SEMISURFACES AND THE EQUATIONS
OF CODAZZI-MAINARDI

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Abstract. In this paper, we shall describe the equations of Codazzi-Mainardi for a surface without any umbilical point, using the gradient vector fields of the mean curvature function and the Gaussian curvature function. In addition, based on this description, we shall obtain a homogeneous polynomial, which is an analogue of a Codazzi-Mainardi polynomial obtained in [3], and we shall find a relation between two polynomials.

1. Introduction

Let M be a smooth two-dimensional manifold and g a Riemannian metric on M. Let $\mathscr{D}_1$, $\mathscr{D}_2$ be two smooth one-dimensional distributions on M. A Riemannian manifold (M, g) equipped with $(\mathscr{D}_1, \mathscr{D}_2)$ is called a *semisurface* if $\mathscr{D}_1$ and $\mathscr{D}_2$ are orthogonal to each other at any point of M with respect to g; if $(M, g, \mathscr{D}_1, \mathscr{D}_2)$ is a semisurface, then a triplet $(g, \mathscr{D}_1, \mathscr{D}_2)$ is called a *semisurface structure* of M. For example, a surface S in $\mathbf{R}^3$ without any umbilical point is considered as a semisurface: the first fundamental form of S and two principal distributions on S form a semisurface structure of S. Let $(M, g, \mathscr{D}_1, \mathscr{D}_2)$ be a semisurface. For each point $p \in M$, there exist local coordinates (u, v) on a neighborhood U_p of p satisfying $\partial/\partial u \in \mathscr{D}_1$ and $\partial/\partial v \in \mathscr{D}_2$ on U_p. Such coordinates are said to be *compatible with* $(\mathscr{D}_1, \mathscr{D}_2)$. The Riemannian metric g is represented as $g = A^2 \, du^2 + B^2 \, dv^2$ on U_p. In [3], we studied a surface S in $\mathbf{R}^3$ with nowhere zero Gaussian curvature K and without any umbilical point,

noticing its semisurface structure. Let k_1, k_2 be distinct two principal curvature functions on S and for each point $q \in S$, let $P_{\mathrm{CM},q}(X_1, X_2)$ be a homogeneous polynomial of degree two defined by

$$P_{\mathrm{CM},q}(X_1, X_2) := c_{20}(q)X_1^2 + c_{11}(q)X_1 X_2 + c_{02}(q)X_2^2,$$

where

$$c_{20} := (\log|K|A^2)_v (\log B)_u - (\log B)_{uv},$$

$$c_{11} := (\log|K|AB)_{uv} - 4(\log A)_v (\log B)_u,$$

$$c_{02} := (\log|K|B^2)_u (\log A)_v - (\log A)_{uv},$$

and (u, v) are local coordinates compatible with principal distributions such that k_1 (respectively, k_2) corresponds to $\partial/\partial u$ (respectively, $\partial/\partial v$). We call $P_{\mathrm{CM},q}$ a *Codazzi-Mainardi polynomial* of S at q. In [3], we proved $P_{\mathrm{CM},q}(k_1(q), k_2(q)) = 0$ for any point q of S. Noticing the equation of Gauss, we see that the coefficients of $P_{\mathrm{CM},q}$ depend only on A, B and their partial derivatives. A Codazzi-Mainardi polynomial at each point is determined by the semisurface structure of S up to a nonzero constant, i.e., for other local coordinates (u', v') compatible with principal distributions such that k_1 (respectively, k_2) corresponds to $\partial/\partial u'$ (respectively, $\partial/\partial v'$), the corresponding Codazzi-Mainardi polynomial $P'_{\mathrm{CM},q}$ is represented by $P_{\mathrm{CM},q}$ up to a nonzero constant for each point q. Therefore we may define a *Codazzi-Mainardi polynomial* of a semisurface $(M, g, \mathscr{D}_1, \mathscr{D}_2)$ at each point of M up to a nonzero constant, if the curvature of (M, g) is nowhere zero.

According to the fundamental theorem of the theory of surfaces, a surface may be considered as a two-dimensional Riemannian manifold (M, g) equipped with a smooth tensor field W of type $(1, 1)$ which is connected with g by the equations of Gauss and Codazzi-Mainardi (then g and W give the first fundamental form and the Weingarten map of the surface, respectively). The author is interested in the semisurface structure of a surface without any umbilical point, because he intends to consider a surface without any umbilical point as a two-dimensional Riemannian manifold (M, g) equipped with an orthogonal pair of two smooth one-dimensional distributions $(\mathscr{D}_1, \mathscr{D}_2)$ which is connected with g by some good relation (then g gives the first fundamental form and $(\mathscr{D}_1, \mathscr{D}_2)$ gives a pair of principal distributions, which give the one-dimensional eigenspaces of W at each point). The finding of Codazzi-Mainardi polynomials motivated him to adopt this view. The two principal curvatures k_1, k_2 (the eigenvalues of the Weingarten map W) satisfy $k_1 k_2 = K$ (the equation of Gauss) and in addition, as

was mentioned in the previous paragraph, if K is nowhere zero, then k_1 and k_2 satisfy $P_{\mathrm{CM},q}(k_1, k_2) = 0$. Since $P_{\mathrm{CM},q}$ is determined by the semisurface structure up to a nonzero constant at each point q, we may represent each of k_1 and k_2 by quantities determined by the semisurface structure, if $P_{\mathrm{CM},q} \not\equiv 0$ (for a concrete representation, see [3]). For a semisurface $(M, g, \mathscr{D}_1, \mathscr{D}_2)$ with nowhere zero curvature K and smooth functions k_1, k_2 satisfying $k_1 k_2 = K$ and $P_{\mathrm{CM}}(k_1, k_2) = 0$, whether M may be locally and isometrically immersed in $\boldsymbol{R}^3$ so that $(\mathscr{D}_1, \mathscr{D}_2)$ (respectively, (k_1, k_2)) gives a pair of principal distributions (respectively, principal curvature functions) depends on whether there exists a good relation between g and $(\mathscr{D}_1, \mathscr{D}_2)$. We may consider the equations of Codazzi-Mainardi

$$(k_1)_v = -(\log A)_v (k_1 - k_2), \quad (k_2)_u = (\log B)_u (k_1 - k_2), \tag{1}$$

where (u, v) are compatible with principal distributions such that k_1 (respectively, k_2) corresponds to $\partial/\partial u$ (respectively, $\partial/\partial v$), as a basic representation of the good relation: whether (1) holds for a pair (k_1, k_2) satisfying $k_1 k_2 = K$ and $P_{\mathrm{CM}}(k_1, k_2) = 0$ determines whether there exists a good relation between g and $(\mathscr{D}_1, \mathscr{D}_2)$. In [3], we showed that if $(M, g, \mathscr{D}_1, \mathscr{D}_2)$ is a semisurface with nowhere zero curvature and everywhere zero Codazzi-Mainardi polynomial, then there exists a good relation between g and $(\mathscr{D}_1, \mathscr{D}_2)$, that is, M may be locally and isometrically immersed in $\boldsymbol{R}^3$ so that $(\mathscr{D}_1, \mathscr{D}_2)$ gives a pair of principal distributions. In addition, we may obtain a more concrete representation than (1) of the good relation between g and $(\mathscr{D}_1, \mathscr{D}_2)$ of a parallel curved surface: for a surface S with nowhere zero Gaussian curvature and without any umbilical point, a neighborhood of each point of S is a canonical parallel curved surface if and only if S satisfies $P_{\mathrm{CM},q} \equiv 0$ for any $q \in S$ and the condition that the integral curves of one principal distribution on S are geodesics; for a semisurface $(M, g, \mathscr{D}_1, \mathscr{D}_2)$ with nowhere zero curvature and everywhere zero Codazzi-Mainardi polynomial such that the integral curves of one of $\mathscr{D}_1$ and $\mathscr{D}_2$ are geodesics, M may be locally and isometrically immersed in $\boldsymbol{R}^3$ as a canonical parallel curved surface so that $(\mathscr{D}_1, \mathscr{D}_2)$ gives a pair of principal distributions ([3]). We may also obtain a more concrete representation than (1) of the good relation between g and $(\mathscr{D}_1, \mathscr{D}_2)$ of a surface with constant mean curvature: if S is with constant mean curvature H_0, then on a neighborhood of each point of S, there exist isothermal coordinates (u, v) compatible with principal distributions and a smooth, positive-valued function A satisfying $g = A^2(du^2 + dv^2)$ and $\Delta \log A + H_0^2 - 1/A^4 = 0$, where Δ is the Laplacian on S (if $P_{\mathrm{CM}} \not\equiv 0$, then the converse holds ([3])); for a semisurface $(M, g, \mathscr{D}_1, \mathscr{D}_2)$ satisfying $g = A^2(du^2 + dv^2)$ and $\Delta \log A + H_0^2 - 1/A^4 = 0$, where (u, v) are compatible with principal dis-

tributions, M may be locally and isometrically immersed in $\boldsymbol{R}^3$ as a surface with constant mean curvature H_0 so that $(\mathscr{D}_1, \mathscr{D}_2)$ gives a pair of principal distributions, and such a surface is determined by the semisurface structure up to a motion of $\boldsymbol{R}^3$ ([4, pp. 22], [6], [8]).

REMARK. As was mentioned in the previous paragraph, a semisurface $(M, g, \mathscr{D}_1, \mathscr{D}_2)$ with nowhere zero curvature and everywhere zero Codazzi-Mainardi polynomial may be locally and isometrically immersed so that $(\mathscr{D}_1, \mathscr{D}_2)$ gives a pair of principal distributions. Then we may obtain plural surfaces which have the same semisurface structure $(g, \mathscr{D}_1, \mathscr{D}_2)$ such that arbitrarily distinct two of the surfaces are not congruent with each other in $\boldsymbol{R}^3$: for each $p \in M$ and each pair of numbers $(k_1^{(0)}, k_2^{(0)})$ satisfying $k_1^{(0)} k_2^{(0)} = K(p)$, there exist a neighborhood U_p of p and an isometric immersion Φ_p of U_p into $\boldsymbol{R}^3$ satisfying (a) $(\mathscr{D}_1, \mathscr{D}_2)$ gives a pair of principal distributions on $\Phi_p(U_p)$, (b) $k_1^{(0)}$ and $k_2^{(0)}$ are the principal curvatures of $\Phi_p(U_p)$ at $\Phi_p(p)$; such an immersion as Φ_p is determined by a pair $(k_1^{(0)}, k_2^{(0)})$ up to a motion of $\boldsymbol{R}^3$. In [5], Kishimura described relations between two canonical parallel curved surfaces with nowhere zero Gaussian curvature which have the same semisurface structure, in terms of generating pairs (each canonical parallel curved surface is determined by a generating pair, which is a pair of two simple curves C_b, C_g in $\boldsymbol{R}^3$ with a unique intersection $p_{(C_b, C_g)}$ and contained in planes P_b, P_g, respectively such that we may choose as P_g the plane normal to C_b at $p_{(C_b, C_g)}$). In addition, he showed that for two generating pairs with the relations, the corresponding canonical parallel curved surfaces have the same semisurface structure.

