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HOCHSCHILD COHOMOLOGY OF ALGEBRAS WITH HOMOLOGICAL IDEALS

By

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Abstract. Let $\varphi : A \to B$ be a homological epimorphism of k-algebras. We investigate the relationship of the Hochschild cohomologies $H^i(A)$ and $H^i(B)$ of A and B, and show that they can be connected by a long exact sequence. In particular, if A is a quasi-hereditary algebra and B is the quotient of A by a minimal heredity ideal, then the long exact sequence provides information on $H^i(A)$, $H^i(B)$ and the extension groups between costandard modules and standard modules, thus one can actually compute $H^i(A)$ inductively. As a consequence, we obtain the Hochschild cohomology of all non-semisimple Temperley-Lieb algebras and representation-finite Schur algebras.

1. Introduction

Let A be a finite dimensional k-algebra over a field k. We consider the enveloping algebra A^e of A which is, by definition, the tensor product $A \otimes_k A^{op}$ of algebras A and A^{op}. Note that the category of left A^e-modules is the same as the category of A-A bimodules. Hence in the sequel we shall not distinguish the left A^e-modules from the A-A-bimodules. Recall that if X is an A-A-bimodule, then we have the following well-known formula for the Hochschhild cohomology of A with coefficients in X:

$$H^n(A, X) = \mathrm{Ext}^n_{A^e}(A, X).$$

If $X = A$ we obtain the n-th Hochschild cohomology of A:

$$H^n(A) = \mathrm{Ext}^n_{A^e}(A, A).$$

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The aim of this work is to present some results that may be useful for the computation of the Hochschild cohomologies of an algebra. For this purpose, we consider a k-algebra homomorphism $\varphi : A \to B$ and the induced embedding functor $\varphi^* : \text{mod } B \to \text{mod } A$, where $\text{mod } A$ stands for the category of all finitely generated left A-modules. The ring morphism $\varphi^e : A^e \to B^e$ induces maps $\varphi_n^e : H^n(B) \to \text{Ext}^n_{A^e}(B, B)$. Recall that φ is an *epimorphism* of algebras if φ^* is a full embedding and, following [9], φ is a *homological epimorphism* if the induced functor of derived categories

$$D^b(\varphi^*) : D^b(\text{mod } B) \to D^b(\text{mod } A)$$

is a full embedding. We shall show that for a homological epimorphism $\varphi : A \to B$ the induced maps $\varphi_n^e : H^n(B) \to \text{Ext}^n_{A^e}(B, B)$ are isomorphisms.

The main example of the above situation arises when J is an idempotent ideal of A which is projective as left A-module. In this case, the quotient $A \to B := A/J$ is a homological epimorphism. Furthermore, if J is a *heredity ideal* of A, that is, J is generated by an idempotent element f in A, and projective as a left A-module with $fAf \simeq k$, then we get a long exact sequence

$$\begin{aligned} 0 \to Z(A) \cap J \to H^0(A) \to H^0(B) \to \text{Ext}^1_A(D(fA), Af) \to H^1(A) \to H^1(B) \\ \to \cdots \to \text{Ext}^n_A(D(fA), Af) \to H^n(A) \to H^n(B) \to \cdots \end{aligned}$$

which is helpful in the calculation of the groups $H^n(A)$. In particular, the existence of this sequence generalizes previous results in [10, 14]. Moreover, our results can be applied to get the Hochschild cohomology of certain quasi-hereditary algebras. In particular, we determine the Hochschild cohomology of Temperley-Lieb algebras and representation-finite q-Schur algebras.

2. Homological Epimorphisms

In this section we recall definitions and elementary results on homological epimorphisms and deduce also basic facts which are needed in the sequel.

Let $\varphi : A \to B$ be a morphism of k-algebras. The natural embedding $\varphi^* : \text{mod } B \to \text{mod } A$ allows to identify B-modules as A-modules. Recall that φ is an *epimorphism* if for all k-algebra morphisms $\psi, \chi : B \rightrightarrows C$, the equation $\psi\varphi = \chi\varphi$ implies $\psi = \chi$. Well-known examples are the canonical epimorphisms $A \to A/I$, the inclusion of the algebra of triangular matrices into the full matrix algebra and the canonical morphism $A \to S^{-1}A$ if A is commutative and $S \subset A$ is a multiplicative subset.

