ON QUASI-EINSTEIN SPACETIMES

By

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Abstract. The notion of quasi-Einstein manifolds arose during the study of exact solutions of the Einstein field equations as well as during considerations of quasi-umbilical hypersurfaces. For instance, the Robertson-Walker spacetimes are quasi-Einstein manifolds. The object of the present paper is to study *quasi-Einstein spacetimes*. Some basic geometric properties of such a spacetime are obtained. The applications of quasi-Einstein spacetimes in general relativity and cosmology are investigated. Finally, the existence of such spacetimes are ensured by several interesting examples.

1. Introduction

It is well known that a connected Riemannian manifold (M^n, g) $(n > 2)$ is Einstein if its Ricci tensor S of type $(0, 2)$ is of the form

$$S = \alpha g,$$

where α is a constant, which turns into

$$S = \frac{r}{n} g,$$

r being the scalar curvature (constant) of the manifold.

The notion of quasi-Einstein manifolds arose during the study of exact solutions of the Einstein field equations as well as during considerations of quasi-umbilical hypersurfaces. For instance, the Robertson-Walker spacetimes are quasi-Einstein manifolds. Let (M^n, g), $n \geq 3$ be a semi-Riemannian manifold. Let

$U_S = \{x \in M : S \neq \frac{r}{n}g \text{ at } x\}$. The manifold (M^n, g) is said to be a quasi-Einstein manifold ([3], [5], [6], [7], [8], [9]) if on $U_S \subset M$, we have

$$(1.1) \qquad\qquad S - \alpha g = \beta A \otimes A,$$

where A is an 1-form on U_S and α, β are some functions on U_S. It is clear that the 1-form A, as well as the function β is non-zero at every point of U_S. An n-dimensional manifold of this kind is denoted by $(QE)_n$. The scalars α, β are known as the associated scalars. From the above definition it follows that every Einstein manifold is quasi-Einstein. In particular, every Ricci flat manifold (e.g. the Schwarzschild spacetime) is quasi-Einstein.

An n-dimensional Lorentzian manifold M is a smooth connected para-compact Hausdorff manifold with a Lorentzian metric g, that is, M admits a smooth symmetric tensor field g of type $(0, 2)$ such that for each point $p \in M$, the tensor $g_p : T_pM \times T_pM \to R$ is a non-degenerate inner product of signature $(+, +, +, \ldots, +, -)$, where T_pM denotes the tangent vector space of M at p and R is the real number space. A non-zero vector $v \in T_pM$ is said to be timelike (resp. non-spacelike, null, spacelike) if it satisfies $g_p(v, v) < 0$ (resp. ≤ 0, $= 0$, > 0) ([2], [18]).

Recently Shaikh et. al. [24] studied the Lorentzian quasi-Einstein manifolds (briefly, $(LQE)_n$) and obtained its several properties with the existence and found its applications to the general relativity and cosmology. A Lorentzian quasi-Einstein manifold is a $(QE)_n$ with the generator ρ as the unit timelike vector field such that $g(\rho, \rho) = -1$. A spacetime is a connected 4-dimensional Lorentzian manifold, and a quasi-Einstein spacetime is a connected $(LQE)_4$.

The present paper deals with a study of quasi-Einstein spacetimes. In general relativity the matter content of the spacetime is described by the energy momentum tensor T which is to be determined from physical considerations dealing with the distribution of matter and energy. Since the matter content of the universe is assumed to behave like a perfect fluid in the standard cosmological models, the physical motivation for studying Lorentzian manifolds is the assumption that a gravitational field may be effectively modeled by some Lorentzian metric defined on a suitable four dimensional manifold M. The Einstein equations are fundamental in the construction of cosmological models which imply that the matter determines the geometry of the spacetime and conversely the motion of matter is determined by the metric tensor of the space which is non-flat.

The physical motivation for studying various types of spacetime models in cosmology is to obtain the information of different phases in the evolution of the

universe, which may be classified into three phases, namely, the initial phase, the intermediate phase and the final phase. The initial phase is just after the Big Bang when the effects of both viscosity and heat flux were quite pronounced. The intermediate phase is that when the effect of viscosity was no longer significant but the heat flux was till not negligible. The final phase, which extends to the present state of the universe when both the effects of viscosity and heat flux have become negligible and the matter content of the universe may be assumed to be perfect fluid. The study of $(LQE)_n$ is important because such spacetime represents the third phase in the evolution of the universe. Consequently, the investigations of quasi-Einstein manifolds helps us to have a deeper understanding of the global character of the universe including the topology, because the nature of the singularities can be defined from a differential geometric stand point.

It is well known that a locally symmetric manifold is Ricci parallel and the converse holds for dimension three. By the decomposition of the covariant derivative ∇S of the Ricci tensor S of type $(0, 2)$, A. Gray [11] introduced two important classes $\mathscr{A}$, $\mathscr{B}$, which lie between the class of Ricci-parallel manifolds and the manifolds of constant scalar curvature, namely (i) the class $\mathscr{A}$ is the class of manifolds whose Ricci tensor is cyclic parallel and (ii) the class $\mathscr{B}$ is the class of manifolds whose Ricci tensor is of Codazzi type. In the present paper both the classes of quasi-Einstein spacetimes are classified.

Section 2 of the paper is devoted to the study of quasi-Einstein spacetime with cyclic parallel Ricci tensor. In a quasi-Einstein spacetime with cyclic parallel Ricci tensor, the scalar curvature is always constant. In a quasi-Einstein spacetime with constant scalar curvature, the associated scalars α and β are not necessarily constants. However, if α and β are constants then such a spacetime is of constant scalar curvature. It is proved that a quasi-Einstein spacetime with cyclic parallel Ricci tensor, the associated scalars are constants. Also it is shown that in a quasi-Einstein spacetime with cyclic parallel Ricci tensor, the energy-momentum tensor is cyclic parallel. In a quasi-Einstein spacetime with cyclic parallel Ricci tensor the nature of the generator ρ is determined and proved that it is a Killing vector field.

