TSUKUBA J. MATH.
Vol. 32 No. 2 (2008), 349–360

REPRESENTATIONS OF NATURAL NUMBERS AS THE SUM OF A PRIME AND A k-TH POWER

By

Jörg Brüdern

Abstract. Subject to the Riemann hypothesis for Dirichlet L-functions an asymptotic formula is obtained for the number of representations of a natural number n as the sum of a prime and a k-th power, valid for almost n. Estimates for the error term in the asymptotic formula as well as for the size of the exceptional set are of a smaller order of magnitude than was known previously.

1. Introduction

We return to the questions investigated in collaboration with Perelli [2], and reexamine sums of a prime and a k-th power where $k \geq 2$ is a fixed integer. Here we are concerned with the number of representations in this form. When n is a natural number, let

$$r_k(n) = \sum_{p+x^k=n} \log p$$

where the sum is over all primes p and all natural numbers x. Likewise, let

$$r_k^*(n) = \sum_{p_1+p_2^k=n} (\log p_1)(\log p_2)$$

where now the sum is over all primes p_1, p_2. According to a widely accepted philosophy, one expects asymptotic formulae for $r_k(n)$ and $r_k^*(n)$ that coincide with the major arc contributions in a formal application of the circle method. In order to describe the main terms in these asymptotics precisely, we require some notation. Let $\varrho(n, p)$ denote the number of incongruent solutions of the congruence

$$x^k \equiv n \bmod p, \tag{1}$$

Received April 2, 2008.

and let $\varrho^*(n,p)$ denote the number of those solutions of (1) where $p \nmid x$. The numbers $\varrho(n,p)$ and $\varrho^*(n,p)$ of course also depend on k, but this is suppressed here for notational simplicity. Let $\mathscr{I}_k$ denote the the set of all natural numbers n for which the polynomial $x^k - n$ is irredicible over the rationals. Note that when n is a natural number, then $x^k - n$ is reducible if and only if $n = m^p$ for some prime $p|k$ (see, for example, Lang [7] Chapter VI, Theorem 9.1), and consequently $\mathscr{I}_k$ contains all but $O(N^{1/2})$ of the natural numbers not exceeding N. Only when $n \in \mathscr{I}_k$, we may expect that $r_k(n)$ is large. Indeed, when $n \in \mathscr{I}_k$, the singular products

$$\mathfrak{S}_k(n) = \prod_p \frac{1 - \frac{\varrho(n,p)}{p}}{1 - \frac{1}{p}}, \quad \mathfrak{S}_k^*(n) = \prod_p \frac{1 - \frac{\varrho^*(n,p)}{p-1}}{1 - \frac{1}{p}} \tag{2}$$

converge as a consequence of the prime ideal theorem, and the expected asymptotic formulae take the shape

$$r_k(n) = \mathfrak{S}_k(n) n^{1/k}(1 + o(1)), \quad r_k^*(n) = \mathfrak{S}_k^*(n) n^{1/k}(1 + o(1)). \tag{3}$$

However, we are far from being able to prove these formulae. All what is currently known is that

$$r_k(n) = \mathfrak{S}_k(n) n^{1/k}\left(1 + O\left(\frac{\log\log N}{\log N}\right)\right)$$

holds for all $n \in \mathscr{I}_k$ with at most $O(N/(\log N)^A)$ exceptions $n \le N$; here $A > 0$ is any fixed real number. This much follows from Miech [8] and Kawada [5], or Perelli and Zaccagnini [9], and a corresponding result for $r_k^*(n)$ is at least implicit in these sources. Because of the possible existence of Siegel zeros, there is little hope for improvements. However, if we assume that the Generalized Riemann Hypothesis (hereafter abbreviated to GRH) is true (that is, all non-trivial zeros of Dirichlet L-functions lie on the line $\operatorname{Re} s = \frac{1}{2}$), then both the error term and the exceptional set may be significantly reduced.

