus_{h\in H} S_h$, where S_h is a simple right R_τ-module. Let L be a non-zero R-submodule of S_h. Then, $S_h=LR_\tau$ and hence L is a τ-dense submodule of S_h. On the other hand, since τ is a perfect torsion theory, S is τ-injective τ-completely decomposable and so is $S\oplus T_\tau(M)$.

(ii) $\Rightarrow$ (i). Let N be a τ-injective τ-torsion R-module. Since every τ-injective submodule of N is a direct summand, N is τ-completely decomposable and $\mathcal{L}_\tau$ satisfies the ascending chain condition. Let K be a right R_τ-module. Since K is a τ-injective R-module it contains an essential τ-completely decomposable R-submodule. In view of [3, p. 186 Corollary] we have that this submodule is a socle of the right R_τ-module K. Hence R_τ is right semi-Artinian.

REMARK. Assume τ_G is the Goldie torsion theory. Put $\mathcal{C}_{\tau_G}=\{$right ideal I of $R \mid R/I$ is τ_G-torsion free$\}$. If $\mathcal{C}_{\tau_G}$ and $\mathcal{L}_{\tau_G}$ satisfy the ascending chain condition, then every injective right R-module is a direct sum of indecomposable submodules in view of Theorem 1 and [7, Theorem 1.2] and hence R is right Noetherian. When R is right non-singular, this is a case of Yamagata [9, Theorem 9].

LEMMA 3. *Let M be a τ-torsion free right R-module. Assume $M=\sum_{i\in I} M_i$, where M_i is a τ-injective hull of its every non-zero submodule. Then, there exists a subset J of I such that $M=\bigoplus_{j\in J} M_j$.*

PROOF. Let $\{M_j\}_{j\in J}$ be a maximal independent subset of the class $\{M_i\}_{i\in I}$. For every $i\in I$ $M_i \cap \bigoplus_{j\in J} M_j$ contains a non-zero element x, say. Therefore, there exists a finite subset $\{j_1, \cdots, j_n\}$ of J so that $E_\tau(xR)$ is contained in $\bigoplus_{k=1}^{n} M_{j_k}$. Let $0\neq y\in E_\tau(xR)+M_i$. Put $y=y_1+y_2$, $y_1\in E_\tau(xR)$, $y_2\in M_i$. Then, there exist $L_1, L_2\in \mathcal{L}_\tau$ such that $y_1L_1\subset xR$ and $y_2L_2\subset xR$. So $0\neq y(L_1\cap L_2)\subset xR$. This implies xR is (an essential) τ-dense submodule of $E_\tau(xR)+M_i$. Since M_i is a τ-injective hull of xR, too, we have $E_\tau(E_\tau(xR)+M_i)=E_\tau(xR)=M_i$. Hence M_i is a submodule of $\bigoplus_{j\in J} M_j$.

THEOREM 3. *Let $M=\bigoplus_{i\in I} M_i$ be a τ-complete decomposition. If N is a τ-torsion free direct summand of M, then N is quasi-injective.*

PROOF. Let $0\neq x\in N$. $E_\tau(xR)$ is contained in a sum of finite number of M_i, $i\in I$. Let $\pi: M\to N$ be the projection. Then, the restriction $\pi \mid E_\tau(xR)$ is a monomorphism. This implies we may assume $E_\tau(xR)$ is contained in N. Now, $E_\tau(xR)$ is isomorphic to a direct sum of finite number of M_i, $i\in I$, by the same method as in the proof of Lemma 2. Therefore, $N=\sum_{x\in N} E_\tau(xR)$ is τ-completely decomposable by Lemma 3. Let $N=\bigoplus_{h\in H} N_h$ be the τ-complete decomposition and $E(N)$ the injective hull of N. To see that N is quasi-injective, it sufficies to show $f(N)\subset N$ for every $f\in \mathrm{End}(E(N)_R)$. Let f_n be the restriction $f|N_h$. Suppose $f_h\neq 0$. Then f_h is monic (cf. [3, Proposition 18.2]) and hence $\mathrm{Im} f_h$ is a τ-injective hull of its every non-zero submodule. Since $\bigoplus_{h\in H} N_h$ is an essential submodule of a τ-torsion free module $E(N)$, it is easy to check that $\mathrm{Im} f_h\subset \bigoplus_{h\in H} N_h$ from the proof of Lemma 3. This proves the Theorem.

Remark. In [2] it is proved that if M is an injective right R-module which is a direct sum of indecomposable modules, then so is its direct summand. Now, let M be a right R-module such that $M=\bigoplus_{i\in I} M_i$, where every proper factor module of each M_i is τ-torsion. Assume N is a τ-injective direct summand of M. Then by the same method as in [8, Lemma 2], it is not hard to see that there is a submodule N' of M such that $M=N\oplus N'$ and $N'=\bigoplus_{i\in I} M_i'$, where $M_i'\subset M_i$ $(i\in I)$. Especially, when $\bigoplus_{i\in I} M_i$ is a τ-complete decomposition, there exists a subset J of I such that $N'=\bigoplus_{j\in J} M_j$. Hence N has a τ-complete decomposition, too.

References

[1] Azumaya, G., Corrections and supplements to my paper concerning Krull-Remak-Schmidt's theorem. Nagoya Math. J., 1 (1950) 117-124.

[2] Faith, C. and Walker, E. A., Direct sum representations of injective modules. J. Algebra **5** (1967) 203-221.

[3] Golan, J. S., Localization of Non-commutative Rings. Marcel Dekker, New York (1975).

[4] Kutami, M. and Oshiro, K., Direct sums of non-singular indecomposable injective modules. Math. J. Okayama Univ. **20** (1978) 91-99.

[5] Matlis, E., Injective modules over Noetherian rings. Pac. J. Math. **8** (1958) 514-528.

[6] Stenström, B., Rings of Quotients. Grundlehren Math. Wiss. Vol **217**, Springer-Verlag Berlin-Heiderberg-New York.

[7] Teply, M., Torsion free injective modules. Pac. J. Math. **28** (1969) 441-453.

[8] Warfield Jr, R. B., Decompositions of injective modules. Pac. J. Math. **31** (1969) 263-276.

[9] Yamagata, K., Non-singular rings and Matlis' problem. Sci. Rep. Tokyo Kyoiku Daigaku A **11** (1971) 114-121.

Department of Mathematics
Tokyo Gakugei University