In the present paper, we shall study a surface S in $\boldsymbol{R}^3$ without any umbilical point, noticing another semisurface structure: the metric is given by the first fundamental form; the two distributions are given by H-distributions, i.e., smooth one-dimensional distributions on S which give directions such that the normal curvatures are equal to the mean curvature of S. In Section 3, we shall study smooth vector fields on a semisurface $(M, g, \mathscr{D}_1, \mathscr{D}_2)$ such that the divergences of them with respect to the Levi-Civita connection are equal to the curvature of (M, g), and in particular, we shall define the *canonical pre-divergence* V_K of a semisurface $(M, g, \mathscr{D}_1, \mathscr{D}_2)$, which is one of such vector fields and determined by the semisurface structure $(g, \mathscr{D}_1, \mathscr{D}_2)$. In Section 4, we shall describe the equations of Codazzi-Mainardi for a surface S in $\boldsymbol{R}^3$ without any umbilical point, using the gradient vector fields of the mean curvature function H and the Gaussian curvature function K, and the canonical pre-divergence of S: we shall prove

THEOREM 1.1. *The equations of Codazzi-Mainardi are represented as*

$$2K \operatorname{grad}(H) = W(\operatorname{grad}(K) + 4(H^2 - K)V_K), \tag{2}$$

where W *and* V_K *are the Weingarten map and the canonical pre-divergence of* S, *respectively.*

Suppose that S is oriented. In Section 5, computing the rotations of the both hand sides of (2), we shall obtain an analogue of a Codazzi-Mainardi polynomial: we shall prove

THEOREM 1.2. *If* $K(q) \neq 0$ *for* $q \in S$, *then*

$$P_{II,q}(H(q), \sqrt{H(q)^2 - K(q)}) = 0 \tag{3}$$

holds, where

$$P_{II,q}(Y_1, Y_2) := c_{II20}(q) Y_1^2 + c_{II11}(q) Y_1 Y_2 + c_{II02}(q) Y_2^2,$$

$$c_{II20} := -\frac{1}{2}\{U_1 U_1(\log|K|) - U_2 U_2(\log|K|)\}$$

$$-\frac{3}{2}\{U_1(\log|K|)U_1(\log B) - U_2(\log|K|)U_2(\log A)\},$$

$$c_{II11} := -2\operatorname{rot}(V_K) - 2\{U_1(\log|K|)U_2(\log A) - U_2(\log|K|)U_1(\log B)\},$$

$$c_{II02} := \frac{1}{2}\{U_1 U_1(\log|K|B^4) - U_2 U_2(\log|K|A^4)$$

$$- U_1(\log|K|B^4)U_1(\log B) + U_2(\log|K|A^4)U_2(\log A)\},$$

$$U_1 := \frac{1}{A}\frac{\partial}{\partial u}, \quad U_2 := \frac{1}{B}\frac{\partial}{\partial v},$$

and (u, v) *are local coordinates which are compatible with H-distributions and give the orientation of* S.

We call $P_{II,q}$ the *second Codazzi-Mainardi polynomial* of S at q. We see that $P_{II,q}$ is determined by the semisurface structure and the orientation of S. Therefore if M is oriented, then we may define the *second Codazzi-Mainardi polynomial* of a semisurface $(M, g, \mathscr{D}_1, \mathscr{D}_2)$ with nowhere zero curvature. We set

$$P_{I,q} := \frac{1}{A(q)B(q)} P_{CM,q}$$

and we call $P_{\mathrm{I},q}$ the *first Codazzi-Mainardi polynomial* of $(M, g, \mathscr{D}_1, \mathscr{D}_2)$ at q. We see that $P_{\mathrm{I},q}$ is determined by the semisurface structure $(g, \mathscr{D}_1, \mathscr{D}_2)$ and the orientation of M. In Section 5, we shall prove

THEOREM 1.3.   *Let $(g, \mathscr{D}_1^+, \mathscr{D}_2^+)$ and $(g, \mathscr{D}_1^\times, \mathscr{D}_2^\times)$ be two semisurface structures of M such that the angle between $\mathscr{D}_i^\times$ and $\mathscr{D}_i^+$ is equal to $\pi/4$ at any point of M. Suppose that the curvature of (M, g) is nowhere zero. Let P_{I}^+ be the first Codazzi-Mainardi polynomial of $(M, g, \mathscr{D}_1^+, \mathscr{D}_2^+)$ and $P_{\mathrm{II}}^\times$ the second Codazzi-Mainardi polynomial of $(M, g, \mathscr{D}_1^\times, \mathscr{D}_2^\times)$. Then*

$$P_{\mathrm{I},q}^+(X_1, X_2) = P_{\mathrm{II},q}^\times(Y_1, Y_2) \tag{4}$$

holds for $q \in M$ and X_1, X_2, Y_1, $Y_2 \in \mathbf{R}$ satisfying $X_1 = Y_1 + Y_2$ and $X_2 = Y_1 - Y_2$.

The following is an analogue of Theorem 1.3 in [3].

THEOREM 1.4.   *Let $(M, g, \mathscr{D}_1, \mathscr{D}_2)$ be a semisurface with nowhere zero curvature satisfying $P_{\mathrm{II}} \equiv 0$ for any point of M. Then for each point $p \in M$ and each number $H^{(0)}$ satisfying $(H^{(0)})^2 - K(p) > 0$, there exists an isometric immersion of a neighborhood U_p of p into $\mathbf{R}^3$ satisfying the following*:

(a) *$(\mathscr{D}_1, \mathscr{D}_2)$ gives a pair of two H-distributions*;

(b) *the mean curvature at p is given by $H^{(0)}$.*

Such an immersion of U_p into $\mathbf{R}^3$ is uniquely determined by $H^{(0)}$ up to a motion of $\mathbf{R}^3$.

We may prove Theorem 1.4, using Theorem 1.3 or referring to the proof of Theorem 1.3 in [3].

In Section 6, we shall study semisurface structures of surfaces with constant mean curvature, surfaces with constant Gaussian curvature and surfaces of revolution.

2. Preliminaries

2.1. The Divergence and the Rotation of a Smooth Vector Field

Let M be a smooth two-dimensional manifold. Let g be a Riemannian metric on M and ∇ the covariant differentiation with respect to the Levi-Civita con-

nection of a Riemannian manifold (M, g). Let Γ_{ij}^l $(i, j, l \in \{1, 2\})$ be the Christoffel symbols of ∇ with respect to local coordinates (u, v), i.e., smooth functions defined by

$$\nabla_{\partial/\partial u} \frac{\partial}{\partial u} = \Gamma_{11}^1 \frac{\partial}{\partial u} + \Gamma_{11}^2 \frac{\partial}{\partial v}, \quad \nabla_{\partial/\partial u} \frac{\partial}{\partial v} = \Gamma_{12}^1 \frac{\partial}{\partial u} + \Gamma_{12}^2 \frac{\partial}{\partial v},$$

$$\nabla_{\partial/\partial v} \frac{\partial}{\partial u} = \Gamma_{21}^1 \frac{\partial}{\partial u} + \Gamma_{21}^2 \frac{\partial}{\partial v}, \quad \nabla_{\partial/\partial v} \frac{\partial}{\partial v} = \Gamma_{22}^1 \frac{\partial}{\partial u} + \Gamma_{22}^2 \frac{\partial}{\partial v}.$$

For local coordinates (u, v), suppose that the metric g is locally represented as $g = A^2 \, du^2 + B^2 \, dv^2$, where A, B are smooth, positive-valued functions. Then the following hold:

$$\Gamma_{11}^1 = (\log A)_u, \quad \Gamma_{11}^2 = -\frac{AA_v}{B^2}, \quad \Gamma_{12}^1 = \Gamma_{21}^1 = (\log A)_v, \tag{5}$$

$$\Gamma_{22}^2 = (\log B)_v, \quad \Gamma_{22}^1 = -\frac{BB_u}{A^2}, \quad \Gamma_{12}^2 = \Gamma_{21}^2 = (\log B)_u. \tag{6}$$

Let V be a smooth vector field on M and τ_V a smooth tensor field on M of type $(1, 1)$ defined by $\tau_V(w) := \nabla_w V$ for each tangent vector w at each point of M. Then the trace of τ_V is denoted by $\mathrm{div}(V)$ and called the *divergence* of V with respect to the Levi-Civita connection. In the following, suppose that M is oriented. Let $(\boldsymbol{u}_1, \boldsymbol{u}_2)$ be an ordered orthonormal basis of the tangent plane at a fixed point of M such that $(\boldsymbol{u}_1, \boldsymbol{u}_2)$ gives the (positive) orientation of M. Then a value

$$g(\tau_V(\boldsymbol{u}_1), \boldsymbol{u}_2) - g(\tau_V(\boldsymbol{u}_2), \boldsymbol{u}_1)$$

is independent of the choice of an ordered orthonormal basis $(\boldsymbol{u}_1, \boldsymbol{u}_2)$ and uniquely determined by V and the metric g. This value is denoted by $\mathrm{rot}(V)$ and called the *rotation* of V with respect to the metric g.

REMARK. A value

$$g(\tau_V(\boldsymbol{u}_1), \boldsymbol{u}_2) + g(\tau_V(\boldsymbol{u}_2), \boldsymbol{u}_1)$$

depends on the choice of $(\boldsymbol{u}_1, \boldsymbol{u}_2)$.