Theorem. *Suppose that GRH holds. Then, for any $k \ge 2$ the asymptotic formulae*

$$r_k(n) = \mathfrak{S}_k(n) n^{1/k} + O(n^{1/k - 1/(6000k^2)})$$

$$r_k^*(n) = \mathfrak{S}_k^*(n) n^{1/k} + O(n^{1/k - 1/(6000k^2)})$$

hold for all but $O(N^{1-1/(6000k^2)})$ of the natural numbers $n \in \mathscr{I}_k$ not exceeding N.

Kawada [5] observed that for $n \in \mathscr{I}_k$ one has $\mathfrak{S}_k(n) \gg (\log n)^{-k}$. Hence the theorem implies that all but $O(N^{1-1/(6000k^2)})$ of the n not exceeding N are the sum of a prime and a k-th power. This is weaker than Theorem 1 of Brüdern and Perelli [2] that asserts that subject to GRH, no more than $O(N^{1-1/(25k)})$ of the natural numbers not exceeding N are not the sum of a prime and a k-th power. However, our result is in line with the general observation, familiar from Waring's problem, that the price for the step from representations to asymptotics is a factor k.

The singular product $\mathfrak{S}_k^*(n)$ of course vanishes occasionally due to local obstructions. Let $w_k = \prod_{p-1 \mid k} p$, and let

$$\mathscr{H}_k = \{n \in \mathscr{I}_k : (n-1, w_k) = 1\}.$$

Only when $n \in \mathscr{H}_k$, one may expect $r_k^*(n)$ to be large. Of course $\mathscr{H}_k$ has positive density, and for $n \in \mathscr{H}_k$ one again has $\mathfrak{S}_k^*(n) \gg (\log n)^{-k}$. We now derive from the theorem that all but $O(N^{1-1/(6000k^2)})$ of the numbers $n \in \mathscr{H}_k$ not exceeding N have a representation $n = p_1 + p_2^k$ with primes p_1, p_2. This improves Theorem 2 of Brüdern and Perelli [2] where an estimate $O(N^{1-c/(k^2 \log k)})$ with some $c > 0$ was obtained for the size of the exceptional set in this representation problem.

The methods that we use are not dissimilar from the general framework of [2], but we shall have occasion to refer to two sources that were not available when [2] was written. The work of Kawada [6] deals with the singular product $\mathfrak{S}_k(n)$, and is very relevant for us. Moreover, work of Ford [3] plays a role in a crucial pruning process within the circle method work in § 3. We explain the mechanism behind this at a later stage.

Notation is standard or otherwise introduced when appropriate. Occassionally we make use of the ε-convention: whenever ε appears in a statement, it is asserted that the statement is true for all real $\varepsilon > 0$; implicit constants may depend on ε. We also write $e(\alpha)$ as an abbreviation for $\exp(2\pi i\alpha)$. The letter p always denotes a prime.

2. A Mean Square Estimate

We prove the theorem with the aid of the circle method. The beginning is conventional. Let

$$f(\alpha) = \sum_{p \le N} (\log p) e(\alpha p).$$

We write $P = N^{1/k}$ and then also put

$$g(\alpha) = \sum_{x \le P} e(\alpha x^k), \quad g^*(\alpha) = \sum_{p \le P} (\log p) e(\alpha p^k).$$

By orthogonality, it follows that whenever $1 \le n \le N$, one has

$$r_k(n) = \int_0^1 f(\alpha) g(\alpha) e(-\alpha n)\, d\alpha. \tag{4}$$

The same formula holds for $r_k^*(n)$ if $g(\alpha)$ is replaced by $g^*(\alpha)$. We now approximate $r_k(n)$ and $r_k^*(n)$, in mean square, by the major arc contribution. Before we can do this, we need to define the Gauss sums

$$S(q,a) = \sum_{x=1}^{q} e\left(\frac{ax^k}{q}\right), \quad S^*(q,a) = \sum_{\substack{x=1 \\ (x,q)=1}}^{q} e\left(\frac{ax^k}{q}\right)$$

as well as the sums

$$v(\beta) = \sum_{x \le N} e(\beta x), \quad w(\beta) = \frac{1}{k} \sum_{x \le P} e(\beta x) x^{1/k-1}.$$

The following lemma is useful.