It is well-known that $\varphi : A \to B$ is an epimorphism if and only if every A-linear map ${}_BX \to {}_BY$ is also B-linear, that is, if $\varphi^* : \operatorname{mod} B \to \operatorname{mod} A$ is a full embedding. Another condition is that φ is an epimorphism if and only if the multiplication map $B \otimes_A B \to B$ is a bimodule isomorphism. The reader is referred to [8] for further information on epimorphisms of finite dimensional k-algebras, for instance, among other examples, it is shown that the path algebra $k\vec{\mathbf{A}}_n$ of the quiver

$$\vec{\mathbf{A}}_n : 1 \to 2 \to 3 \to \cdots \to n$$

accepts $\frac{1}{n+2}\binom{2n+2}{n+1}$ epiclasses.

If $\varphi : A \to B$ is a k-algebra homomorphism, then for any two B-modules X, Y we get a natural map $\varphi_0 : \operatorname{Hom}_B(X, Y) \to \operatorname{Hom}_A(X, Y)$.

Consider the derived category $D^b(\operatorname{mod} A)$ of $\operatorname{mod} A$ equipped with the translation functor T given by $T(X^\bullet)^n = X^{n+1}$ and $(Td_X^\bullet)^n = -d_X^{n+1}$ for an object $X^\bullet \in D^b(\operatorname{mod} A)$ with chain maps $d_X^n : X^n \to X^{n+1}$. Recall that there is a natural identification of $\operatorname{mod} A$ with a full subcategory of $D^b(\operatorname{mod} A)$ and for any two modules $X, Y \in \operatorname{mod} A$ we have

$$\operatorname{Ext}_A^n(X, Y) = \operatorname{Hom}_{D^b(\operatorname{mod} A)}(X, T^n Y).$$

For concepts related with derived categories see [11].

Consider the induced functor of derived categories

$$D^b(\varphi^*) : D^b(\operatorname{mod} B) \to D^b(\operatorname{mod} A).$$

Then for modules $X, Y \in \operatorname{mod} B$ there are natural maps

$$\varphi_n : \operatorname{Ext}_B^n(X, Y) = \operatorname{Hom}_{D^b(\operatorname{mod} B)}(X, T^n Y) \to \operatorname{Hom}_{D^b(\operatorname{mod} A)}(X, T^n Y) = \operatorname{Ext}_A^n(X, Y).$$

Following [9], we say that $\varphi : A \to B$ is a *homological epimorphism* if $D^b(\varphi^*) : D^b(\operatorname{mod} B) \to D^b(\operatorname{mod} A)$ is a full embedding, equivalently, if for every $X, Y \in \operatorname{mod} B$ and $n \geq 0$ the morphisms $\varphi_n : \operatorname{Ext}_B^n(X, Y) \to \operatorname{Ext}_A^n(X, Y)$ are isomorphisms.

To see whether an epimorphism is a homological epimorphism, we have the following statements proved in [9].

Lemma 2.1. *Assume that $\varphi : A \to B$ is an epimorphism. Then the following statements are equivalent*:

(0) *φ is a homological epimorphism.*

(1) *For all $i \geq 1$, $\operatorname{Tor}_i^A(B, B) = 0$.*

(2) *For all $i \geq 1$ and module ${}_BY$, $\operatorname{Tor}_i^A(B, Y) = 0$.*

(2′) *For all* $i \geq 1$ *and module* X_B, $\mathrm{Tor}_i^A(X, B) = 0$.
(3) *For all* $i \geq 1$ *and modules* X_B, ${}_BY$, $\mathrm{Tor}_i^A(X, Y) \xrightarrow{\sim} \mathrm{Tor}_i^B(X, Y)$.
(4) *For all* $i \geq 1$, $\mathrm{Ext}_A^i(B, B) = 0$.
(5) *For all* $i \geq 1$ *and* $Y \in \mathrm{mod}\, B$, $\mathrm{Ext}_A^i(B, Y) = 0$.
(6) *For all* $i \geq 1$ *and* $X, Y \in \mathrm{mod}\, B$, $\mathrm{Ext}_B^i(X, Y) \xrightarrow{\sim} \mathrm{Ext}_A^i(X, Y)$.
(0^{op}) $\varphi^{op} : A^{op} \to B^{op}$ *is a homological epimorphism.* ■

The following observation is a variation of some of the above statements.