Let (u, v) be local coordinates on a neighborhood U of each point of M which satisfy $g = A^2 \, du^2 + B^2 \, dv^2$ and give the orientation of M. A smooth vector field V is locally represented as $V = a\partial/\partial u + b\partial/\partial v$, where a, b are smooth functions on U. Then by (5) together with (6), we obtain

$$\nabla_{\partial/\partial u} V = \nabla_{\partial/\partial u}\left(a\frac{\partial}{\partial u} + b\frac{\partial}{\partial v}\right)$$

$$= \left\{\frac{\partial a}{\partial u} + V(\log A)\right\}\frac{\partial}{\partial u} + \left\{\frac{\partial b}{\partial u} - a\frac{AA_v}{B^2} + b(\log B)_u\right\}\frac{\partial}{\partial v}, \qquad (7)$$

$$\nabla_{\partial/\partial v} V = \nabla_{\partial/\partial v}\left(a\frac{\partial}{\partial u} + b\frac{\partial}{\partial v}\right)$$

$$= \left\{\frac{\partial a}{\partial v} - b\frac{BB_u}{A^2} + a(\log A)_v\right\}\frac{\partial}{\partial u} + \left\{\frac{\partial b}{\partial v} + V(\log B)\right\}\frac{\partial}{\partial v}. \qquad (8)$$

We set

$$\boldsymbol{U}_1 := \frac{1}{A}\frac{\partial}{\partial u}, \quad \boldsymbol{U}_2 := \frac{1}{B}\frac{\partial}{\partial v}. \qquad (9)$$

Then we may rewrite (7) and (8) into

$$\nabla_{\boldsymbol{U}_1} V = \left\{\frac{\partial a}{\partial u} + V(\log A)\right\}\boldsymbol{U}_1 + \left\{\frac{B}{A}\frac{\partial b}{\partial u} - a\frac{A_v}{B} + b\frac{B_u}{A}\right\}\boldsymbol{U}_2$$

$$= \{\boldsymbol{U}_1(a_0) + b_0\boldsymbol{U}_2(\log A)\}\boldsymbol{U}_1 + \{\boldsymbol{U}_1(b_0) - a_0\boldsymbol{U}_2(\log A)\}\boldsymbol{U}_2,$$

$$\nabla_{\boldsymbol{U}_2} V = \left\{\frac{A}{B}\frac{\partial a}{\partial v} + a\frac{A_v}{B} - b\frac{B_u}{A}\right\}\boldsymbol{U}_1 + \left\{\frac{\partial b}{\partial v} + V(\log B)\right\}\boldsymbol{U}_2$$

$$= \{\boldsymbol{U}_2(a_0) - b_0\boldsymbol{U}_1(\log B)\}\boldsymbol{U}_1 + \{\boldsymbol{U}_2(b_0) + a_0\boldsymbol{U}_1(\log B)\}\boldsymbol{U}_2,$$

respectively, where $a_0 := Aa$, $b_0 := Bb$ (notice $V = a_0\boldsymbol{U}_1 + b_0\boldsymbol{U}_2$). Therefore we obtain

$$[\tau_V(\boldsymbol{U}_1), \tau_V(\boldsymbol{U}_2)] = [\boldsymbol{U}_1, \boldsymbol{U}_2](X(V) + Y(V)), \qquad (10)$$

where $X(V) := (x_{ij}(V))$ and $Y(V) := (y_{ij}(V))$ are symmetric and alternating matrices, respectively, defined by

$$x_{11}(V) := \boldsymbol{U}_1(a_0) + b_0\boldsymbol{U}_2(\log A),$$

$$x_{12}(V) = x_{21}(V) := \frac{\boldsymbol{U}_1(b_0) + \boldsymbol{U}_2(a_0) - b_0\boldsymbol{U}_1(\log B) - a_0\boldsymbol{U}_2(\log A)}{2},$$

$$x_{22}(V) := \boldsymbol{U}_2(b_0) + a_0\boldsymbol{U}_1(\log B),$$

$$y_{11}(V) = y_{22}(V) := 0,$$

$$y_{21}(V) = -y_{12}(V) := \frac{\boldsymbol{U}_1(b_0) - \boldsymbol{U}_2(a_0) + b_0\boldsymbol{U}_1(\log B) - a_0\boldsymbol{U}_2(\log A)}{2}.$$

Therefore we obtain the following:

$$\text{div}(V) = x_{11}(V) + x_{22}(V), \quad \text{rot}(V) = 2y_{21}(V). \tag{11}$$

REMARK. For a smooth function f on M, let $\text{grad}(f)$ be the gradient vector field of f with respect to the metric g. Then the following hold:

$$\text{div}(\text{grad}(f)) = \Delta f, \quad \text{rot}(\text{grad}(f)) = 0, \tag{12}$$

where Δ is the Laplacian on M with respect to g.

REMARK. For a smooth vector field V on M, let $V^{\perp}$ be a smooth vector field on M such that for a point p of M where V is not zero, $(V/|V|, V^{\perp}/|V|)$ is an ordered orthonormal basis of the tangent plane at p which gives the orientation of M, where $|V| := \sqrt{g(V, V)}$. If V is locally represented as $V = a_0 U_1 + b_0 U_2$, then $V^{\perp}$ is locally represented as $V^{\perp} = -b_0 U_1 + a_0 U_2$. The following hold:

$$\text{div}(V^{\perp}) = -\text{rot}(V), \quad \text{rot}(V^{\perp}) = \text{div}(V). \tag{13}$$

Let (M, g) be a two-dimensional Riemannian manifold and (u, v) local coordinates such that the metric g is locally represented as $g = A^2 \, du^2 + B^2 \, dv^2$. Then the curvature K of (M, g) is represented as follows:

$$K = -\frac{1}{AB} \left\{ \left(\frac{A_v}{B} \right)_v + \left(\frac{B_u}{A} \right)_u \right\}. \tag{14}$$

A smooth vector field V on M is called a *pre-divergence* of a two-dimensional Riemannian manifold (M, g) if the divergence $\text{div}(V)$ of V with respect to the Levi-Civita connection is equal to the curvature K.

REMARK. Let (u, v) be local coordinates on an open set U of M. The metric g is locally represented as

$$g = E \, du^2 + 2F \, dudv + G \, dv^2,$$

where E, F, G are smooth functions on U satisfying $E > 0$, $G > 0$ and $EG - F^2 > 0$. Liouville showed that the curvature K may be represented in the divergence form: he proved

$$K = \frac{1}{\sqrt{EG - F^2}} \left\{ \frac{\partial}{\partial u} \left(-\frac{\sqrt{EG - F^2}}{E} \Gamma_{12}^2 \right) + \frac{\partial}{\partial v} \left(\frac{\sqrt{EG - F^2}}{E} \Gamma_{11}^2 \right) \right\}$$

$$= \frac{1}{\sqrt{EG - F^2}} \left\{ \frac{\partial}{\partial u} \left(\frac{\sqrt{EG - F^2}}{G} \Gamma_{22}^1 \right) + \frac{\partial}{\partial v} \left(-\frac{\sqrt{EG - F^2}}{G} \Gamma_{12}^1 \right) \right\},$$

where Γ_{ij}^l $(i, j, l \in \{1, 2\})$ are the Christoffel symbols of ∇ with respect to local coordinates (u, v) ([7, pp. 114]).

2.2. Incompressibility and Irrotationality of a Smooth Vector Field

Let M be an oriented, smooth two-dimensional manifold and g a Riemannian metric on M. A smooth vector field V on M is said to be *incompressible* if V satisfies $\mathrm{div}(V) = 0$ on M; V is said to be *irrotational* if V satisfies $\mathrm{rot}(V) = 0$ on M. Suppose that V is irrotational. Then on a neighborhood of each point of M, there exists a smooth function f satisfying $\mathrm{grad}(f) = V$. In addition, suppose that V is incompressible. Then $\Delta f = 0$ holds, i.e., f is harmonic. By (13), we obtain $\mathrm{div}(V^\perp) = 0$ and $\mathrm{rot}(V^\perp) = 0$, i.e., we see that $V^\perp$ is incompressible and irrotational. Therefore we see that on a neighborhood of each point of M, there exists a smooth function $f^\perp$ satisfying $\mathrm{grad}(f^\perp) = V^\perp$ and $\Delta f^\perp = 0$. Let F be a smooth, complex-valued function on a neighborhood of each point of M defined by $F := f + \sqrt{-1} f^\perp$. We call F a (*local*) *complex potential* of V. We see that F satisfies the *equations of Cauchy-Riemann*:

$$\frac{1}{A} \frac{\partial f}{\partial u} = \frac{1}{B} \frac{\partial f^\perp}{\partial v}, \quad \frac{1}{B} \frac{\partial f}{\partial v} = -\frac{1}{A} \frac{\partial f^\perp}{\partial u}, \tag{15}$$

where (u, v) are local coordinates which give the orientation of M and satisfy $g = A^2 \, du^2 + B^2 \, dv^2$. In general, a complex potential F may not be necessarily extended on M as a single-valued function.

Let F be a complex-valued function on M. Then there exist real-valued functions f, $f^\perp$ satisfying $F = f + \sqrt{-1} f^\perp$ on M. Suppose that F is smooth, i.e., f and $f^\perp$ are smooth. Then F is said to be *holomorphic* if F satisfies (15). We see that a complex potential of an incompressible and irrotational vector field is holomorphic. Whether F is holomorphic or not depends only on the conformal structure of (M, g): it depends on neither the choice of local coordinates satisfying the above conditions nor the choice of a metric conformal to g. For a holomorphic function $F = f + \sqrt{-1} f^\perp$ on M, vector fields $V := \mathrm{grad}(f)$ and $V^\perp := \mathrm{grad}(f^\perp)$ on M are incompressible and irrotational.

2.3. The Exterior Derivative and the Co-Derivative of a 1-Form

Let M be an oriented, smooth two-dimensional manifold and $C^\infty(M)$ the set of smooth functions on M. We know that $C^\infty(M)$ is considered as a ring. For each $k \in \{0, 1, 2\}$, let $\Lambda^k(M)$ be the set of k-forms on M. We know that $\Lambda^0(M)$ is just $C^\infty(M)$ and that for $k \in \{1, 2\}$, $\Lambda^k(M)$ is a $C^\infty(M)$-module. Let g be a Riemannian metric on M. A 1-form $\theta \in \Lambda^1(M)$ is locally represented as $\theta = a_0\theta_1 + b_0\theta_2$, where $\theta_1 := A\,du$, $\theta_2 := B\,dv$, and (u, v) are local coordinates which satisfy $g = A^2\,du^2 + B^2\,dv^2$ and give the orientation of M. Then the *exterior derivative* of θ is locally represented as

$$d\theta = \{\boldsymbol{U}_1(b_0) - \boldsymbol{U}_2(a_0) + b_0\boldsymbol{U}_1(\log B) - a_0\boldsymbol{U}_2(\log A)\}\Omega, \qquad (16)$$

where $\Omega \in \Lambda^2(M)$ is the area element of (M, g) and locally represented as $\Omega = \theta_1 \wedge \theta_2$. Let $*$ denote Hodge's $*$-operator on M with respect to g, i.e., a $C^\infty(M)$-homomorphism from $\Lambda^k(M)$ onto $\Lambda^{2-k}(M)$ for $k \in \{0, 1, 2\}$ defined as follows:

(i)   for a 0-form $1 \in \Lambda^0(M)$, which is a function identically equal to one on M, $*(1)$ corresponds with the area element Ω;

(ii) for a 1-form $\theta \in \Lambda^1(M)$, $*(\theta)$ is locally represented as $*(\theta) = -b_0\theta_1 + a_0\theta_2$;

(iii) for the area element $\Omega \in \Lambda^2(M)$, $*(\Omega)$ corresponds with 1.