LEMMA 1. *Assume GRH. Let* $(a,q) = 1$. *Then*

$$f\left(\frac{a}{q} + \beta\right) = \frac{\mu(q)}{\varphi(q)} v(\beta) + O((qN)^{1/2+\varepsilon}(1 + N|\beta|)).$$

Moreover, if $k \ge 2$ *and* q *is square-free, then*

$$g\left(\frac{a}{q} + \beta\right) = \frac{S(q,a)}{q} w(\beta) + O(q^{1/2+\varepsilon}(1 + N|\beta|)^{1/2})$$

and

$$g^*\left(\frac{a}{q} + \beta\right) = \frac{S^*(q,a)}{\varphi(q)} w(\beta) + O((qP)^{1/2+\varepsilon}(1 + N|\beta|)^{1/2}).$$

PROOF. The formula for $g\left(\frac{a}{q} + \beta\right)$ follows from Theorem 4.1 of Vaughan [10]. The formulae for $f\left(\frac{a}{q} + \beta\right)$ and $g^*\left(\frac{a}{q} + \beta\right)$ are essentially contained in Lemmata 2 and 3 of Brüdern and Perelli [2]. However, in [2] the sums $f(\alpha)$ and $g^*(\alpha)$ are over a dyadic range (like $N < p \le 2N$), and the sums $v(\beta)$, $w(\beta)$ are replaced by their integral analogues. Yet, it is clear that the methods of [2] in conjunction with Euler's summation formula completely cover the needs to establish the current version of Lemma 1.

Let $1 \le X \le \frac{1}{2}\sqrt{N}$, and let $\mathfrak{M}(X)$ denote the union of the intervals

$$\{\alpha \in [0,1] : |q\alpha - a| \le XN^{-1}\} \tag{5}$$

with $(a,q) = 1$, $1 \le q \le X$. Note that these intervals are pairwise disjoint. For simplicity, we write

$$\mathfrak{M} = \mathfrak{M}(N^{1/4}), \quad \mathfrak{m} = [0,1]\backslash\mathfrak{M}.$$

For later use, we record here that the argument on p. 518 of Brüdern and Perelli [2] gives

$$\sup_{\alpha \in \mathfrak{m}} |f(\alpha)| \ll N^{7/8+\varepsilon}. \tag{6}$$

We are now prepared to introduce a first approximation to $r_k(n)$ and $r_k^*(n)$. When $\alpha \in \mathfrak{M}$ is in the interval defined by (5) with $X = N^{1/4}$, let

$$V(\alpha) = \frac{\mu(q)S(q,a)}{q\varphi(q)} v\left(\alpha - \frac{a}{q}\right) w\left(\alpha - \frac{a}{q}\right),$$

$$V^*(\alpha) = \frac{\mu(q)S^*(q,a)}{\varphi(q)^2} v\left(\alpha - \frac{a}{q}\right) w\left(\alpha - \frac{a}{q}\right),$$

and write

$$t_k(n) = \int_{\mathfrak{M}} V(\alpha)e(-\alpha n)\,d\alpha, \quad t_k^*(\alpha) = \int_{\mathfrak{M}} V^*(\alpha)e(-\alpha n)\,d\alpha. \tag{7}$$

Lemma 2. *Assume GRH. Then, for all $k \ge 2$ one has*

$$\sum_{n \le N} |r_k(n) - t_k(n)|^2 \ll P^2 N^{1-1/(50k^2)}.$$

The same is true if r_k, t_k are replaced by r_k^*, t_k^*.