Proposition 2.2. *Let* J *be an ideal of* A *and let* $\varphi : A \to B$ *be the canonical epimorphism with* $B = A/J$. *Then*

(a) $\varphi : A \to B$ *is a homological epimorphism if and only if* $\mathrm{Tor}_n^A(J, B) = 0$ *for all* $n \geq 0$. *In this case,* J *is idempotent.*
(b) *An idempotent ideal* J *of* A *is homological if and only if* $\mathrm{Ext}_A^n(J, A/J) = 0$ *for all* $n \geq 0$.
(c) *If* J *is a projective* A*-module and* $J^2 = J$, *then* φ *is a homological epimorphism.*

Proof. (a) By applying the functor $- \otimes_A B$ to the exact sequence $0 \to J \xrightarrow{j} A \to B \to 0$ we get a new exact sequence

$$0 \to \mathrm{Tor}_1^A(B, B) \to J \otimes_A B \to A \otimes_A B \to B \otimes_A B \to 0$$

and $\mathrm{Tor}_n^A(B, B) \xrightarrow{\sim} \mathrm{Tor}_{n-1}^A(J, B)$ for $n \geq 2$.

Since $J \otimes_A B \simeq J/J^2$ and $B \simeq B \otimes_A B$, we have $\mathrm{Tor}_1^A(B, B) \simeq J/J^2$. Hence we get the desired condition.

The proof of (b) is similar to (a) and (c) follows from (a). ■

An ideal J of A will be called a *homological ideal* in A if the quotient $A \to A/J$ is a homological epimorphism. Observe that such ideals were called *strong idempotent ideals* in [1].

Since our consideration involves also the enveloping algebras, we need also the following result.

Proposition 2.3. *Let* $\varphi : A \to B$ *be an epimorphism of* k*-algebras. Then the following hold*:

(1) $\varphi^e : A^e \to B^e$ *is an epimorphism.*
(2) $\varphi_0^e : H^0(B) \to \mathrm{Hom}_{A^e}(B, B)$ *is an isomorphism.*

Moreover, if φ *is a homological epimorphism, then*

(3) φ^e *is a homological epimorphism.*

(4) $\varphi_n^e : H^n(B) \to \mathrm{Ext}^n_{A^e}(B,B)$ *is an isomorphism for each* $n \geq 0$.

Proof. In our proof we shall make use of the following identities shown in Cartan-Eilenberg's book for any three k-algebras Λ, Γ, Σ and modules $X_{\Lambda-\Gamma}$, ${}_{\Lambda}Y_{\Sigma}$, ${}_{\Gamma-\Sigma}Z$:

(a) $(X \otimes_\Lambda Y) \otimes_{\Gamma\otimes\Sigma} Z \xrightarrow{\sim} X \otimes_{\Lambda\otimes\Gamma} (Y \otimes_\Sigma Z)$ [4, IX.2.1],

(b) if $\mathrm{Tor}_n^\Lambda(X,Y) = 0 = \mathrm{Tor}_n^\Sigma(Y,Z)$ for all $n > 0$, then

$$\mathrm{Tor}_n^{\Gamma\otimes\Sigma}(X \otimes_\Lambda Y, Z) \xrightarrow{\sim} \mathrm{Tor}_n^{\Lambda\otimes\Gamma}(X, Y \otimes_\Sigma Z) \text{ [4, IX.2.8].}$$

Moreover, since $\varphi : A \to B$ is an epimorphism, we have

(c) for any B-module X, there is an isomorphism $B \otimes_A X \xrightarrow{m} X$, $b \otimes x \mapsto bx$. In particular, the algebras $B \otimes_A B^{op}$ and B^{op} are isomorphic.