We set $\delta := *d*$ (notice the sign). Then for a 1-form θ on M, $\delta(\theta)$ is a smooth function on M. This function is called the *co-derivative* of θ. By (16), we see that $d(*(\theta))$ is locally represented as

$$d(*(\theta)) = d(-b_0\theta_1 + a_0\theta_2)$$

$$= \{\boldsymbol{U}_1(a_0) + \boldsymbol{U}_2(b_0) + a_0\boldsymbol{U}_1(\log B) + b_0\boldsymbol{U}_2(\log A)\}\Omega.$$

Therefore we see that the co-derivative $\delta(\theta)$ of θ is locally represented as follows:

$$\delta(\theta) = \boldsymbol{U}_1(a_0) + \boldsymbol{U}_2(b_0) + a_0\boldsymbol{U}_1(\log B) + b_0\boldsymbol{U}_2(\log A). \qquad (17)$$

For a 1-form θ on M, let V_θ be a smooth vector field on M satisfying $g(V_\theta, w) = \theta(w)$ for any tangent vector w at any point of M. We see that such a vector field is uniquely determined by θ and that V_θ is locally represented as $V_\theta = a_0\boldsymbol{U}_1 + b_0\boldsymbol{U}_2$. Therefore noticing (11), (16) and (17), we obtain the following:

$$\delta(\theta) = \mathrm{div}(V_\theta), \quad d\theta = \mathrm{rot}(V_\theta)\Omega. \qquad (18)$$

For a smooth vector field V on M, let θ_V be a 1-form on M satisfying $g(V, w) = \theta_V(w)$ for any tangent vector w at any point of M. We see that such a 1-form is uniquely determined by V and we obtain

$$\delta(\theta_V) = \operatorname{div}(V), \quad d\theta_V = \operatorname{rot}(V)\Omega. \tag{19}$$

REMARK. Let V be a smooth vector field on M. Then from (19), we see that if V is incompressible (respectively, irrotational), then the corresponding 1-form θ_V satisfies $\delta(\theta_V) = 0$ (respectively, $d\theta_V = 0$). In particular, we see that if V is incompressible and irrotational, then θ_V satisfies $\delta(\theta_V) = 0$ and $d\theta_V = 0$, i.e., θ_V is harmonic. Let θ be a 1-form on M. Then we see from (18) that if θ satisfies $\delta(\theta) = 0$ (respectively, $d\theta = 0$), then the corresponding vector field V_θ is incompressible (respectively, irrotational). In particular, we see that if θ is harmonic, then V_θ is incompressible and irrotational.

REMARK. Let f be a smooth function on M. Then by (18) or (19), we obtain a rewrite of (12): $\delta(df) = \Delta f$ and $d(df) = 0$.

2.4. Conformal Semisurfaces

Let M be a smooth two-dimensional manifold and for a Riemannian metric g on M, let $\mathscr{C}_g$ denote the conformal structure of (M, g), i.e., the set of metrics on M conformal to g. Let $\mathscr{D}_1$, $\mathscr{D}_2$ be two smooth one-dimensional distributions on M. Then $(M, \mathscr{C}_g, \mathscr{D}_1, \mathscr{D}_2)$ is called a *conformal semisurface* if $\mathscr{D}_1$ and $\mathscr{D}_2$ are perpendicular to each other at any point of M with respect to a metric in $\mathscr{C}_g$. Whether $\mathscr{D}_1$ and $\mathscr{D}_2$ are perpendicular to each other does not depend on the choice of a metric in $\mathscr{C}_g$. The triplet $(\mathscr{C}_g, \mathscr{D}_1, \mathscr{D}_2)$ is called a *conformal semisurface structure* of M.

In the sequel, we call $(M, g, \mathscr{D}_1, \mathscr{D}_2)$ (respectively, $(g, \mathscr{D}_1, \mathscr{D}_2)$) a *Riemannian semisurface* (respectively, a *Riemannian semisurface structure*) if $(M, g, \mathscr{D}_1, \mathscr{D}_2)$ is a semisurface in the sense of the beginning of Section 1.

Suppose that M is oriented. Let $(M, \mathscr{C}, \mathscr{D}_1, \mathscr{D}_2)$ be a conformal semisurface and (u, v) local coordinates which are compatible with $(\mathscr{D}_1, \mathscr{D}_2)$ and give the orientation of M. A Riemannian metric g on M in $\mathscr{C}$ is locally represented as $g = A^2\, du^2 + B^2\, dv^2$. We set $\phi := \log(B/A)$. Then a 2-form $\Theta := \phi_{uv}\, du \wedge dv$ does not depend on the choice of local coordinates (u, v) satisfying the above conditions. In addition, Θ does not depend on the choice of a metric g in $\mathscr{C}$, either. Thus Θ is defined on M by a given conformal semisurface structure $(\mathscr{C}, \mathscr{D}_1, \mathscr{D}_2)$ of M. We call Θ the *distorsion 2-form* of a conformal semisurface $(M, \mathscr{C}, \mathscr{D}_1, \mathscr{D}_2)$.

The following holds:

PROPOSITION 2.1. *Let $(M, \mathscr{C}, \mathscr{D}_1, \mathscr{D}_2)$ be a conformal semisurface. Then the following are mutually equivalent*:

(a) *the distorsion 2-form of $(M, \mathscr{C}, \mathscr{D}_1, \mathscr{D}_2)$ is identically zero*;

(b) *on a neighborhood of each point of M, there exists a nowhere zero, incompressible and irrotational vector field V_i with respect to $g \in \mathscr{C}$ satisfying $V_i \in \mathscr{D}_i$;*

(c) *on a neighborhood of each point of M, there exist isothermal coordinates with respect to $g \in \mathscr{C}$ compatible with $(\mathscr{D}_1, \mathscr{D}_2)$.*

3. Pre-Divergences of a Riemannian Semisurface

Let M be an oriented, smooth two-dimensional manifold and g a Riemannian metric on M. It is said that a smooth one-dimensional distribution $\mathscr{D}$ on M is *parallel* with respect to a vector field V on M if for a smooth unit vector field U on a neighborhood of each point of M satisfying $U \in \mathscr{D}$, $\nabla_V U$ is identically zero. Let $\mathscr{D}_1$, $\mathscr{D}_2$ be two smooth one-dimensional distributions on M such that g, $\mathscr{D}_1$ and $\mathscr{D}_2$ form a Riemannian semisurface structure of M. A smooth vector field V on M is called a *pre-divergence* of a Riemannian semisurface $(M, g, \mathscr{D}_1, \mathscr{D}_2)$ if V satisfies the following:

(i) V is a pre-divergence of a Riemannian manifold (M, g);

(ii) the rotation $\mathrm{rot}(V)$ of V with respect to g satisfies $\mathrm{rot}(V)\Omega = \Theta$, where Ω is the area element of (M, g) and Θ is the distorsion 2-form of the conformal semisurface $(M, \mathscr{C}_g, \mathscr{D}_1, \mathscr{D}_2)$;

(iii) $\mathscr{D}_1$ and $\mathscr{D}_2$ are parallel with respect to V.

We shall prove

PROPOSITION 3.1. *There exists a pre-divergence of $(M, g, \mathscr{D}_1, \mathscr{D}_2)$.*

PROOF. Let V_K be a smooth vector field on M defined by

$$V_K := -\boldsymbol{U}_1(\log B)\boldsymbol{U}_1 - \boldsymbol{U}_2(\log A)\boldsymbol{U}_2, \tag{20}$$

where $\boldsymbol{U}_1$ and $\boldsymbol{U}_2$ are as in (9) and (u, v) are local coordinates which are compatible with $(\mathscr{D}_1, \mathscr{D}_2)$ and give the orientation of M. The definition of V_K

does not depend on the choice of local coordinates (u, v) satisfying the above conditions. We shall show that V_K is a pre-divergence of $(M, g, \mathscr{D}_1, \mathscr{D}_2)$. Noticing (14), we obtain $\mathrm{div}(V_K) = K$, i.e., we see that V_K is a pre-divergence of a Riemannian manifold (M, g). By direct computation, we obtain

$$\mathrm{rot}(V_K) = \frac{1}{AB}(\log(B/A))_{uv}. \tag{21}$$

Therefore we obtain $\mathrm{rot}(V_K)\Omega = \Theta$. By (5) together with (6), we obtain

$$\nabla_{V_K} U_1 = -\frac{U_1(\log B)}{A}\nabla_{\partial/\partial u}\left(\frac{1}{A}\frac{\partial}{\partial u}\right) - \frac{U_2(\log A)}{B}\nabla_{\partial/\partial v}\left(\frac{1}{A}\frac{\partial}{\partial u}\right)$$

$$= -\frac{U_1(\log B)}{A}\left\{-\frac{(\log A)_u}{A}\frac{\partial}{\partial u} + \frac{1}{A}\left((\log A)_u\frac{\partial}{\partial u} - \frac{AA_v}{B^2}\frac{\partial}{\partial v}\right)\right\}$$

$$- \frac{U_2(\log A)}{B}\left\{-\frac{(\log A)_v}{A}\frac{\partial}{\partial u} + \frac{1}{A}\left((\log A)_v\frac{\partial}{\partial u} + (\log B)_u\frac{\partial}{\partial v}\right)\right\}$$

$$= 0.$$

Therefore we see that $\nabla_{V_K} U_1$ is identically zero. In the same way, we may show that $\nabla_{V_K} U_2$ is identically zero. Therefore $\mathscr{D}_1$ and $\mathscr{D}_2$ are parallel with respect to V_K. Hence we see that V_K is a pre-divergence of $(M, g, \mathscr{D}_1, \mathscr{D}_2)$. □

Noticing (11), (13) and (20), we obtain

PROPOSITION 3.2. *The following are mutually equivalent*:

(a) V_K *is contained in* $\mathscr{D}_1$ *(respectively,* $\mathscr{D}_2$*)*;

(b) U_1 *(respectively,* U_2*) is irrotational, i.e.,* $\mathrm{rot}(U_1) \equiv 0$ *(respectively,* $\mathrm{rot}(U_2) \equiv 0$*)*;

(c) U_2 *(respectively,* U_1*) is incompressible, i.e.,* $\mathrm{div}(U_2) \equiv 0$ *(respectively,* $\mathrm{div}(U_1) \equiv 0$*)*;

(d) *The integral curves of* $\mathscr{D}_1$ *(respectively,* $\mathscr{D}_2$*) are geodesics.*

We shall prove

PROPOSITION 3.3. *For a vector field V on M, the following are mutually equivalent*:

(a) *one of $\mathcal{D}_1$ and $\mathcal{D}_2$ is parallel with respect to V;*

(b) *both of $\mathcal{D}_1$ and $\mathcal{D}_2$ are parallel with respect to V;*