Section 3 deals with perfect fluid quasi-Einstein spacetimes with the generator ρ of the spacetime as the flow vector field of the fluid and proved that such a spacetime can not contain pure matter.

Section 4 is concerned with the study of perfect fluid quasi-Einstein spacetimes with Codazzi type energy-momentum tensor. It is proved that in a quasi-Einstein spacetime, the energy-momentum tensor is of Codazzi type if and only if the Ricci tensor is of Codazzi type. Again, it is shown that if the energy-

momentum tensor of a perfect fluid quasi-Einstein spacetime is of Codazzi type then both the energy density and isotropic pressure of the spacetimes are constant over a hypersurface orthogonal to the flow vector field. Again it is shown that if the energy-momentum tensor of a perfect fluid quasi-Einstein spacetime is of Codazzi type, then the possible local cosmological structure of such a spacetime are of Petrov type I, D or O. It is proved that if a perfect fluid quasi-Einstein spacetime with Codazzi type energy-momentum tensor admits a conformal Killing vector field, then such a spacetime is either conformally flat or of Petrov type N.

Section 5 deals with conformally flat quasi-Einstein spacetimes and proved that such a spacetime is infinitesimally spatially isotropic relative to the unit timelike vector field ρ. Finally the last section deals with some non trivial examples of quasi-Einstein spacetimes.

2. Quasi-Einstein Spacetimes with Cyclic Parallel Ricci Tensor

Let us consider a quasi-Einstein spacetime with cyclic parallel Ricci tensor. Then we have

$$(2.1) \qquad (\nabla_X S)(Y, Z) + (\nabla_Y S)(X, Z) + (\nabla_Z S)(X, Y) = 0.$$

From (1.1) it follows that

$$(2.2) \qquad\qquad r = 4\alpha - \beta$$

and

$$(2.3) \qquad\qquad S(\rho, \rho) = \beta - \alpha.$$

Again from (1.1), we obtain

$$(2.4) \qquad (\nabla_Z S)(X, Y) = d\alpha(Z)g(X, Y) + d\beta(Z)A(X)A(Y)$$
$$+ \beta[(\nabla_Z A)(X)A(Y) + A(X)(\nabla_Z A)(Y)].$$

In view of (2.4), (2.1) yields

$$(2.5) \qquad d\alpha(X)g(Y, Z) + d\beta(X)A(Y)A(Z) + d\alpha(Y)g(Z, X)$$
$$+ d\beta(Y)A(Z)A(X) + d\alpha(Z)g(X, Y) + d\beta(Z)A(X)A(Y)$$
$$+ \beta[(\nabla_X A)(Y)A(Z) + A(Y)(\nabla_X A)(Z) + (\nabla_Y A)(Z)A(X)$$
$$+ A(Z)(\nabla_Y A)(X) + (\nabla_Z A)(X)A(Y) + A(X)(\nabla_Z A)(Y)] = 0.$$

Setting $Y = Z = \rho$ in (2.5), we obtain

$$(2.6) \qquad 2\beta(\nabla_\rho A)(X) = d\beta(X) - d\alpha(X) + 2A(X)[d\alpha(\rho) - d\beta(\rho)].$$

Let $\{e_i : i = 1, 2, 3, 4\}$ be an orthonormal basis of the tangent space at each point of the spacetime. Setting $Y = Z = e_i$ in (2.5) and then taking summation for $1 \leq i \leq 4$, we get

$$(2.7) \qquad 6\, d\alpha(X) - d\beta(X) + 2A(X)\, d\beta(\rho)$$

$$+ 2\beta\left[(\nabla_\rho A)(X) + \sum_{i=1}^{4} \varepsilon_i(\nabla_{e_i} A)(e_i)A(X)\right] = 0,$$

where $\varepsilon_i = g(e_i, e_i)$.

Now from (2.2) we have

$$(2.8) \qquad dr(X) = 4\, d\alpha(X) - d\beta(X).$$

In a quasi-Einstein spacetime with cyclic parallel Ricci tensor, the scalar curvature r is always constant and hence

$$(2.9) \qquad dr(X) = 0 \quad \text{for all } X.$$

By virtue of (2.9) we have from (2.8) that

$$(2.10) \qquad d\alpha(X) = \frac{1}{4}\, d\beta(X).$$

Using (2.10) in (2.6) we have

$$(2.11) \qquad 2\beta(\nabla_\rho A)(X) = \frac{3}{4}\, d\beta(X) - \frac{3}{2}A(X)\, d\beta(\rho).$$

In view of (2.10) and (2.11), (2.7) yields

$$(2.12) \qquad 2\beta\sum_{i=1}^{4} \varepsilon_i(\nabla_{e_i} A)(e_i)A(X) = -\frac{5}{4}\, d\beta(X) - \frac{1}{2}A(X)\, d\beta(\rho).$$

Setting $X = \rho$ in (2.11), we obtain

$$(2.13) \qquad 2\beta\sum_{i=1}^{4} \varepsilon_i(\nabla_{e_i} A)(e_i) = \frac{3}{4}\, d\beta(\rho).$$