In the interest of expository simplicity, we present a detailed proof only of the version with r_k^*, t_k^* and leave the (simple) changes required for the other part of Lemma 2 to the reader. The main difficulty is that the major arcs $\mathfrak{M}$ are extraordinarily "thick" for a k-th power. Hence, the bulk of the work is a pruning of the major arcs that we perform in section 4, after removing the minor arcs in the next section.

3. Proof of Lemma 2: First Steps

Consider the function, defined on $[0,1]$, that is given by $f(\alpha)g^*(\alpha)$ when $\alpha \in \mathfrak{m}$, and by $f(\alpha)g^*(\alpha) - V^*(\alpha)$ when $\alpha \in \mathfrak{M}$. Its n-th Fourier coefficient is $r_k^*(n) - t_k^*(n)$, as a consequence of (4) and (7). Hence, by Bessel's inequality,

$$\sum_{n \le N} |r_k^*(n) - t_k^*(n)|^2 \le \int_{\mathfrak{m}} |f(\alpha)g^*(\alpha)|^2 \, d\alpha + \int_{\mathfrak{M}} |f(\alpha)g^*(\alpha) - V^*(\alpha)|^2 \, d\alpha. \quad (8)$$

In this section, we estimate the minor arc part of the right hand side of (8). Let $\delta > 0$ be a fixed real number; we assume throughout that δ is small. Take

$$s = [2 - \log \delta] k^2. \quad (9)$$

Then, by combining a classical version of Vinogradov's mean value theorem (Vaughan [10], Theorem 5.1) with Theorem 1 of Ford [3], we obtain for each integer m with $1 \le m \le k$ the estimate

$$\int_0^1 |g(\alpha)|^{2s} \, d\alpha \ll P^{2s-k+\Delta/m}$$

where

$$\Delta = \frac{1}{2} k^2 \exp(-(2s - 2k - m(m-1))/(2k^2)) < \delta k^2.$$

We take $m = k$ and then deduce that

$$\int_0^1 |g(\alpha)|^{2s} \ll P^{2s} N^{\delta - 1}. \quad (10)$$

On considering the underlying diophantine equations, we conclude that

$$\int_0^1 |g^*(\alpha)|^{2s} \, d\alpha \ll (\log N)^{2s} \int_0^1 |g(\alpha)|^{2s} \, d\alpha \ll (P \log N)^{2s} N^{\delta - 1}. \quad (11)$$

We now apply Hölder's inequality in the form

$$\int_{\mathfrak{m}} |f(\alpha)g^*(\alpha)|^2 \, d\alpha \le \sup_{\alpha \in \mathfrak{m}} |f(\alpha)|^{2/s} \left(\int_0^1 |f(\alpha)|^2 \, d\alpha \right)^{1-1/s} \left(\int_0^1 |g^*(\alpha)|^{2s} \, d\alpha \right)^{1/s}.$$

By Parseval's identity, the first integral on the right hand side here does not exceed $O(N \log N)$. Hence, by (6) and (11), we now infer the bound

$$\int_{\mathfrak{m}} |f(\alpha)g^*(\alpha)|^2\, d\alpha \ll N^{7/4s}(N \log N)^{1-1/s}(P \log N)^2 N^{(1/s)(\delta-1)}$$

$$\ll N^{1-(1/s)(1/4-\delta)}(\log N)^3 P^2.$$

We may now take $\delta = e^{-8}$ and $s = 10k^2$ in accordance with (9) to confirm the estimate

$$\int_{\mathfrak{m}} |f(\alpha)g^*(\alpha)|^2\, d\alpha \ll N^{1-1/(50k^2)} P^2 \tag{12}$$

that suffices for our purposes.