(c) *V and V_K are linearly dependent at any point of M.*

PROOF.   Let (u,v) be local coordinates on a neighborhood of each point of M which are compatible with $(\mathcal{D}_1, \mathcal{D}_2)$. A vector field V is locally represented as $V = a\partial/\partial u + b\partial/\partial v$. Then by (5) together with (6), we obtain

$$\nabla_V U_1 = a\nabla_{\partial/\partial u}\left(\frac{1}{A}\frac{\partial}{\partial u}\right) + b\nabla_{\partial/\partial v}\left(\frac{1}{A}\frac{\partial}{\partial u}\right) = \left(-a\frac{A_v}{B} + b\frac{B_u}{A}\right)U_2, \qquad (22)$$

$$\nabla_V U_2 = a\nabla_{\partial/\partial u}\left(\frac{1}{B}\frac{\partial}{\partial v}\right) + b\nabla_{\partial/\partial v}\left(\frac{1}{B}\frac{\partial}{\partial v}\right) = -\left(-a\frac{A_v}{B} + b\frac{B_u}{A}\right)U_1. \qquad (23)$$

If one of $\nabla_V U_1$ and $\nabla_V U_2$ is identically zero, then we obtain $aA_v/B \equiv bB_u/A$. This implies that V and V_K are linearly dependent at any point of M. If V and V_K are linearly dependent at any point of M, then from (22) and (23), we see that both $\nabla_V U_1$ and $\nabla_V U_2$ are identically zero. Hence we obtain Proposition 3.3. $\qquad\square$

COROLLARY 3.4.   *Let $(M, g, \mathcal{D}_1, \mathcal{D}_2)$ be a Riemannian semisurface. Then the following are mutually equivalent*:

(a) *$\mathcal{D}_i$ is parallel with respect to any vector field on M;*

(b) *on a neighborhood of each point of M, a smooth unit vector field $U_i \in \mathcal{D}_i$ is incompressible and irrotational;*

(c) *V_K is identically zero;*

(d) *the integral curves of $\mathcal{D}_1$ and $\mathcal{D}_2$ are geodesics.*

In addition, if one of the above (a)~(d) holds, then the following hold:

(e) *the curvature of (M, g) is identically zero;*

(f) *the distorsion 2-form of $(M, \mathcal{C}_g, \mathcal{D}_1, \mathcal{D}_2)$ vanishes;*

(g) *any incompressible and irrotational vector field is a pre-divergence of $(M, g, \mathcal{D}_1, \mathcal{D}_2)$.*

Let V_1 and V_2 be smooth vector fields on M satisfying the following:

(i) $\operatorname{div}(V_1) = \operatorname{div}(V_2)(=: d_0)$;

(ii) $\operatorname{rot}(V_1) = \operatorname{rot}(V_2)(=: r_0)$;

(iii) V_1 and V_2 are linearly dependent at any point of M.

Suppose that for a point $p \in M$, $V_1(p)$ is not zero, and let U_p be a neighborhood of p such that V_1 is nowhere zero on U_p and λ a smooth function satisfying $V_2 = \lambda V_1$ on U_p. Then by (i) together with (ii), we obtain

$$(\lambda - 1)d_0 = -V_1(\lambda), \quad (\lambda - 1)r_0 = V_1^{\perp}(\lambda). \tag{24}$$

We see that $\lambda \equiv 1$ is always a solution of this overdetermined system for a given V_1. Let (u, v) be local coordinates on a neighborhood of p such that V_1 and $V_1^{\perp}$ are locally represented as

$$V_1 = f(u, v)\frac{\partial}{\partial u}, \quad V_1^{\perp} = g(u, v)\frac{\partial}{\partial v},$$

respectively, where f and g are smooth, positive-valued functions. Then we see that there exists another solution of (24) than $\lambda \equiv 1$ if and only if $(d_0/f)_v + (r_0/g)_u \equiv 0$, i.e.,

$$V_1^{\perp}(d_0) + V_1(r_0) = d_0 V_1^{\perp}(\log f) + r_0 V_1(\log g)$$

holds and that if there exists another solution λ, then $\lambda(p) \neq 1$ holds and λ is uniquely determined by the initial value at p. Suppose $V_1 = V_K$. Then it is possible that there exists another solution of (24) than $\lambda \equiv 1$. Therefore noticing Proposition 3.3, we see that it is possible that there exist plural pre-divergences of $(M, g, \mathscr{D}_1, \mathscr{D}_2)$, even if $(M, g, \mathscr{D}_1, \mathscr{D}_2)$ is not any Riemannian semisurface satisfying one of (a)~(d) in Corollary 3.4. In Section 4, we shall prove (2). Equation (2) implies that V_K defined as in (20) should be considered as a special pre-divergence. In the following, we call V_K the *canonical pre-divergence* of $(M, g, \mathscr{D}_1, \mathscr{D}_2)$.

The following hold:

PROPOSITION 3.5.   *Let $(M, g, \mathscr{D}_1, \mathscr{D}_2)$ be a Riemannian semisurface and V_K the canonical pre-divergence of this semisurface. Then V_K is incompressible and irrotational if and only if the curvature and the distorsion 2-form vanish. In addition, if V_K is incompressible and irrotational, then the metric g is locally represented as $g = e^{2h}(du^2 + dv^2)$, where h is a harmonic function and (u, v) are local coordinates compatible with $(\mathscr{D}_1, \mathscr{D}_2)$.*

PROPOSITION 3.6.   *Let $(M, g, \mathscr{D}_1, \mathscr{D}_2)$ be a Riemannian semisurface and V_K the canonical pre-divergence of this semisurface. Then V_K is represented by $\boldsymbol{U}_1 + \boldsymbol{U}_2$ up to a constant at each point of M if and only if on a neighborhood of each point of M, there exists a smooth function f satisfying $g = f_u^2\, du^2 + f_v^2\, dv^2$, where (u, v) are local coordinates which are compatible with $(\mathscr{D}_1, \mathscr{D}_2)$ and satisfy $f_u > 0$ and $f_v > 0$. In addition, if $(M, g, \mathscr{D}_1, \mathscr{D}_2)$ satisfies one of these conditions, then the curvature K is locally represented as follows*:

$$K = -\frac{1}{f_u f_v}\,(\log f_u f_v)_{uv}. \tag{25}$$

Let $(M, g, \mathscr{D}_1, \mathscr{D}_2)$ be a Riemannian semisurface and θ_K a 1-form on M defined by

$$\theta_K := -(\log B)_u\, du - (\log A)_v\, dv, \tag{26}$$

where (u, v) are local coordinates which are compatible with $(\mathscr{D}_1, \mathscr{D}_2)$ and give the orientation of M. The definition of θ_K in (26) does not depend on the choice of local coordinates (u, v) satisfying the above conditions. For any tangent vector w at any point of M, the following holds:

$$\theta_K(w) = g(V_K, w). \tag{27}$$

Therefore by (19), we obtain

PROPOSITION 3.7.   *The following hold*:

$$\delta(\theta_K) = K, \quad d\theta_K = \Theta.$$

COROLLARY 3.8.   *Let $(M, g, \mathscr{D}_1, \mathscr{D}_2)$ be a Riemannian semisurface. Then the following are mutually equivalent*:

(a) *one of (a)$\sim$(c) in Proposition 2.1 holds, where $\mathscr{C} = \mathscr{C}_g$*;

(b) *θ_K is closed*;

(c) *V_K is locally represented as the gradient vector field of a smooth function.*

REMARK.   Suppose that $(M, g, \mathscr{D}_1, \mathscr{D}_2)$ satisfies one of the conditions in Proposition 3.6. Then the following holds:

$$\theta_K = -\frac{f_{uv}}{f_u f_v}\, df.$$

REMARK. For a 1-form $\theta \in \Lambda^1(M)$, let θ' be a 1-form defined by $\theta' := b_0\theta_1 + a_0\theta_2$. We see that θ' is determined by the Riemannian semisurface structure. For a smooth vector field V on M, the following holds:

$$2x_{12}(V^\perp) = *(d\theta'_V + 2\theta_V \wedge \theta'_K), \tag{28}$$

where x_{12} is defined by (U_1, U_2) such that local coordinates (u, v) are compatible with $(\mathscr{D}_1, \mathscr{D}_2)$ and give the orientation of M. We shall use (28) in Section 5.

4. The Equations of Codazzi-Mainardi

Let S be an oriented surface in $\mathbf{R}^3$ without any umbilical point. Then for each point p of S, there exist just two one-dimensional subspaces L_1, L_2 of the tangent plane to S at p such that the normal curvatures of S at p with respect to L_1 and L_2 are equal to the mean curvature. Such a one-dimensional subspace as L_i is called an *H-direction* of S at p. There exist two smooth one-dimensional distributions $\mathscr{D}_1$, $\mathscr{D}_2$ on S which give the two H-directions at each point of S. Such a distribution as $\mathscr{D}_i$ is called an *H-distribution* on S. We see that $\mathscr{D}_1$ and $\mathscr{D}_2$ are perpendicular to each other at any point with respect to the first fundamental form I of S. This implies that I, $\mathscr{D}_1$ and $\mathscr{D}_2$ form a Riemannian semisurface structure of S. Let (u, v) be local coordinates compatible with $(\mathscr{D}_1, \mathscr{D}_2)$. Such coordinates are also said to be *compatible with H-distributions*. The first fundamental form I of S is locally represented as $\mathrm{I} = A^2\, du^2 + B^2\, dv^2$, where A and B are smooth, positive-valued functions. Then the second fundamental form II of S is locally represented as

$$\mathrm{II} = HA^2\, du^2 \pm 2\varepsilon AB\, dudv + HB^2\, dv^2, \tag{29}$$

where $\varepsilon := \sqrt{H^2 - K}$, and K and H are the Gaussian and the mean curvatures of S, respectively. In the following, *we suppose that the sign of the second term of the right hand side of* (29) *is positive*. The equation of Gauss is given by (14). By (5) together with (6), we see that the equations of Codazzi-Mainardi are represented as follows:

$$(HA^2)_v - (\varepsilon AB)_u = (\log A)_v \cdot HA^2 + (\log B/A)_u \cdot \varepsilon AB + \frac{AA_v}{B^2} \cdot HB^2, \tag{30}$$

$$(\varepsilon AB)_v - (HB^2)_u = -\frac{BB_u}{A^2} \cdot HA^2 + (\log B/A)_v \cdot \varepsilon AB - (\log B)_u \cdot HB^2. \tag{31}$$