Again setting $X = \rho$ in (2.11) we get

$$(2.14) \qquad d\beta(\rho) = 0.$$

By virtue of (2.14), it follows from (2.11) and (2.13) that

(2.15)
$$2\beta(\nabla_\rho A)(X) = \frac{3}{4}\,d\beta(X)$$

and

(2.16)
$$\sum_{i=1}^{4} \varepsilon_i(\nabla_{e_i}A)(e_i) = 0.$$

Using (2.10) and (2.14)–(2.16) in (2.7) we get

(2.17)
$$d\beta(X) = 0 \quad \text{for all } X$$

and

(2.18)
$$d\alpha(X) = 0 \quad \text{for all } X,$$

that is, α and β are constants. This leads to the following:

THEOREM 2.1.   *In a quasi-Einstein spacetime with cyclic parallel Ricci tensor, the associated scalars are constants.*

The general relativity flows from Einstein's equation which is given by

(2.19)
$$S(X, Y) - \frac{r}{2}g(X, Y) + \lambda g(X, Y) = kT(X, Y)$$

for all vector fields X, Y, where S is the Ricci tensor of type $(0, 2)$, r is the scalar curvature, k is the gravitational constant, λ is the cosmological constant and T is the energy-momentum tensor of type $(0, 2)$. The matter content of the spacetime is described by the energy-momentum tensor T which is to be determined from physical considerations dealing with the distribution of matter and energy. Now from (2.19) we have

(2.20)
$$(\nabla_Z S)(X, Y) = \frac{1}{2}\,dr(Z)g(X, Y) + k(\nabla_Z T)(X, Y).$$

By virtue of (2.9) we have from (2.20) that

(2.21)
$$(\nabla_Z S)(X, Y) = k(\nabla_Z T)(X, Y)$$

and hence from (2.1) we obtain

(2.22)
$$(\nabla_X T)(Y, Z) + (\nabla_Y T)(X, Z) + (\nabla_Z T)(X, Y) = 0,$$

that is, the energy-momentum tensor is cyclic parallel. Hence we can state the following:

THEOREM 2.2. *In a quasi-Einstein spacetime with cyclic parallel Ricci tensor, the energy-momentum tensor is cyclic parallel.*

Again if (2.22) holds, then in view of (2.21) we obtain

$$(2.23) \qquad (\nabla_X S)(Y, Z) + (\nabla_Y S)(X, Z) + (\nabla_Z S)(X, Y)$$

$$= \frac{1}{2}[dr(X)g(Y, Z) + dr(Y)g(Z, X) + dr(Z)g(X, Y)].$$

Again in a quasi-Einstein spacetime the relation (2.8) holds. Hence if α and β are constants, then by virtue of (2.8) we obtain from the above relation that the Ricci tensor is cyclic parallel. Thus we can state the following:

THEOREM 2.3. *If in a quasi-Einstein spacetime with constant associated scalars, the energy-momentum tensor is cyclic parallel, then the Ricci tensor is cyclic parallel.*

Next in view of (2.17) and (2.18), (2.4) implies that

$$(2.24) \quad (\nabla_X A)(Y)A(Z) + A(Y)(\nabla_X A)(Z) + (\nabla_Y A)(Z)A(X)$$

$$+ A(Z)(\nabla_Y A)(X) + (\nabla_Z A)(X)A(Y) + A(X)(\nabla_Z A)(Y) = 0.$$

In a quasi-Einstein spacetime with cyclic parallel Ricci tensor, α and β are constants, and hence (2.11) yields

$$(\nabla_\rho A)(X) = 0 \quad \text{for all } X.$$

Setting $Z = \rho$ in (2.24) we obtain by virtue of above that

$$(2.25) \qquad\qquad (\nabla_X A)(Y) + (\nabla_Y A)(X) = 0,$$

which yields

$$(2.26) \qquad\qquad g(Y, \nabla_X \rho) + g(X, \nabla_Y \rho) = 0,$$

which implies that ρ is a Killing vector field. This leads to the following:

THEOREM 2.4. *In a quasi-Einstein spacetime with cyclic parallel Ricci tensor, the generator ρ is a Killing vector field.*

3. Perfect Fluid Quasi-Einstein Spacetimes

We now consider that the matter distribution of a non-flat quasi-Einstein spacetime is perfect fluid. Then the Einstein's field equation without cosmological constant is given by

$$(3.1) \qquad S(X, Y) - \frac{r}{2}g(X, Y) = kT(X, Y)$$

for all vector fields X, Y, where S is the Ricci tensor of type $(0, 2)$, r is the scalar curvature, k is the gravitational constant and T is the energy-momentum tensor of type $(0, 2)$.

In a perfect fluid spacetime, the energy-momentum tensor is of the following form [18]:

$$(3.2) \qquad T(X, Y) = pg(X, Y) + (\sigma + p)A(X)A(Y),$$

where σ, p are respectively the energy density, isotropic pressure and ρ is the unit timelike flow vector field of the fluid such that $A(X) = g(X, \rho)$ for all X.

In view of (3.2), the relation (3.1) can be written as

$$(3.3) \qquad S(X, Y) - \frac{r}{2}g(X, Y) = k[pg(X, Y) + (\sigma + p)A(X)A(Y)].$$

Taking a frame field and contracting (3.3) over X and Y, we get

$$(3.4) \qquad r = k(\sigma - 3p).$$

By virtue of (3.4), (3.3) yields

$$(3.5) \qquad S(X, Y) = k\left[(\sigma + p)A(X)A(Y) + \frac{1}{2}(\sigma - p)g(X, Y)\right]$$

and hence

$$(3.6) \qquad S(QX, Y) = k\left[(\sigma + p)A(QX)A(Y) + \frac{1}{2}(\sigma - p)S(X, Y)\right].$$

Taking contraction on (3.6) over X and Y, we obtain

$$(3.7) \qquad \|Q\|^2 = k\left[(\sigma + p)S(\rho, \rho) + \frac{1}{2}(\sigma - p)r\right].$$

Using (2.3) and (3.4) in (3.7) we have

$$(3.8) \qquad \|Q\|^2 = k\left[(\sigma + p)(\beta - \alpha) + \frac{1}{2}k(\sigma - p)(\sigma - 3p)\right].$$

Setting $X = Y = \rho$ in (3.5) and using (2.3) we get

$$(3.9) \qquad \beta - \alpha = \frac{k}{2}(\sigma + 3p).$$

Since the quasi-Einstein spacetime under consideration is non-flat, we have $\beta - \alpha \neq 0$ and hence (3.9) implies that $(\sigma + 3p) \neq 0$ and $k \neq 0$.