We now turn our attention to the second term on the right hand side of (8). It is straightforward to estimate the contribution from the "pruned arcs"

$$\mathfrak{N} = \mathfrak{M}(P^{1/8}).$$

In fact, by Lemma 1 we see that for $\alpha \in \mathfrak{N}$ one has

$$f(\alpha)g^*(\alpha) - V^*(\alpha) \ll NP^{5/8+\varepsilon},$$

and hence, since $\mathfrak{N}$ has measure $O(P^{1/4}N^{-1})$,

$$\int_{\mathfrak{N}} |f(\alpha)g^*(\alpha) - V^*(\alpha)|^2\, d\alpha \ll NP^{3/2+\varepsilon}. \tag{13}$$

Note that the bound (13) is much smaller than the right hand side of (12). Hence, it now remains to consider the "intermediate arcs" $\mathfrak{M}\backslash\mathfrak{N}$. This is often the most difficult part, as is the case here. We use the elementary inequality

$$\int_{\mathfrak{M}\backslash\mathfrak{N}} |f(\alpha)g^*(\alpha) - V^*(\alpha)|^2\, d\alpha \ll \int_{\mathfrak{M}\backslash\mathfrak{N}} |f(\alpha)g^*(\alpha)|^2\, d\alpha + \int_{\mathfrak{M}\backslash\mathfrak{N}} |V^*(\alpha)|^2\, d\alpha, \tag{14}$$

and estimate the two terms on the right hand side of (14) in the next section.

4. A Pruning Exercise

The treatment of the set $\mathfrak{M}\backslash\mathfrak{N}$ depends on our pruning lemma [1] for which we now prepare the scene. When $\alpha \in \mathfrak{M}$ is in the interval (5) (with $X = N^{1/4}$), let

$$\Upsilon(\alpha) = (q + N|q\alpha - a|)^{-1}. \tag{15}$$

This defines $\Upsilon(\alpha)$ for $\alpha \in \mathfrak{M}$. For these α, Lemma 1 combined with elementary estimates shows

$$f(\alpha) \ll \varphi(q)^{-1}N\left(1+N\left|\alpha-\frac{a}{q}\right|\right)^{-1}+\sqrt{qN}(\log N)^2\left(1+N\left|\alpha-\frac{a}{q}\right|\right)$$
$$\ll N^{1+\varepsilon}\Upsilon(\alpha). \tag{16}$$

It is perhaps worth pointing out that at this point GRH is indispensable, at the current state of knowledge. Unconditional versions of (16) would involve $\Upsilon(\alpha)^{1/2}$ rather than $\Upsilon(\alpha)$, and this is too weak to be of use in the argument below.

We cover the set $\mathfrak{M}\backslash\mathfrak{N}$ by $O(\log N)$ sets $\mathfrak{M}(2X)\backslash\mathfrak{M}(X)$ with $P^{1/8} \le X \le N^{1/4}$. For $\alpha \in \mathfrak{M}(2X)\backslash\mathfrak{M}(X)$ it follows from (16) that one has $|f(\alpha)|^2 \ll N^{2+\varepsilon}X^{-1}\Upsilon(\alpha)$, and hence, for some X in the aforementioned range

$$\int_{\mathfrak{M}\backslash\mathfrak{N}} |f(\alpha)g^*(\alpha)|^2\,d\alpha \ll N^{2+\varepsilon}X^{-1}\int_{\mathfrak{M}(2X)} \Upsilon(\alpha)|g^*(\alpha)|^2\,d\alpha.$$

By Lemma 2 of Brüdern [1] (with $\Psi(\alpha)=|g^*(\alpha)|^2$), we find that

$$\int_{\mathfrak{M}(2X)} \Upsilon(\alpha)|g^*(\alpha)|^2\,d\alpha \ll N^{\varepsilon-1}\left(X\int_0^1 |g^*(\alpha)|^2\,d\alpha+|g^*(0)|^2\right)$$
$$\ll N^{2\varepsilon-1}(XP+P^2),$$

and therefore, with X as before,

$$\int_{\mathfrak{M}\backslash\mathfrak{N}} |f(\alpha)g^*(\alpha)|^2\,d\alpha \ll N^{1+\varepsilon}(P+X^{-1}P^2) \ll N^{1+\varepsilon}P^{15/8}. \tag{17}$$