We see that (30) and (31) are equivalent to

$$(\varepsilon B^2)_u = ABH_v, \tag{32}$$

$$(\varepsilon A^2)_v = ABH_u, \tag{33}$$

respectively. In addition, by $\varepsilon = \sqrt{H^2 - K}$, we see that (32) and (33) are equivalent to

$$H_v = \frac{BH}{A\varepsilon} H_u + \frac{B}{A\varepsilon} \left\{ (\log B^2)_u \varepsilon^2 - \frac{K_u}{2} \right\}, \tag{34}$$

$$H_u = \frac{AH}{B\varepsilon} H_v + \frac{A}{B\varepsilon} \left\{ (\log A^2)_v \varepsilon^2 - \frac{K_v}{2} \right\}, \tag{35}$$

respectively. We see that (34) and (35) hold if and only if the following hold:

$$2K \begin{pmatrix} \boldsymbol{U}_1(H) \\ \boldsymbol{U}_2(H) \end{pmatrix} = \begin{pmatrix} Hc_1(\varepsilon) + \varepsilon c_2(\varepsilon) \\ Hc_2(\varepsilon) + \varepsilon c_1(\varepsilon) \end{pmatrix} = \begin{pmatrix} H & \varepsilon \\ \varepsilon & H \end{pmatrix} \begin{pmatrix} c_1(\varepsilon) \\ c_2(\varepsilon) \end{pmatrix}, \tag{36}$$

where

$$c_1(\varepsilon) := \boldsymbol{U}_1(K) - 4\varepsilon^2 \boldsymbol{U}_1(\log B), \quad c_2(\varepsilon) := \boldsymbol{U}_2(K) - 4\varepsilon^2 \boldsymbol{U}_2(\log A). \tag{37}$$

Let W be the Weingarten map of S. Then the following holds:

$$[\mathrm{W}(\boldsymbol{U}_1), \mathrm{W}(\boldsymbol{U}_2)] = [\boldsymbol{U}_1, \boldsymbol{U}_2] \begin{pmatrix} H & \varepsilon \\ \varepsilon & H \end{pmatrix}.$$

We set $V_0 := c_1(\varepsilon)\boldsymbol{U}_1 + c_2(\varepsilon)\boldsymbol{U}_2$. Then (36) is represented as $2K \operatorname{grad}(H) = \mathrm{W}(V_0)$. The following holds:

$$V_0 = \operatorname{grad}(K) + 4\varepsilon^2 V_K,$$

where V_K denotes the canonical pre-divergence of a Riemannian semisurface $(S, \mathrm{I}, \mathscr{D}_1, \mathscr{D}_2)$. Therefore we obtain Theorem 1.1.

The following holds:

$$\mathrm{II} = H\mathrm{I} + \varepsilon AB(du \otimes dv + dv \otimes du), \tag{38}$$

where (u, v) are local coordinates compatible with H-distributions. Therefore by (2), (27) and (38), we see that for any tangent vector w at any point of S, the following hold:

$$2K \, dH(w) = 2K\mathrm{I}(\mathrm{grad}(H), w)$$

$$= \mathrm{I}(\mathrm{W}(\mathrm{grad}(K) + 4\varepsilon^2 V_K), w)$$

$$= \mathrm{II}(\mathrm{grad}(K) + 4\varepsilon^2 V_K, w)$$

$$= H\mathrm{I}(\mathrm{grad}(K) + 4\varepsilon^2 V_K, w)$$

$$+ \varepsilon AB(du \otimes dv + dv \otimes du)(\mathrm{grad}(K) + 4\varepsilon^2 V_K, w)$$

$$= H \, dK(w) + 4H\varepsilon^2 \theta_K(w)$$

$$+ \varepsilon\left(\frac{B}{A} K_u \, dv + \frac{A}{B} K_v \, du\right)(w) - 4\varepsilon^3\left(\frac{B_u}{A} \, dv + \frac{A_v}{B} \, du\right)(w).$$

Therefore we obtain

COROLLARY 4.1. *The equations of Codazzi-Mainardi are represented as follows*:

$$2K \, dH = H \, dK + 4H\varepsilon^2 \theta_K$$

$$+ \varepsilon\left(\frac{B}{A} K_u \, dv + \frac{A}{B} K_v \, du\right) - 4\varepsilon^3\left(\frac{B_u}{A} \, dv + \frac{A_v}{B} \, du\right), \qquad (39)$$

where (u, v) are local coordinates compatible with H-distributions.

5. The Second Codazzi-Mainardi Polynomial

Let $(\mathscr{D}_1, \mathscr{D}_2)$ be a pair of H-distributions such that I, $\mathscr{D}_1$ and $\mathscr{D}_2$ form a Riemannian semisurface structure of S. For a smooth vector field V on S, we set $V^- := a_0 U_1 - b_0 U_2$, where (u, v) are compatible with $(\mathscr{D}_1, \mathscr{D}_2)$ and give the orientation of M. We see that V^- is determined by the Riemannian semisurface structure of M.

PROOF OF THEOREM 1.2. The following holds:

$$\mathrm{W}(\mathrm{grad}(K) + 4(H^2 - K)V_K)$$

$$= H \, \mathrm{grad}(K) + 4H\varepsilon^2 V_K + \varepsilon(U_2(K)U_1 + U_1(K)U_2)$$

$$- 4\varepsilon^3(U_2(\log A)U_1 + U_1(\log B)U_2). \qquad (40)$$

By (2) together with (40), we obtain

$2 \operatorname{rot}(K \operatorname{grad}(H))$

$$= -4H\varepsilon^2 g(V_K^{\perp}, \operatorname{grad}(\log|K|)) + 4\varepsilon^3 g(V_K^-, \operatorname{grad}(\log|K|))$$

$$+ (H^2 - \varepsilon^2)\varepsilon g(\operatorname{grad}(\log|K|)^-, \operatorname{grad}(\log|K|)). \tag{41}$$

Noticing $\varepsilon^2 = H^2 - K$, we obtain

$4 \operatorname{rot}(H\varepsilon^2 V_K)$

$$= 4H\varepsilon^2 \operatorname{rot}(V_K) - 6H\varepsilon^2 g(V_K^{\perp}, \operatorname{grad}(\log|K|))$$

$$- 2(2H^2 + \varepsilon^2)\varepsilon g(V_K^-, \operatorname{grad}(\log|K|)) - \frac{8(2H^2 + \varepsilon^2)\varepsilon^3}{K} g(V_K^-, V_K), \tag{42}$$

$\operatorname{rot}(\varepsilon(\boldsymbol{U}_2(K)\boldsymbol{U}_1 + \boldsymbol{U}_1(K)\boldsymbol{U}_2))$

$$= \varepsilon \operatorname{rot}(\boldsymbol{U}_2(K)\boldsymbol{U}_1 + \boldsymbol{U}_1(K)\boldsymbol{U}_2)$$

$$+ \frac{(H^2 - \varepsilon^2)\varepsilon}{2} g(\operatorname{grad}(\log|K|)^-, \operatorname{grad}(\log|K|))$$

$$- 2H\varepsilon^2 g(V_K^{\perp}, \operatorname{grad}(\log|K|)) + 2H^2 \varepsilon g(V_K^-, \operatorname{grad}(\log|K|)), \tag{43}$$

$-4 \operatorname{rot}(\varepsilon^3(\boldsymbol{U}_2(\log A)\boldsymbol{U}_1 + \boldsymbol{U}_1(\log B)\boldsymbol{U}_2))$

$$= -4\varepsilon^3 \operatorname{rot}(\boldsymbol{U}_2(\log A)\boldsymbol{U}_1 + \boldsymbol{U}_1(\log B)\boldsymbol{U}_2)$$

$$+ 6H\varepsilon^2 g(V_K^{\perp}, \operatorname{grad}(\log|K|))$$

$$+ 6\varepsilon^3 g(V_K^-, \operatorname{grad}(\log|K|)) + \frac{24H^2\varepsilon^3}{K} g(V_K^-, V_K). \tag{44}$$

Using (2), (40)~(44) and

$$\operatorname{rot}(H \operatorname{grad}(K)) = -\operatorname{rot}(K \operatorname{grad}(H)),$$

we obtain

$$-4H\varepsilon^2 g(V_K^{\perp}, \operatorname{grad}(\log|K|)) + 2(H^2 + \varepsilon^2)\varepsilon g(V_K^-, \operatorname{grad}(\log|K|))$$

$$+ (H^2 - \varepsilon^2)\varepsilon g(\operatorname{grad}(\log|K|)^-, \operatorname{grad}(\log|K|)) - 4H\varepsilon^2 \operatorname{rot}(V_K)$$

$$- \varepsilon \operatorname{rot}(\boldsymbol{U}_2(K)\boldsymbol{U}_1 + \boldsymbol{U}_1(K)\boldsymbol{U}_2)$$

$$- 4\varepsilon^3\{2g(V_K^-, V_K) - \operatorname{rot}(\boldsymbol{U}_2(\log A)\boldsymbol{U}_1 + \boldsymbol{U}_1(\log B)\boldsymbol{U}_2)\}$$

$$= 0. \tag{45}$$

The following hold:

$$\mathrm{rot}(\boldsymbol{U}_2(K)\boldsymbol{U}_1 + \boldsymbol{U}_1(K)\boldsymbol{U}_2)$$

$$= (H^2 - \varepsilon^2)\{\boldsymbol{U}_1\boldsymbol{U}_1(\log|K|) - \boldsymbol{U}_2\boldsymbol{U}_2(\log|K|)$$

$$+ g(\mathrm{grad}(\log|K|)^-, \mathrm{grad}(\log|K|))\}$$

$$- (H^2 - \varepsilon^2)g(V_K^-, \mathrm{grad}(\log|K|)), \tag{46}$$

$$2g(V_K^-, V_K) - \mathrm{rot}(\boldsymbol{U}_2(\log A)\boldsymbol{U}_1 + \boldsymbol{U}_1(\log B)\boldsymbol{U}_2)$$

$$= -\boldsymbol{U}_1\boldsymbol{U}_1(\log B) + \boldsymbol{U}_1(\log B)^2 + \boldsymbol{U}_2\boldsymbol{U}_2(\log A) - \boldsymbol{U}_2(\log A)^2$$

$$= 2x_{12}(V_K^\perp). \tag{47}$$

Applying (46) and (47) to (45), and noticing

$$2x_{12}(\mathrm{grad}(\log|K|)^\perp)$$

$$= \boldsymbol{U}_1\boldsymbol{U}_1(\log|K|) - \boldsymbol{U}_2\boldsymbol{U}_2(\log|K|) + g(V_K^-, \mathrm{grad}(\log|K|)),$$

we obtain

$$\{-x_{12}(\mathrm{grad}(\log|K|)^\perp) + 2g(V_K^-, \mathrm{grad}(\log|K|))\}H^2\varepsilon$$

$$- 2\{\mathrm{rot}(V_K) + g(V_K^\perp, \mathrm{grad}(\log|K|))\}H\varepsilon^2$$

$$+ \{x_{12}(\mathrm{grad}(\log|K|)^\perp) - 4x_{12}(V_K^\perp)\}\varepsilon^3$$

$$= 0. \tag{48}$$

Noticing $\varepsilon \neq 0$, we see that (48) is equivalent to $P_{\mathrm{II}}(H, \sqrt{H^2 - K}) = 0$. Hence we obtain Theorem 1.2. □