So by virtue of (3.9) we obtain from (3.8) that

$$(3.10) \qquad \|Q\|^2 = k^2(\sigma^2 + 3p^2).$$

Let us suppose that the length of the Ricci operator of the perfect fluid non-flat quasi-Einstein spacetime be $\frac{1}{3}r^2$, where r is the scalar curvature of the spacetime. Then from (3.10) we have

$$\frac{1}{3}r^2 = k^2(\sigma^2 + 3p^2),$$

which yields by virtue of (3.4) that

$$(3.11) \qquad k^2(\sigma + 3p)\sigma = 0.$$

Since $(\sigma + 3p) \neq 0$ and $k \neq 0$, it follows from (3.11) that $\sigma = 0$, which is not possible as when the pure matter exists, σ is always greater than zero. Hence the spacetime under consideration can not contain pure matter.

Now we determine the sign of pressure in such a spacetime without pure matter. Hence for $\sigma = 0$, (3.4) yields

$$(3.12) \qquad p = -\frac{r}{3k}.$$

Again by virtue of (1.1), for $\sigma = 0$ the relation (3.3) yields

$$(3.13) \qquad \left(\alpha - \frac{r}{2}\right)g(X, Y) + \beta A(X)A(Y) = kp[A(X)A(Y) + g(X, Y)].$$

Contracting (3.13) over X and Y, we get

$$r = \frac{1}{2}(4\alpha - \beta - 3kp),$$

and hence (3.12) reduces to

$$(3.14) \qquad p = \frac{1}{3k}(\beta - 4\alpha),$$

which implies that $p > 0$ if $\alpha < \frac{\beta}{4}$ and $p < 0$ if $\alpha > \frac{\beta}{4}$. This leads to the following:

THEOREM 3.1. *If a perfect fluid non-flat quasi-Einstein spacetime obeys Einstein equation without cosmological constant and the square of the length of the Ricci operator is $\frac{1}{3}r^2$, then the spacetime can not contain pure matter. Also in such a spacetime without pure matter the pressure of the fluid is positive or negative according as $\alpha < \frac{\beta}{4}$ or $\alpha > \frac{\beta}{4}$.*

4. Perfect Fluid Quasi-Einstein Spacetimes with Codazzi Type Energy-Momentum Tensor

We now consider the perfect fluid quasi-Einstein spacetime with Codazzi type energy-momentum tensor [10]. In a perfect fluid quasi-Einstein spacetime the energy-momentum tensor is of the form (3.2).

Since the energy-momentum tensor T is of Codazzi type [10], we have

$$(4.1) \qquad (\nabla_X T)(Y, Z) = (\nabla_Z T)(Y, X).$$

Then from (2.20) we get

$$(4.2) \qquad (\nabla_X S)(Y, Z) - (\nabla_Z S)(X, Y) - \frac{1}{2}[dr(X)g(Y, Z) - dr(Z)g(X, Y)]$$
$$= k[(\nabla_X T)(Y, Z) - (\nabla_Z T)(X, Y)].$$

In view of (4.1), (4.2) yields

$$(4.3) \qquad (\nabla_X S)(Y, Z) - (\nabla_Z S)(X, Y) - \frac{1}{2}[dr(X)g(Y, Z) - dr(Z)g(X, Y)] = 0.$$

Now contracting (4.3) over Y and Z we get

$$(4.4) \qquad dr(X) = 0 \quad \text{for all } X,$$

which shows that the scalar curvature is constant.

Using (4.4) in (4.3) we obtain

$$(4.5) \qquad (\nabla_X S)(Y, Z) = (\nabla_Z S)(Y, X),$$

that is, the Ricci tensor is of Codazzi type. Again taking contraction on (4.5) over Y and Z, we get the relation (4.4). Then by virtue of (4.4) and (4.5), (4.2) yields the relation (4.1). Hence we can state the following:

THEOREM 4.1 [24]. *In a quasi-Einstein spacetime the energy-momentum tensor is of Codazzi type if and only if its Ricci tensor is of Codazzi type.*

Let $T(X, Y) = g(\overset{*}{T}X, Y)$. Then from (2.19) it follows that

$$(4.6) \qquad QX = \frac{1}{2}rX + k\overset{*}{T}X - \lambda X,$$

where Q is the Ricci operator and hence (4.1) can be written as

$$(4.7) \qquad (\nabla_X \overset{*}{T})Y = (\nabla_Y \overset{*}{T})X.$$

From (3.2) we have

$$(4.8) \qquad \overset{*}{T}Y = (\sigma + p)A(Y)\rho + pY.$$

Differentiating (4.8) covariantly we get

$$(4.9) \qquad (\nabla_X \overset{*}{T})Y = \{(X\sigma) + (Xp)\}A(Y)\rho + (\sigma + p)(\nabla_X A)(Y)\rho$$
$$+ (\sigma + p)A(Y)\nabla_X \rho + (Xp)Y.$$