The treatment of $V^*(\alpha)$ is elementary. By Lemma 5 of Hua [4] one has $S^*(q,a) \ll q^{1/2+\varepsilon}$ when $(a,q)=1$, and hence, by Lemma 2.6 of Vaughan [10] it follows that

$$V^*(\alpha) \ll N^{1+\varepsilon}P\Upsilon(\alpha)^{1+1/k}$$

holds for $\alpha \in \mathfrak{M}$. A routine argument involving only straightforward estimates then yields

$$\int_{\mathfrak{M}\backslash\mathfrak{N}} |V^*(\alpha)|^2\,d\alpha \ll N^{1+\varepsilon}P^{2-1/(4k)}. \tag{18}$$

By (18), (17), (8), (12) and (14), the part of Lemma 2 relating to $r_k^*(n)$ now follows. Almost all of the above estimates remain valid with the stars removed, and in that case the work of § 3 may even be simplified somewhat. This completes the proof of Lemma 2.

5. A Further Preparation of the Main Term

On writing

$$H(q,n) = \sum_{\substack{a=1 \\ (a,q)=1}}^{q} S(q,a)e\left(-\frac{an}{q}\right), \quad H^*(q,n) = \sum_{\substack{a=1 \\ (a,q)=1}}^{q} S^*(q,a)e\left(-\frac{an}{q}\right),$$

we define the truncated singular series by

$$\mathfrak{S}_k(n,M) = \sum_{q \le M} \frac{\mu(q)}{q\varphi(q)} H(q,n), \quad \mathfrak{S}_k^*(n,M) = \sum_{q \le M} \frac{\mu(q)}{\varphi(q)^2} H^*(q,n),$$

and may then annouce a counterpart of Lemma 2.

Lemma 3. *One has*

$$\sum_{n \le N} |t_k(n) - \mathfrak{S}_k(n, N^{1/4})n^{1/k}|^2 \ll N^{1+\varepsilon}P^{3/2},$$

and the same is true with t_k and $\mathfrak{S}_k$ replaced by t_k^* and $\mathfrak{S}_k^*$.

Note that Lemma 3 is independent of GRH. We again give a detailed proof only for the star version of Lemma 3, and leave (most) alterations for the other case to the reader. We begin by defining the intervals

$$\left\{\alpha \in [0,1] : \left|\alpha - \frac{a}{q}\right| \le N^{-2/3}\right\}$$

and note that these intervals with $0 \le a \le q \le N^{1/4}$, $(a,q) = 1$ are pairwise disjoint. Their union is denoted by $\mathfrak{K}$. Then $\mathfrak{M} \subset \mathfrak{K}$, and the definition of the functions V and V^* on $\mathfrak{M}$ extend naturally to $\mathfrak{K}$. We then put

$$u_k(n) = \int_{\mathfrak{K}} V(\alpha)e(-\alpha n)\, d\alpha, \quad u_k^*(n) = \int_{\mathfrak{K}} V^*(\alpha)e(-\alpha n)\, d\alpha.$$

Then

$$u_k^*(n) - t_k^*(n) = \int_{\mathfrak{K}\setminus\mathfrak{M}} V^*(\alpha)e(-\alpha n)\, d\alpha,$$

and hence, by Bessel's inequality,

$$\sum_{n\le N}|u_k^*(n)-t_k^*(n)|^2 \le \int_{\mathfrak{K}\setminus\mathfrak{M}}|V^*(\alpha)|^2\,d\alpha.$$