REMARK. We shall obtain another representation of (3): we shall prove

$$\{-d(d(\log|K|))' + 2d(\log|K|) \wedge \theta_K'\}H^2$$

$$+ 4\{-\Theta + d(\log|K|) \wedge \theta_K\}H\varepsilon$$

$$+ \{d(d(\log|K|))' + 2d(\log|K|) \wedge \theta_K' - 4[d\theta_K' + 2\theta_K \wedge \theta_K']\}\varepsilon^2$$

$$= 0 \tag{49}$$

(noticing (28), we see that (49) is equivalent to (48)). By (39), we obtain

$$2d(K\,dH) = 4H\varepsilon^2 d(\log|K|) \wedge \theta_K + 4\varepsilon^3 d(\log|K|) \wedge \theta_K'$$
$$+ (H^2 - \varepsilon^2)\varepsilon d(\log|K|) \wedge (d(\log|K|))', \tag{50}$$

$$4d(H\varepsilon^2\theta_K) = 4H\varepsilon^2\Theta - 2(2H^2 + \varepsilon^2)\varepsilon d(\log|K|) \wedge \theta_K'$$
$$+ 6H\varepsilon^2 d(\log|K|) \wedge \theta_K - \frac{8(2H^2 + \varepsilon^2)\varepsilon^3}{K}\theta_K \wedge \theta_K', \tag{51}$$

$$d\left(\varepsilon\left(\frac{B}{A}K_u\,dv + \frac{A}{B}K_v\,du\right)\right)$$
$$= \frac{(H^2 - \varepsilon^2)\varepsilon}{2}d(\log|K|) \wedge (d(\log|K|))' + 2H^2\varepsilon d(\log|K|) \wedge \theta_K'$$
$$+ 2H\varepsilon^2 d(\log|K|) \wedge \theta_K + \varepsilon d(dK)', \tag{52}$$

$$-4d\left(\varepsilon^3\left(\frac{B_u}{A}\,dv + \frac{A_v}{B}\,du\right)\right) = 6\varepsilon^3 d(\log|K|) \wedge \theta_K' + 24\frac{H^2\varepsilon^3}{K}\theta_K \wedge \theta_K'$$
$$- 6H\varepsilon^2 d(\log|K|) \wedge \theta_K + 4\varepsilon^3\,d\theta_K'. \tag{53}$$

Using (39), (50)$\sim$(53) and $d(H\,dK) = -d(K\,dH)$, we obtain (49).

PROOF OF THEOREM 1.3. Let (u^+, v^+) (respectively, $(u^\times, v^\times)$) be local coordinates on an open set U of M which are compatible with $(\mathscr{D}_1^+, \mathscr{D}_2^+)$ (respectively, $(\mathscr{D}_1^\times, \mathscr{D}_2^\times)$) and give the orientation of M. Then the metric g is represented as

$$g = (A^+)^2(du^+)^2 + (B^+)^2(dv^+)^2 = (A^\times)^2(du^\times)^2 + (B^\times)^2(dv^\times)^2.$$

The following holds:

$$P_{\mathrm{I}}^+(Y_1 + Y_2, Y_1 - Y_2)$$
$$= (c_{\mathrm{I}20}^+ + c_{\mathrm{I}11}^+ + c_{\mathrm{I}02}^+)Y_1^2 + 2(c_{\mathrm{I}20}^+ - c_{\mathrm{I}02}^+)Y_1 Y_2 + (c_{\mathrm{I}20}^+ - c_{\mathrm{I}11}^+ + c_{\mathrm{I}02}^+)Y_2^2, \tag{54}$$

where $c_{\mathrm{I}ij}^+ := c_{ij}^+/A^+B^+$. We set

$$\boldsymbol{U}_1^+ := \frac{1}{A^+}\frac{\partial}{\partial u^+}, \quad \boldsymbol{U}_2^+ := \frac{1}{B^+}\frac{\partial}{\partial v^+}, \quad \boldsymbol{U}_1^\times := \frac{1}{A^\times}\frac{\partial}{\partial u^\times}, \quad \boldsymbol{U}_2^\times := \frac{1}{B^\times}\frac{\partial}{\partial v^\times}.$$

We may suppose

$$\boldsymbol{U}_1^+ = \frac{1}{\sqrt{2}}(\boldsymbol{U}_1^\times + \boldsymbol{U}_2^\times), \quad \boldsymbol{U}_2^+ = \frac{1}{\sqrt{2}}(-\boldsymbol{U}_1^\times + \boldsymbol{U}_2^\times).$$

Then we obtain

$$
\begin{aligned}
c_{\mathrm{I}20}^+ &+ c_{\mathrm{I}11}^+ + c_{\mathrm{I}02}^+ \\
&= \frac{1}{A^+ B^+}\{(\log|K|)_{u^+}(\log A^+)_{v^+} + (\log|K|)_{v^+}(\log B^+)_{u^+} + (\log|K|)_{u^+ v^+}\} \\
&= \frac{1}{4}(\boldsymbol{U}_1^\times + \boldsymbol{U}_2^\times)(-\boldsymbol{U}_1^\times + \boldsymbol{U}_2^\times)(\log|K|) \\
&\quad + \frac{1}{4}(-\boldsymbol{U}_1^\times + \boldsymbol{U}_2^\times)(\boldsymbol{U}_1^\times + \boldsymbol{U}_2^\times)(\log|K|) \\
&\quad + \frac{3}{4}\{-\boldsymbol{U}_1^\times(\log|K|) + \boldsymbol{U}_2^\times(\log|K|)\}\{\boldsymbol{U}_1^\times(\log B^+) + \boldsymbol{U}_2^\times(\log B^+)\} \\
&\quad + \frac{3}{4}\{\boldsymbol{U}_1^\times(\log|K|) + \boldsymbol{U}_2^\times(\log|K|)\}\{-\boldsymbol{U}_1^\times(\log A^+) + \boldsymbol{U}_2^\times(\log A^+)\} \\
&= \frac{1}{2}\{-\boldsymbol{U}_1^\times \boldsymbol{U}_1^\times(\log|K|) + \boldsymbol{U}_2^\times \boldsymbol{U}_2^\times(\log|K|)\} \\
&\quad + \frac{3}{4}\{-\boldsymbol{U}_1^\times(\log|K|) + \boldsymbol{U}_2^\times(\log|K|)\}\{\boldsymbol{U}_1^\times(\log B^\times) + \boldsymbol{U}_2^\times(\log A^\times)\} \\
&\quad + \frac{3}{4}\{\boldsymbol{U}_1^\times(\log|K|) + \boldsymbol{U}_2^\times(\log|K|)\}\{-\boldsymbol{U}_1^\times(\log B^\times) + \boldsymbol{U}_2^\times(\log A^\times)\} \\
&= \frac{1}{2}\{-\boldsymbol{U}_1^\times \boldsymbol{U}_1^\times(\log|K|) + \boldsymbol{U}_2^\times \boldsymbol{U}_2^\times(\log|K|)\} \\
&\quad + \frac{3}{2}\{-\boldsymbol{U}_1^\times(\log|K|)\boldsymbol{U}_1^\times(\log B^\times) + \boldsymbol{U}_2^\times(\log|K|)\boldsymbol{U}_2^\times(\log A^\times)\} \\
&= c_{\mathrm{II}20}^\times.
\end{aligned}
\tag{55}
$$

Similarly, we obtain

$$
\begin{aligned}
c_{\mathrm{I}20}^+ &- c_{\mathrm{I}02}^+ \\
&= \frac{1}{A^+ B^+}\{(\log|K|)_{v^+}(\log B^+)_{u^+} - (\log|K|)_{u^+}(\log A^+)_{v^+} - (\log B^+/A^+)_{u^+ v^+}\} \\
&= -\mathrm{rot}(V_K^\times) - g((V_K^\times)^\perp, \mathrm{grad}(\log|K|)) \\
&= c_{\mathrm{II}11}^\times/2,
\end{aligned}
$$

$$c_{I20}^+ - c_{I11}^+ + c_{I02}^+$$

$$= \frac{1}{A^+ B^+} \{ (\log|K|(A^+)^4)_{v^+} (\log B^+)_{u^+} + (\log|K|(B^+)^4)_{u^+} (\log A^+)_{v^+}$$

$$- (\log|K|(A^+)^2 (B^+)^2)_{u^+ v^+} \}$$

$$= x_{12}^\times (\mathrm{grad}(\log|K|))^\perp) - 4 x_{12}^\times ((V_K^\times)^\perp)$$

$$= c_{II02}^\times.$$

Hence we obtain Theorem 1.3. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

6.   Riemannian Semisurface Structures of Some Surfaces

In this section, we suppose the following:

(i)   any smooth two-dimensional manifold is connected and oriented;

(ii)   there exists no umbilical point on any surface;

(iii)   on any Riemannian semisurface $(M, g, \mathscr{D}_1, \mathscr{D}_2)$, local coordinates are always compatible with $(\mathscr{D}_1, \mathscr{D}_2)$ and give the orientation of M; on any surface S, local coordinates are always compatible with H-distributions and give the orientation of S.

6.1.   Surfaces with Constant Mean Curvature

Let S be a surface with constant mean curvature $H_0 \in \mathbf{R}$. Then the left hand side of (2) is identically zero. If the Gaussian curvature function K is identically zero, then S is part of a cylinder. In the following, suppose $K \neq 0$. Then we see that $V_0 = \mathrm{grad}(K) + 4\varepsilon^2 V_K$ is identically zero. This is equivalent to

$$4V_K = \mathrm{grad}(\log(H_0^2 - K)). \qquad\qquad (56)$$

From (56), we obtain

$$\boldsymbol{U}_1(\log((H_0^2 - K)B^4)) = 0, \quad \boldsymbol{U}_2(\log((H_0^2 - K)A^4)) = 0. \qquad (57)$$

From (57), we see that $(H_0^2 - K)B^4$ (respectively, $(H_0^2 - K)A^4$) is of one-variable v (respectively, u). We set

$$e^{4f(u)} := (H_0^2 - K)A^4, \quad e^{4g(v)} := (H_0^2 - K)B^4,$$

$$\tilde{A} := e^{-f(u)}A = e^{-g(v)}B = (H_0^2 - K)^{-1/4}.$$

Let $\tilde{u}$ (respectively, $\tilde{v}$) be a smooth function of u (respectively, v) satisfying

$$\frac{d\tilde{u}}{du} = e^{f(u)} \quad \left(\text{respectively,} \ \frac{d\tilde{v}}{dv} = e^{g(v)} \right).$$

Then we see that $(\tilde{u}, \tilde{v})$ are isothermal coordinates: we may consider $\tilde{A}$ as a function of $\tilde{u}$ and $\tilde{v}$, and we obtain $I = \tilde{A}^2(d\tilde{u}^2 + d\tilde{v}^2)$. We see that $V_K = -\mathrm{grad}(\log \tilde{A})$ and $K = H_0^2 - 1/\tilde{A}^4$ hold.