Using (4.9) in (4.7) and then setting $Y = \rho$, we obtain

$$(4.10) \qquad -\{(X\sigma) + (Xp)\}\rho - (\sigma + \rho)\nabla_X \rho + (Xp)\rho$$
$$= \{(\rho\sigma) + (\rho p)\}A(X)\rho + (\sigma + p)(\nabla_\rho A)(X)\rho + (\sigma + p)\nabla_\rho \rho + (\rho p)X.$$

In a subsequent paper [24] Shaikh et. al. obtained the following:

THEOREM 4.2. *If the energy-momentum tensor of a perfect fluid quasi-Einstein spacetime is of Codazzi type, then the integral curves of the flow vector field are geodesics.*

PROOF. If the energy-momentum tensor is of Codazzi type then by virtue of Theorem 4.1, the Ricci tensor is of Codazzi type and hence the relations (4.4) and (4.5) hold. Again using (1.1) in the relation (4.4), we get

$$(4.11) \qquad d\alpha(X)g(Y, Z) + d\beta(X)A(Y)A(Z) + \beta[(\nabla_X A)(Y)A(Z) + A(Y)(\nabla_X A)(Z)]$$
$$= d\alpha(Z)g(X, Y) + d\beta(Z)A(X)A(Y)$$
$$+ \beta[(\nabla_Z A)(X)A(Y) + A(X)(\nabla_Z A)(Y)].$$

Let $\{e_i : i = 1, 2, 3, 4\}$ be an orthonormal frame field at any point of the quasi-Einstein spacetime. Setting $Y = Z = e_i$ in (4.11) and then taking summation for $1 \le i \le 4$, we obtain

$$(4.12) \quad 3\,d\alpha(X) - d\beta(X) = d\beta(\rho)A(X) + \beta\left\{(\nabla_\rho A)(X) + A(X)\sum_{i=1}^{4} \varepsilon_i(\nabla_{e_i}A)(e_i)\right\},$$

where $\varepsilon_i = g(e_i, e_i)$. Again setting $Y = Z = \rho$ in (4.11) we get

$$(4.13) \qquad d\beta(X) - d\alpha(X) = d\alpha(\rho)A(X) - d\beta(\rho)A(X) - \beta(\nabla_\rho A)(X).$$

Adding (4.12) and (4.13) we have

$$(4.14) \qquad\qquad 2\,d\alpha(X) = d\alpha(\rho)A(X) + \beta A(X)\sum_{i=1}^{4} \varepsilon_i(\nabla_{e_i}A)(e_i).$$

Setting $X = \rho$ in (4.14) we get

$$(4.15) \qquad\qquad\qquad -3\,d\alpha(\rho) = \beta\sum_{i=1}^{4} \varepsilon_i(\nabla_{e_i}A)(e_i).$$

By virtue of (4.14) and (4.15) we obtain

$$(4.16) \qquad\qquad\qquad d\alpha(X) + d\alpha(\rho)A(X) = 0.$$

Since in a quasi-Einstein spacetime the scalar curvature r is given by $r = 4\alpha - \beta$, so by virtue of (4.4) we get $d\alpha(X) = \frac{1}{4}\,d\beta(X)$, and hence (4.16) yields

$$(4.17) \qquad\qquad\qquad d\beta(X) + d\beta(\rho)A(X) = 0.$$

In view of (4.15), (4.16) and (4.17), we obtain from (4.12) that

$$(4.18) \qquad\qquad\qquad (\nabla_\rho A)(X) = 0$$

for all X, which implies that

$$(4.19) \qquad\qquad\qquad \nabla_\rho \rho = 0$$

and hence the integral curves of ρ are geodesics.

Again substituting Z by ρ in (4.11) and then using (4.17) and (4.18), we obtain

$$(4.20) \qquad\qquad \beta(\nabla_X A)(Y) = d\alpha(X)A(Y) - d\alpha(\rho)g(X, Y).$$

From (4.16) and (4.20), it follows that

(4.21) $$(\nabla_X A)(Y) = f\{g(X, Y) + A(X)A(Y)\},$$

where f is a non-vanishing scalar given by

$$f = -\frac{d\alpha(\rho)}{\beta} = -\frac{d\beta(\rho)}{4\beta}.$$

By virtue of (4.18), (4.19) and (4.21) it follows from (4.10) that

(4.22) $$-\{(X\sigma) + (Xp)\}\rho - (\sigma + \rho)f[X + A(X)\rho] + (Xp)\rho$$

$$= \{(\rho\sigma) + (\rho p)\}A(X)\rho + (\rho p)X.$$

Taking contraction on (4.6) we have

(4.23) $$r = 4\lambda + k(\sigma - 3p).$$

Again differentiating (4.23) covariantly, we have

(4.24) $$dr(X) = k\{(X\sigma) - 3(Xp)\}.$$

Since the spacetime under consideration has Codazzi type energy-momentum tensor, we have the relation (4.4) and hence by virtue of (4.4) we get from (4.24) that

(4.25) $$(Xp) = \frac{1}{3}(X\sigma).$$

Using (4.25) in (4.22) we obtain

(4.26) $$-(X\sigma)\rho - (\sigma + \rho)f[X + A(X)\rho] = \frac{4}{3}(\rho\sigma)A(X)\rho + \frac{1}{3}(\rho\sigma)X.$$