The bound for $V^*(\alpha)$ used in §4 is still valid on $\mathfrak{K}$, and so,

$$\begin{aligned}\sum_{n\le N}|u_k^*(n)-t_k^*(n)|^2 &\ll N^{2+\varepsilon}P^2\int_{\mathfrak{K}\setminus\mathfrak{M}}\Upsilon(\alpha)^{2+2/k}\,d\alpha\\ &\ll N^{\varepsilon}\sum_{q\le N^{1/4}}q^{-1-2/k}\int_{q^{-1}N^{3/4}}^{\infty}\beta^{-2-2/k}\,d\beta\\ &\ll (NP^2)^{3/4}N^{1/4+\varepsilon}\ll N^{1+\varepsilon}P^{3/2}.\end{aligned}\tag{19}$$

A similar estimate holds for $u_k(n)-t_k(n)$, and the proof is essentially identical.

Next, we evaluate $u_k(n)$ and $u_k^*(n)$. Recalling the definitions of V and V^*, we have

$$u_k(n)=\mathfrak{S}_k(n,N^{1/4})\int_{-N^{-2/3}}^{N^{-2/3}}v(\beta)w(\beta)e(-\beta n)\,d\beta,$$

and this is also true with u_k^*, $\mathfrak{S}_k^*$ in place of u_k, $\mathfrak{S}_k$. By orthogonality, we have

$$\int_{-1/2}^{1/2}v(\beta)w(\beta)e(-\beta n)\,d\beta=\frac{1}{k}\sum_{m\le n}m^{1/k-1}=n^{1/k}+O(1),$$

and

$$\int_{N^{-2/3}}^{1/2}v(\beta)w(\beta)e(-\beta n)\,d\beta\ll\int_{N^{-2/3}}^{\infty}\beta^{-1-1/k}\ll N^{-2/(3k)}.$$

It follows that

$$u_k(n)=\mathfrak{S}_k(n,N^{1/4})(n^{1/k}+O(N^{-2/(3k)})),$$

and again the star version of this evaluation also holds. From (31), (34) and (37) of Brüdern and Perelli [2] we have $H^*(q,n)\ll q^{1+\varepsilon}$ whenever q is square-free. Likewise, the bound $H(q,n)\ll q^{1+\varepsilon}$ follows from Kawada [6]. Consequently, $\mathfrak{S}_k(n,N^{1/4})\ll N^{\varepsilon}$, and likewise for $\mathfrak{S}_k^*$. It follows that

$$\sum_{n\le N}|u_k(n)-n^{1/k}\mathfrak{S}_k(n,N^{1/4})|^2\ll N^{1-2/(3k)+\varepsilon}P^2,$$

and again this also holds with stars attached. Lemma 3 now follows on combining this with (19) and its analogue with the stars removed.

6. The Singular Series

Recall that $\mathscr{I}_k$ is the set of all positive integers for which $x^k - n$ is irreducible in $\mathbf{Z}[x]$, and that the products $\mathfrak{S}_k(n)$ and $\mathfrak{S}_k^*(n)$ converge for $n \in \mathscr{I}_k$. Indeed, Kawada [6] observed this for $\mathfrak{S}_k(n)$, but when $p \nmid n$, one has $\varrho(n,p) = \varrho^*(n,p)$, and hence, the convergence of $\mathfrak{S}(n)$ implies that of $\mathfrak{S}^*(n)$. As a special case of Corollary 1 of Kawada [6] (take $\delta = \frac{1}{2k}$ and $H = N$ in Kawada's notation), we also have

$$\mathfrak{S}(n, N^{1/4}) = \mathfrak{S}(n) + O(N^{-1/(6000k^2)}) \tag{20}$$

for all but $O(N^{1-1/(6000k^2)})$ of the integers $n \in \mathscr{I}_k$ that lie in the interval $\frac{1}{2}N < n \le N$.