Let $(M, g, \mathscr{D}_1, \mathscr{D}_2)$ be a Riemannian semisurface. Then (56) holds for a real number $H_0 \in \boldsymbol{R}$ if and only if on a neighborhood of each point of M, there exist isothermal coordinates (u, v) and a smooth, positive-valued function $\tilde{A}$ satisfying $g = \tilde{A}^2(du^2 + dv^2)$ and $K = H_0^2 - 1/\tilde{A}^4$ for some $H_0 \in \boldsymbol{R}$. Suppose that one of these conditions holds. Then by (2) together with the fundamental theorem of the theory of surfaces, we see that M may be locally and isometrically immersed in $\boldsymbol{R}^3$ as a surface with constant mean curvature H_0 so that $(\mathscr{D}_1, \mathscr{D}_2)$ gives a pair of H-distributions and that such a surface is determined by the Riemannian semisurface structure up to a motion of $\boldsymbol{R}^3$.

REMARK. Let S be a surface in $\boldsymbol{R}^3$ with nowhere zero Gaussian curvature K. Suppose that on a neighborhood of each point of S, there exist isothermal coordinates (u, v) and a smooth, positive-valued function $\tilde{A}$ satisfying $I = \tilde{A}^2(du^2 + dv^2)$ and $K = H_0^2 - 1/\tilde{A}^4$ for some $H_0 \in \boldsymbol{R}$. In addition, suppose that for the function $\tilde{A}$, there exists a nonconstant smooth function $\tilde{\alpha}$ of one variable satisfying $\tilde{A}(u, v) = \tilde{\alpha}(u + v)$. Then there exists a nonconstant smooth function $\tilde{h}$ of one variable satisfying

$$K\tilde{h}' = -2(\tilde{h}^2 - H_0^2)(\tilde{h} + \sqrt{\tilde{h}^2 - K})(\log \tilde{\alpha})'.$$

Therefore from (2) and the fundamental theorem of the theory of surfaces, we see that it is possible that the mean curvature function on S is not constant.

6.2. Flat Surfaces

Let S be a surface with identically zero Gaussian curvature. Then from (2), we see that $\mathrm{W}(V_K)$ is identically zero. This implies that at any point of S, V_K is in a principal direction such that the corresponding principal curvature is zero. Then we may suppose that at each point of S, V_K is represented by $\boldsymbol{U}_1 + \boldsymbol{U}_2$ up to a constant. Therefore from Proposition 3.6, we see that on a neighborhood of each point of S, there exists a smooth function f satisfying $f_u > 0$, $f_v > 0$,

$I = f_u^2 \, du^2 + f_v^2 \, dv^2$ and $(\log f_u f_v)_{uv} = 0$. Then there exist local coordinates $(\tilde{u}, \tilde{v})$ satisfying $f_{\tilde{u}} f_{\tilde{v}} = 1$ and V_K is represented as

$$V_K = -f_{\tilde{u}\tilde{v}}(\boldsymbol{U}_1 + \boldsymbol{U}_2). \tag{58}$$

Let $(M, g, \mathscr{D}_1, \mathscr{D}_2)$ be a Riemannian semisurface with identically zero curvature such that V_K is represented by $\boldsymbol{U}_1 + \boldsymbol{U}_2$ up to a constant at each point of M. Then by (2), we see that $(M, g, \mathscr{D}_1, \mathscr{D}_2)$ may be locally and isometrically immersed in $\boldsymbol{R}^3$ so that $(\mathscr{D}_1, \mathscr{D}_2)$ gives a pair of H-distributions.

6.3. Surfaces with Nonzero Constant Gaussian Curvature

Let S be a surface with nonzero constant Gaussian curvature K_0. Then from (2), we obtain

$$2K_0 \operatorname{grad}(H) = 4(H^2 - K_0)\mathrm{W}(V_K); \tag{59}$$

from $P_{\mathrm{II}}(H, \sqrt{H^2 - K_0}) = 0$, we obtain

$$H \operatorname{rot}(V_K) + 2\sqrt{H^2 - K_0}x_{12}(V_K^{\perp}) = 0. \tag{60}$$

Let $(M, g, \mathscr{D}_1, \mathscr{D}_2)$ be a Riemannian semisurface with nonzero constant curvature K_0. Then M may be locally and isometrically immersed in $\boldsymbol{R}^3$ so that $(\mathscr{D}_1, \mathscr{D}_2)$ gives a pair of H-distributions if and only if for a smooth function H satisfying (60), (59) holds. If $\operatorname{rot}(V_K) \neq 0$, then there exists at most one function H satisfying (60). Suppose $P_{\mathrm{II},q} \equiv 0$ for any $q \in M$. This condition is equivalent to the condition that on a neighborhood of each point of M, there exists a smooth, positive-valued function A satisfying $g = A^2(du^2 + dv^2)$ and

$$\Delta \log A + K_0 = 0, \tag{61}$$

$$\boldsymbol{U}_1\boldsymbol{U}_1(\log A) - \boldsymbol{U}_2\boldsymbol{U}_2(\log A) - \boldsymbol{U}_1(\log A)^2 + \boldsymbol{U}_2(\log A)^2 = 0. \tag{62}$$

We see that A satisfies (61) and (62) if and only if A satisfies

$$\boldsymbol{U}_1\boldsymbol{U}_1(\log A) + \boldsymbol{U}_2(\log A)^2 + K_0/2 = 0, \tag{63}$$

$$\boldsymbol{U}_2\boldsymbol{U}_2(\log A) + \boldsymbol{U}_1(\log A)^2 + K_0/2 = 0. \tag{64}$$

There exists a smooth function A satisfying (63) and (64). Therefore noticing Theorem 1.4, we see that for each point p of M, there exist plural isometric immersions of a neighborhood of p into $\boldsymbol{R}^3$ such that arbitrarily distinct two of the images by them are not congruent with each other in $\boldsymbol{R}^3$ and have the same Riemannian semisurface structure.

6.4. Surfaces of Revolution

Let S be a surface of revolution. Then at each point of S, $\mathrm{grad}(H)$ is in a principal direction. Therefore from (2), we see that V_0 is in the same principal direction at each point. Since at each point of S, $\mathrm{grad}(K)$ is in the same principal direction, we see that at each point of S, V_K is in the same principal direction. Then we may suppose that at each point of S, V_K is represented by $U_1 + U_2$ up to a constant. From Proposition 3.6, we see that on a neighborhood of each point of M, there exists a smooth function f satisfying $\mathrm{I} = f_u^2\,du^2 + f_v^2\,dv^2$. In addition, we see that there exist isothermal coordinates $(\tilde{u}, \tilde{v})$ and a smooth function $\tilde{f}$ of one variable satisfying $f(\tilde{u}, \tilde{v}) = \tilde{f}(\tilde{u} + \tilde{v})$ and $\tilde{f}' > 0$. Then V_K is represented as follows:

$$V_K = \left(\frac{1}{\tilde{f}'}\right)'(U_1 + U_2).$$

Let $(M, g, \mathscr{D}_1, \mathscr{D}_2)$ be a Riemannian semisurface with nowhere zero curvature such that at any point of M, the direction determined by $U_1 + U_2$ contains both V_K and $\mathrm{grad}(K)$. Then the following hold:

$$g(V_K^{\perp}, \mathrm{grad}(\log|K|)) = 0, \quad g(V_K^{-}, \mathrm{grad}(\log|K|)) = 0.$$

In addition, the following hold:

$$2x_{12}(V_K^{\perp}) = 2x_{12}(U_2(\log A)U_1 - U_1(\log B)U_2)$$

$$= -2x_{12}(V_K^{-})$$

$$= \frac{1}{AB}(\log(B/A))_{uv}$$

$$= \mathrm{rot}(V_K);$$

$$2x_{12}(\mathrm{grad}(\log|K|)^{\perp}) = 2x_{12}(-U_2(\log|K|)U_1 + U_1(\log|K|)U_2)$$

$$= -2x_{12}(\mathrm{grad}(\log|K|)^{-})$$

$$= U_1 U_2(\log|K|) - U_2 U_1(\log|K|)$$

$$= \frac{1}{AB}(\log|K|)_{uv} - U_1(\log B)U_2(\log|K|)$$

$$\quad - \frac{1}{AB}(\log|K|)_{uv} + U_1(\log|K|)U_2(\log A)$$

$$= 0.$$

Therefore we obtain

$$P_{\mathrm{II}}(Y_1, Y_2) = -2\,\mathrm{rot}(V_K)(Y_1 + Y_2)\,Y_2. \tag{65}$$

This implies that $P_{\mathrm{II}} \equiv 0$ is equivalent to $\mathrm{rot}(V_K) = 0$. Therefore noticing Theorem 1.4, we see that if the distorsion 2-form of $(M, \mathscr{C}_g, \mathscr{D}_1, \mathscr{D}_2)$ vanishes, then M may be locally and isometrically immersed in $\mathbf{R}^3$ so that $(\mathscr{D}_1, \mathscr{D}_2)$ gives a pair of two H-distributions. Then at each point, a principal direction contains both V_K and $\mathrm{grad}(K)$.

REMARK. Suppose $P_{\mathrm{II},q} \not\equiv 0$ for $q \in M$, i.e., $\mathrm{rot}(V_K) \neq 0$ at q. Then by Theorem 1.2 together with (65), we see that there exists no neighborhood of q which may be isometrically immersed in $\mathbf{R}^3$ so that $(\mathscr{D}_1, \mathscr{D}_2)$ gives a pair of two H-distributions.

Let S be a surface with nowhere zero Gaussian curvature such that at each point of S, there exists a principal direction which contains both V_K and $\mathrm{grad}(K)$. Then there exists a smooth function f as in Proposition 3.6 on a neighborhood of each point of S. In addition, noticing Proposition 2.1 and the above remark, we see that there exist isothermal coordinates $(\tilde{u}, \tilde{v})$ and a smooth function $\tilde{f}$ of one-variable satisfying $f(\tilde{u}, \tilde{v}) = \tilde{f}(\tilde{u} + \tilde{v})$. We may suppose $\tilde{f}' > 0$. From (25), we see that K is represented as

$$K = -\frac{2}{\tilde{f}'(\tilde{u} + \tilde{v})^2} \left(\log \tilde{f}'(\tilde{u} + \tilde{v})\right)''.$$

Therefore from (2), we see that for the mean curvature function H, there exists a smooth function $\tilde{h}$ of one variable satisfying $H(\tilde{u}, \tilde{v}) = \tilde{h}(\tilde{u} + \tilde{v})$. Then by the fundamental theorem of the theory of surfaces, we see that S is part of a surface of revolution.

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