Taking the inner product on both sides of (4.26) by Y, we get

(4.27) $$-(X\sigma)A(Y) - (\sigma + \rho)f[g(X, Y) + A(X)A(Y)]$$

$$= \frac{1}{3}(\rho\sigma)[4A(X)A(Y) + g(X, Y)].$$

Setting $Y = \rho$ in (4.27) we obtain

(4.28) $$(X\sigma) = -(\rho\sigma)A(X) \quad \text{i.e., } \operatorname{grad} \sigma = -(\rho\sigma)\rho,$$

and hence from (4.25) and (4.28) it follows that

(4.29) $$(Xp) = -(\rho p)A(X) \quad \text{i.e., } \operatorname{grad} p = -(\rho p)\rho.$$

From (4.28) and (4.29) we may conclude that σ and p are constants over a hypersurface orthogonal. Thus we can state the following:

THEOREM 4.3.   *If the energy-momentum tensor of a perfect fluid quasi-Einstein spacetime is of Codazzi type, then both the energy density and isotropic pressure of the fluid are constants over a hypersurface orthogonal to ρ.*

Again since the integral curves of ρ in a quasi-Einstein spacetime with Codazzi type energy-momentum tensor are geodesics, the Roy Choudhury equation [20] for the fluid in a quasi-Einstein spacetime can be written as

$$(4.30) \qquad (\nabla_X A)(Y) = \omega(X, Y) + \tau(X, Y) + f[g(X, Y) + A(X)A(Y)],$$

where ω is the vorticity tensor and τ is the shear tensor respectively.

Comparing (4.21) and (4.30) we get

$$(4.31) \qquad\qquad\qquad \omega(X, Y) + \tau(X, Y) = 0.$$

Also from (4.21) it follows that

$$(4.32) \qquad\qquad\qquad (\nabla_X A)(Y) - (\nabla_Y A)(X) = 0,$$

i.e.,

$$g(\nabla_X \rho, Y) - g(X, \nabla_Y \rho) = 0,$$

which implies that *curl* $\rho = 0$, that is, ρ is irrotational. Hence the vorticity of the fluid vanishes. Therefore $\omega(X, Y) = 0$ and consequently (4.31) implies that $\tau(X, Y) = 0$. Thus we can state the following:

THEOREM 4.4.   *In a perfect fluid quasi-Einstein spacetime with Codazzi type energy-momentum tensor, the fluid has vanishing vorticity and vanishing shear.*

According to Petrov [19] classification a spacetime can be devided into six types denoted by I, II, III, D, N and O. Again Barnes [1] has been proved that if a perfect fluid spacetime is shear free, vorticity free and the velocity vector field of the fluid is hypersurface orthogonal and the energy density is constant over a hypersurface orthogonal to the velocity vector field, then the possible local cosmological structure of the spacetime are of Petrov type I, D or O. Since in a perfect fluid quasi-Einstein spacetime the velocity vector field of the fluid is always hypersurface orthogonal by virtue of Theorem 4.3. and Theorem 4.4., we can state the following:

THEOREM 4.5. *If the energy-momentum tensor of a perfect fluid quasi-Einstein spacetime is of Codazzi type, then the possible local cosmological structure of the spacetime are of Petrov type I, D or O.*

In a spacetime, the divergence of the conformal curvature tensor 'C' is given by

$$(4.33) \qquad (div\ C)(X, Y)Z = \frac{1}{2}\left[(\nabla_X S)(Y, Z) - (\nabla_Z S)(Y, X)\right.$$

$$\left. -\frac{1}{3}\{dr(X)g(Y, Z) - dr(Z)g(Y, X)\}\right].$$

If the energy-momentum tensor is of Codazzi type then by virtue of Theorem 4.1, the Ricci tensor is of Codazzi type and hence the relation (4.5) holds, from which it follows that the scalar curvature r is constant. Consequently from (4.33) we have $div\ C = 0$.

Again Sharma [21] proved that if a spacetime with divergence free conformal curvature admits a conformal Killing vector field, then the spacetime is either conformally flat or of Petrov type N. Since a quasi-Einstein spacetime with Codazzi type energy-momentum tensor is of divergence free conformal curvature tensor, we can state the following:

THEOREM 4.6. *If a perfect fluid quasi-Einstein spacetime with Codazzi type energy-momentum tensor admits a conformal Killing vector field, then the spacetime is either conformally flat or of Petrov type N.*

5. Conformally Flat Quasi-Einstein Spacetimes

Let the quasi-Einstein spacetime be conformally flat. Then the curvature tensor R of type $(1, 3)$ is of the following form:

$$(5.1) \quad R(X, Y)Z = \frac{1}{2}[S(Y, Z)X - S(X, Z)Y + g(Y, Z)QX - g(X, Z)QY]$$

$$-\frac{r}{6}[g(Y, Z)X - g(X, Z)Y],$$

where Q is the Ricci operator. From (1.1) we have

$$(5.2) \qquad\qquad QX = \alpha X + \beta A(X)\rho.$$

Using (1.1), (2.2) and (5.2), we can express (5.1) as follows:

$$(5.3) \quad R(X,Y)Z = \frac{2\alpha+\beta}{6}[g(Y,Z)X - g(X,Z)Y] + \frac{\beta}{6}[A(Y)A(Z)X$$
$$- A(X)A(Z)Y + g(Y,Z)A(X)\rho - g(X,Z)A(Y)\rho].$$

Let $\rho^\perp$ be the 3-dimensional distribution orthogonal to the generator ρ. Then from (5.1) we have

$$(5.4) \quad R(X,Y)Z = \frac{2\alpha+\beta}{6}[g(Y,Z)X - g(X,Z)Y] \quad \text{for all } X,Y,Z \in \rho^\perp$$

and hence

$$(5.5) \quad R(X,\rho)\rho = -\frac{2\alpha+\beta}{6}X \quad \text{for all } X \in \rho^\perp.$$