(1): $B^e \otimes_{A^e} B^e = (B \otimes_k B^{op}) \otimes_{A\otimes A^{op}} B^e \underset{(a)}{\xrightarrow{\sim}} B \otimes_A (B^{op} \otimes_{A^{op}} B^e)$.

Since $B^{op} \otimes_{A^{op}} (B \otimes_k B^{op}) \xrightarrow{\sim} (B^{op} \otimes_{A^{op}} B) \otimes_k B^{op} \underset{(c)^{op}}{\xrightarrow{\sim}} B \otimes_k B^{op}$, we get

$$B^e \otimes_{A^e} (B^e) \xrightarrow{\sim} (B \otimes_A B) \otimes_k B^{op} \underset{(c)}{\xrightarrow{\sim}} B \otimes_k B^{op} = B^e.$$

(2): $H^0(B) = \mathrm{Hom}_{B^e}(B,B) \xrightarrow{\sim} \mathrm{Hom}_{A^e}(B,B)$.

Assume now that $\varphi : A \to B$ is a homological epimorphism.

(3): We shall check that property (1) in 2.1 is satisfied by φ^e. Namely,

$$\mathrm{Tor}_n^{A^e}(B^e, B^e) = \mathrm{Tor}_n^{A\otimes_k A^{op}}(B \otimes_k B^{op}, B^e) \underset{(b)}{\xrightarrow{\sim}} \mathrm{Tor}_n^A(B, B^{op} \otimes_{A^{op}} B^e) \underset{2.1(2)}{=} 0.$$

(4): $H^n(B) = \mathrm{Ext}^n_{B^e}(B,B) \xrightarrow{\sim} \mathrm{Ext}^n_{A^e}(B,B)$. ∎

In the following we shall compare the algebra A^e and B^e. Let $e_1, \dots, e_n$ be a complete set of pairwise orthogonal primitive idempotents for A. Then $P_1 = Ae_1, \dots, P_n = Ae_n$ is a set of representatives of the isomorphism classes of indecomposable projective A-modules.

Consider $A^e = A \otimes_k A^{op}$. The elements $e_i \otimes e_j^{op} \in A^e$ $(1 \leq i,j \leq n)$ form a complete set of pairwise orthogonal primitive idempotents for A^e. Then the indecomposable projective A^e-modules are of the form $A^e(e_i \otimes e_j) = Ae_i \otimes_k e_jA$.

Given an ideal J of A, and the canonical epimorphism $\varphi : A \to B = A/J$, we consider the ideal $I = A \otimes_k J^{op} + J \otimes_k A^{op}$ of A^e, which gives rise to exact sequences

$$0 \to I \to A^e \xrightarrow{\varphi^e} B^e \to 0 \quad \text{and} \quad 0 \to J \otimes_k J^{op} \to A \otimes_k J^{op} \oplus J \otimes_k A^{op} \to I \to 0.$$

The second sequence is exact since $\dim_k(A \otimes_k J^{op} \oplus J \otimes_k A^{op}) - \dim_k I = 2\dim_k A \cdot \dim_k J - (\dim_k A)^2 + (\dim_k A - \dim_k J)^2 = \dim_k J \otimes_k J^{op}$.

We get the following technical remark.

Lemma 2.4. *If an idempotent ideal J in A is a projective left A-module, then*

$$0 \to J \otimes_k J^{op} \to A \otimes_k J^{op} \oplus J \otimes_k A^{op} \to I \to 0$$

is a projective presentation of I as A^e-module. Moreover, the two first terms of the sequence are in $\mathrm{add}(P)$ *for a projective A^e-module P whose trace in A^e is I. In particular,* $\mathrm{proj.dim}_{A^e} B^e \leq 2$.