We also require the star version of (20). One way to do this would be to adjust the arguments in [6] to cover the star situation. Alternatively, we may proceed as follows. By orthogonality, one has

$$H(p,n) = p(\varrho(n,p) - 1), \quad H^*(p,n) = (p-1)(\varrho^*(n,p) - 1) + \varrho^*(n,p).$$

Let $\lambda(n,m)$ and $\lambda^*(n,m)$ be the completely multiplicative functions defined on primes by

$$\lambda(n,p) = \varrho(n,p) - 1, \quad \lambda^*(n,p) = (\varrho^*(n,p) - 1) + \frac{\varrho^*(n,p)}{p-1}.$$

Then we may rewrite the truncated series as

$$\mathfrak{S}(n,M) = \sum_{m \le M} \frac{\mu(m)}{\varphi(m)} \lambda(n,m), \quad \mathfrak{S}^*(n,M) = \sum_{m \le M} \frac{\mu(m)}{\varphi(m)} \lambda^*(n,m).$$

The Dirichlet series

$$Z_n(s) = \sum_{m=1}^{\infty} \frac{\mu(m)\lambda(n,m)}{\varphi(m)m^{s-1}}, \quad Z_n^*(s) = \sum_{m=1}^{\infty} \frac{\mu(m)\lambda^*(n,m)}{\varphi(m)m^{s-1}}$$

converge in $\mathrm{Re}(s) > 1$, but also have Euler products. For $p \nmid n$, we recall $\varrho^*(n,p) = \varrho(n,p) \le k$. Hence $|\lambda(n,p) - \lambda^*(n,p)| \le k/(p-1)$ for these p, and so the Euler product for $Z_n(s)/Z_n^*(s)$ converges in $\mathrm{Re}(s) > \frac{1}{2}$. It now transpires that Kawada's auxiliary estimates for zeros of $Z_n(s)$ remain valid for $Z_n^*(s)$, and hence the star version of (20) also holds, by Kawada's arguments.

It is now easy to establish the theorem. By a standard argument, we deduce from Lemmas 2 and 3 that

$$r_k(n) - \mathfrak{S}_k(n, N^{1/4})n^{1/k} = O(n^{1/k}N^{-1/(150k^2)})$$

holds for all but $O(N^{1-1/(150k^2)})$ of the integers $n \in \mathscr{I}_k$ with $\frac{1}{2}N < n \le N$. Apply (20) to replace $\mathfrak{S}_k(n, N^{1/4})$ by $\mathfrak{S}_k(n)$. Now sum over dyadic ranges for N. This proves the first part of the Theorem, and the star version follows mutatis mutandis.

References

[1] J. Brüdern. A problem in additive number theory. Math. Proc. Cambridge Philos. Soc. **103** (1988), 27–33.
[2] J. Brüdern and A. Perelli. The addition of primes and powers. Can. J. Math. **48** (1996), 512–526.
[3] K. Ford. New estimates for mean values of Weyl sum. Internat. math. Res. Notices (1995), 155–171.
[4] L. K. Hua. Some results in the additive prime number theory. Quart. J. Math. Oxford (2), **9** (1938), 68–80.
[5] K. Kawada. On the asymptotic formula for the number of representations of numbers as the sum of a prime and a k-th power. Proc. Japan Acad. **69A** (1993), 283–286.
[6] K. Kawada. A zero density estimate for Dedekind zeta functions of pure extension fields. Tsukuba J. Math. **22** (1998), 357–369.
[7] S. Lang. Algebra. 3rd ed. Addison-Wesley, New York 1993.
[8] R. J. Miech. On the equation $n = p + x^2$. Trans. Amer. Math. Soc. **130** (1968), 494–512.
[9] A. Perelli and A. Zaccagnini. On the sum of a prime and a kth power. Izv. Ross. Akad. Nauk Ser. Mat. **59** (1995), 185–200.
[10] R. C. Vaughan. The Hardy-Littlewood method. 2nd edn. Cambridge University Press 1997.

Jörg Brüdern
Institut für Algebra und Zahlentheorie
Universität Stuttgart
D-70511 Stuttgart, Germany