According to Karchar [12] a Lorentzian manifold is called infinitesimally spatially isotropic relative to a timelike unit vector field ρ if its curvature tensor R satisfies the relations

$$R(X,Y)Z = l[g(Y,Z)X - g(X,Z)Y] \quad \text{for all } X,Y,Z \in \rho^\perp$$

and

$$R(X,\rho)\rho = mX \quad \text{for all } X \in \rho^\perp,$$

where l, m are real valued functions on the manifold. So by virtue of (5.4) and (5.5), we can state the following:

THEOREM 5.1.   *A conformally flat quasi-Einstein spacetime is infinitesimally spatially isotropic relative to the unit timelike vector field ρ.*

Again using (1.1) in (2.19) we have

$$(5.6) \quad \left(\alpha - \frac{r}{2} + \lambda\right)g(X,Y) + \beta A(X)A(Y) = kT(X,Y).$$

Let us assume that α and β are constants. Then r is also a constant. Also let us assume that the generator ρ be a Killing vector field, that is,

$$(5.7) \quad (\mathcal{L}_\rho g)(X,Y) = 0,$$

where $\mathcal{L}$ denotes the Lie derivative with respect to ρ.

Now from (5.6) we obtain

$$(5.8) \qquad \left(\alpha - \frac{r}{2} + \lambda\right)(\mathscr{L}_\rho g)(X, Y) = k(\mathscr{L}_\rho T)(X, Y).$$

In view of (5.7), (5.6) yields

$$(5.9) \qquad (\mathscr{L}_\rho T)(X, Y) = 0, \quad \text{since } k \neq 0.$$

Thus we can state the following:

THEOREM 5.2. *In a quasi-Einstein spacetime with constant associated scalars obeying Einstein's equation, the generator ρ of the spacetime is a Killing vector field if and only if the Lie derivative of the energy-momentum tensor with respect to ρ is zero.*

6. Some Examples of Quasi-Einstein Spacetimes

This section deals with several proper examples of quasi-Einstein spacetimes.

EXAMPLE 6.1. In 1989 K. Matsumoto [13] introduced the notion of LP-Sasakian manifolds. Then I. Mihai and R. Rosca [15] introduced the same notion independently and later studied by many authors ([14], [16], [23]).

An n-dimensional differentiable manifold M is said to be a LP-Sasakian manifold ([4], [15]) if it admits an (1, 1) tensor field ϕ, a vector field ξ, an 1-form η and a Lorentzian metric g, which satisfy

$$(6.1) \qquad \eta(\xi) = -1,$$

$$(6.2) \qquad \phi^2 X = X + \eta(X)\xi,$$

$$(6.3) \qquad g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y),$$

$$(6.4) \qquad \text{(a)} \quad g(X, \xi) = \eta(X), \quad \text{(b)} \quad \nabla_X \xi = \phi X,$$

$$(6.5) \qquad (\nabla_X \phi)(Y) = g(X, Y)\xi + \eta(Y)X + 2\eta(X)\eta(Y)\xi,$$

where ∇ denotes the operator of covariant differentiation with respect to the Lorentzian metric g. The unit timelike vector field ξ is called the characteristic vector field of the manifold.

In a 4-dimensional connected LP-Sasakian manifold $M^4(\phi, \xi, \eta, g)$, the following relations hold ([4], [16]):

(6.6)   (a) $\phi \circ \xi = 0$,   (b) $\eta \circ \phi = 0$,

(6.7) $R(X, Y)\xi = \eta(Y)X - \eta(X)Y$,

(6.8) $S(X, \xi) = 3\eta(X)$,

where R is the curvature tensor of the manifold and S is the Ricci tensor of type (0, 2).

We consider a conformally flat connected 4-dimensional LP-Sasakian manifold (M^4, g). Since in a conformally flat connected 4-dimensional LP-Sasakian manifold the conformal curvature tensor C vanishes, we have

(6.9) $R(X, Y)Z = \dfrac{1}{2}[g(Y, Z)QX - g(X, Z)QY + S(Y, Z)X - S(X, Z)Y]$

$$-\dfrac{r}{6}[g(Y, Z)X - g(X, Z)Y],$$

Setting $Z = \xi$ in (6.9) we obtain by virtue of (6.7) and (6.8) that

(6.10) $\eta(X)QY - \eta(Y)QX = \left(1 - \dfrac{r}{3}\right)[\eta(Y)X - \eta(X)Y]$.

Replacing Y by ξ in (6.10) and then using (6.1) and (6.8) we get

$$QX = \left(\dfrac{r}{3} - 1\right)X + \left(\dfrac{r}{3} - 4\right)\eta(X)\xi,$$

which can be written as

$$S(X, Y) = \alpha g(X, Y) + \beta \eta(X)\eta(Y),$$

where $\alpha = \left(\frac{r}{3} - 1\right)$ and $\beta = \left(\frac{r}{3} - 4\right)$ are scalars. Hence a conformally flat connected 4-dimensional LP-Sasakian manifold (M^4, g) is a quasi-Einstein spacetime.

EXAMPLE 6.2.   As a generalization of LP-Sasakian manifold, recently Shaikh [22] introduced the notion of Lorentzian concircular structure manifold and proved its existence and also obtained several applications to the general relativity and cosmology. In a Lorentzian manifold (M^n, g) a vector field P defined by

$$g(X, P) = A(X)$$

for any $X \in \chi(M)$ is said to be a concircular vector field if

$$(\nabla_X A)(Y) = \alpha\{g(X, Y) + \omega(X)A(Y)\}$$

where α is a non-zero scalar and ω is a closed 1-form.