Proof. Assume that $J = \bigoplus_{j \in S} Ae_j = AeA$ for an idempotent $e = \sum_{j \in S} e_j$. Consider the unity $1 = \sum_{i=1}^{n} e_i$ of A and set $P = \bigoplus_{i=1}^{n} \bigoplus_{j \in S} [A^e(e_i \otimes e_j^{op}) \oplus A^e(e_j \otimes e_i^{op})]$ which is a projective A^e-module. Then $J \otimes_k A^{op} \simeq \bigoplus_{j \in S} \bigoplus_{i=1}^{n} Ae_j \otimes_k e_k A^{op} \in \mathrm{add}(P)$ and similarly, $J \otimes_k J^{op}$ and $A \otimes_k J^{op} \in \mathrm{add}(P)$.

Finally, the trace of P in A^e is

$$\mathrm{tr}_P(A^e) = \sum_{1 \leq i \leq n} \sum_{j \in S} [A^e(e_i \otimes e_j^{op}) + A^e(e_j \otimes e_i^{op})] = A \otimes_k J^{op} + J \otimes_k A^{op} = I.$$

■

The above Lemma may be used to provide another proof of the fact that $\varphi^e : A^e \to B^e$ is a homological epimorphism as a consequence of the following Theorem shown in [1]:

Let I be a an idempotent ideal of a k-algebra C and $D = C/I$. Assume that there is a projective C-module P such that $I = \mathrm{tr}_P(C)$ and that the minimal projective resolution

$$\cdots \to P_i \to \cdots \to P_1 \to P_0 \to I \to 0$$

has its first $s+1$ terms $P_0, P_1, \ldots, P_s \in \mathrm{add}(P)$. Then for any $X, Y \in \mathrm{mod}\, D$,

$$\mathrm{Ext}^i_D(X, Y) \xrightarrow{\sim} \mathrm{Ext}^i_C(X, Y), \quad 0 \leq i \leq s+1.$$

Let us end this section with some examples of homological epimorphisms and Hochschild cohomology.

Example 1. In this example we shall have a homological epimorphism $\varphi : A \to A/J$ with non-projective ${}_AJ$.

Consider the algebra $A = kQ/I$ with Q and I given by

$$Q : 1 \underset{\beta}{\overset{\alpha}{\rightleftarrows}} 2, \quad I = \langle \beta\alpha\beta \rangle.$$

Let J be the trace in A of the projective module P_1 corresponding to the vertex 1. A projective resolution of J is:

$$\cdots \to P_1 \xrightarrow{f_2} P_1 \xrightarrow{f_1} P_1 \oplus P_1 \xrightarrow{f_0} J \to 0,$$

where $f_0(e_1, 0) = e_1$, $f_0(0, e_1) = \beta$, $f_1(e_1) = (0, \beta\alpha)$, $f_2(e_1) = \beta\alpha, \ldots$. Hence J has infinite projective dimension. Moreover, the ring $B = A/J$ is just k with ${}_AB$ the simple S_2 at the vertex 2. Hence $\mathrm{Ext}^n_A(B, B) = 0$ for all $n \geq 1$ since

$$\cdots \to P_1 \xrightarrow{f_2} P_1 \to P_2 \to B \to 0$$

$$e_1 \mapsto \beta$$

is a projective resolution which remains exact after applying $\mathrm{Hom}_A(S_2, -)$. Therefore $\varphi : A \to B$ is a homological epimorphism.

Example 2. Let A and B be two algebras and let $F : D^b(\mathrm{mod}\, A) \to D^b(\mathrm{mod}\, B)$ be a derived equivalence which sends A to B. Happel has shown [10] that $H^n(A) \xrightarrow{\sim} H^n(B)$ for all $n \geq 0$. In fact, one can establish a derived equivalence $\tilde{F} : D^b(\mathrm{mod}\, A^e) \to D^b(\mathrm{mod}\, B^e)$ sending A to B. Then $H^n(A) = \mathrm{Ext}^n_{A^e}(A, A) = \mathrm{Hom}_{D^b(\mathrm{mod}\, A^e)}(A, T^n A) \cong \mathrm{Hom}_{D^b(\mathrm{mod}\, B^e)}(B, T^n B) = \mathrm{Ext}^n_{B^e}(B, B) = H^n(B)$. Especially, the Hochschild cohomology of an algebra is both tilting-invariant and Morita-invariant.