Let M^n be a Lorentzian manifold admitting a unit timelike concircular vector field ξ, called the characteristic vector field of the manifold. Then we have

(6.11) $$g(\xi, \xi) = -1.$$

Since ξ is a unit concircular vector field, there exists a non-zero 1-form η such that for

(6.12) $$g(X, \xi) = \eta(X)$$

the equation of the following form holds

(6.13) $$(\nabla_X \eta)(Y) = \alpha\{g(X, Y) + \eta(X)\eta(Y)\} \quad (\alpha \neq 0)$$

for all vector fields X, Y where ∇ denotes the operator of covariant differentiation with respect to the Lorentzian metric g and α is a non-zero scalar function satisfies

(6.14) $$\nabla_X \alpha = (X\alpha) = \rho\eta(X),$$

ρ being a certain scalar function.

If we put

(6.15) $$\phi X = \frac{1}{\alpha}\nabla_X \xi,$$

then from (6.13) we have

(6.16) $$\phi X = X + \eta(X)\xi,$$

from which it follows that ϕ is a symmetric (1, 1) tensor. Thus the Lorentzian manifold M^n together with the unit timelike concircular vector field ξ, its associated 1-form η and (1, 1) tensor field ϕ is said to be a Lorentzian concircular structure manifold (briefly $(LCS)_n$-manifold) [22].

Especially, if we take $\alpha = 1$, then we can obtain the LP-Sasakian structure of Matsumoto [13].

In a $(LCS)_4$ manifold, the following relations hold [22]:

(6.17) $$g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y),$$

(6.18) $$R(X, Y)\xi = (\rho - \alpha^2)\{\eta(Y)X - \eta(X)Y\},$$

(6.19) $$S(X, \xi) = 3(\rho - \alpha^2)\eta(X) \quad \text{for any } X, Y.$$

In a conformally flat $(LCS)_4$ manifold, the relation (6.9) holds. Then proceeding similarly as in Example 5.1 we obtain by virtue of (6.17)–(6.19) that in a con-

formally flat $(LCS)_4$ manifold, the Ricci tensor is of the form

$$S(Y,Z) = \gamma g(Y,Z) + \delta \eta(Y)\eta(Z),$$

where $\gamma = \frac{r}{3} - (\rho - \alpha^2)$ and $\delta = \frac{r}{3} - 4(\rho - \alpha^2)$ are non-vanishing scalars. Hence a conformally flat $(LCS)_4$ is a quasi-Einstein spacetime.

EXAMPLE 6.3.   The Robertson-Walker spacetime is a quasi-Einstein spacetime.

EXAMPLE 6.4.   We define a Lorentzian metric g on a 4-dimensional real number space $\boldsymbol{R}^4$ by the formula

$$(6.20) \qquad ds^2 = (x^1)^2 (dx^1)^2 + \sin^2 x^1 (dx^2)^2 + \sin^2 x^1 (dx^3)^2 - \sin^2 x^1 (dx^4)^2,$$

$$0 < x^1 < \frac{\pi}{2}.$$

Then the only non-vanishing components of the Christoffel symbols and the curvature tensor are

$$\Gamma^1_{11} = \frac{1}{x^1}, \quad \Gamma^2_{12} = \Gamma^3_{13} = \Gamma^4_{14} = \cot x^1,$$

$$\Gamma^1_{22} = -\frac{\sin x^1 \cos x^1}{(x^1)^2} = \Gamma^1_{33} = -\Gamma^1_{44},$$

$$R_{1221} = -\frac{x^1 \sin^2 x^1 + \sin x^1 \cos x^1}{x^1} = R_{1331} = -R_{1441},$$

$$R_{2332} = \frac{\sin^2 x^1 \cos^2 x^1}{(x^1)^2} = -R_{2442} = -R_{3443}$$

and the components which can be obtained from these by the symmetry properties. Using the above relations, we can find the non-vanishing components of Ricci tensor as follows:

$$S_{11} = -3\left(1 + \frac{\cot x^1}{x^1}\right),$$

$$S_{22} = \frac{1}{2(x^1)^3}[4x^1 \cos^2 x^1 - 2x^1 \sin^2 x^1 - \sin 2x^1] = S_{33} = -S_{44}.$$

Also it can be easily found that the scalar curvature of the manifold is non-zero. Therefore $\boldsymbol{R}^4$ with the considered metric is a Lorentzian manifold (M^4, g) of non-

vanishing scalar curvature. We shall now show that this M^4 is a quasi-Einstein spacetime, i.e., it satisfies (1.1).

Let us now consider the associated scalars as follows:

$$(6.21) \qquad \alpha = \frac{4x^1 \cos^2 x^1 - 2x^1 \sin^2 x^1 - \sin 2x^1}{2(x^1)^3 \sin^2 x^1}, \quad \beta = -\frac{1}{x^1 \sin^2 x^1}.$$

In terms of local coordinate system, let us consider the components of the 1-form A as follows:

$$(6.22) \qquad A_i(x) = \sqrt{2x^1 + \sin 2x^1} \quad \text{for } i = 1,$$
$$= 0 \quad \text{otherwise.}$$

In terms of local coordinate system, the defining condition (1.1) of a quasi-Einstein spacetime can be written as

$$(6.23) \qquad S_{ij} = \alpha g_{ij} + \beta A_i A_j, \quad i, j = 1, 2, 3, 4.$$

By virtue of (6.21) and (6.22), it can be easily shown that (6.23) holds for $i, j = 1, 2, 3, 4$. Therefore, (M^4, g) is quasi-Einstein spacetime. Hence we can state the following:

THEOREM 6.1. *Let (M^4, g) be a Lorentzian manifold endowed with the metric given in* (6.20). *Then (M^4, g) is a quasi-Einstein spacetime with non-vanishing scalar curvature.*

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