Example 3. Finally, let us remark that there is a formula between Hochschild cohomology and homology, namely, $H^i(A, X) \simeq H_i(A, DX)$ for all A-A-bimodule X, where D is the k-duality. However, this does not help us very much when we calculate Hochschild cohomology $H^i(A)$. For example, it is proved in [20] that for a quasi-hereditary algebra A we always have $H_n(A) = 0$ for all $n \geq 1$, but the Hochschild cohomology $H^n(A)$ may not vanish. An easy example is the Auslander algebra A of $k[x]/(x^n)$, in this case, we obtain $\dim_k H^0(A) = n$, $\dim_k H^1(A) = n - 1$, and $\dim_k H^2(A) = n - 1$, and $H^i(A) = 0$ for all $i \geq 3$ since the global dimension of A is at most 2. (For this and other similar examples see [10].)

3. Hochschild Cohomology of an Algebra with Homological Ideals

In this section we assume that A is a finite dimensional k-algebra and J is a homological ideal of A. Write $B = A/J$. Our results in this section will be useful for calculation of the Hochschild cohomology of an algebra with homological ideals.

Proposition 3.1. (a) *For every B-B-bimodule Y, there is a long exact sequence*

$$0 \to H^1(B, Y) \to H^1(A, Y) \to \mathrm{Ext}^1_{A^e}(J, Y) \to \cdots \to$$

$$\to H^i(B, Y) \to H^i(A, Y) \to \mathrm{Ext}^i_{A^e}(J, Y) \to \cdots.$$

Moreover, $H^0(B) = H^0(A, B)$.

(b) *If* ${}_AJ$ *is projective, then* $\mathrm{Ext}^1_{A^e}(J, Y) = 0$ *for every B-B-bimodule Y. In particular*, $H^1(B) = H^1(A, B)$.

(c) *If J is a projective A^e-module, then* $H^i(B) = H^i(A, B)$ *for* $i \geq 0$.

Proof. Consider the exact sequence $0 \to J \to A \to B \to 0$ of A^e-modules. The sequence in (a) is obtained by applying $\mathrm{Hom}_{A^e}(-, Y)$ to get the long exact sequence

$$0 \to \mathrm{Hom}_{A^e}(B, Y) \to \mathrm{Hom}_{A^e}(A, Y) \to \mathrm{Hom}_{A^e}(J, Y) \to \mathrm{Ext}^1_{A^e}(B, Y) \to \mathrm{Ext}^1_{A^e}(A, Y)$$

$$\to \mathrm{Ext}^1_{A^e}(J, Y) \to \cdots \to \mathrm{Ext}^i_{A^e}(B, Y) \to \mathrm{Ext}^i_{A^e}(A, Y) \to \mathrm{Ext}^i_{A^e}(J, Y) \to \cdots.$$

We show that $\mathrm{Hom}_{A^e}(J, Y) = 0$. Indeed, let $\alpha \in \mathrm{Hom}_{A^e}(J, Y)$ be an element. Since J is idempotent by 2.2, any element $x \in J$ is a linear combination of elements of the form uv with $u, v \in J$. Thus $\alpha(uv) = u\alpha(v) \in JY = 0$ and $\alpha = 0$.

By 2.2, $H^i(B, Y) = \mathrm{Ext}^i_{B^e}(B, Y) = \mathrm{Ext}^i_{A^e}(B, Y)$, for $i \geq 0$. The result follows.

(b): Assume that J is a projective A-module. Since J is an idempotent ideal, we have $J = AfA$ for some idempotent element $f \in A$. Then, the projectivity of J implies that $J \xrightarrow{\sim} Af \otimes_{fAf} fA$. Consider the exact sequence

$$0 \to K \to Af \otimes_k fA \xrightarrow{g} J \simeq Af \otimes_{fAf} fA \to 0,$$

where $g(a \otimes_k b) = a \otimes_{fAf} b$. Then K is the $A - A$-bimodule generated by $\{x \otimes f - f \otimes x \mid x \in fAf\}$.

To show that $\mathrm{Ext}^1_{A^e}(J, Y) = 0$, it is enough to prove that $\mathrm{Hom}_{A^e}(K, Y) = 0$. Indeed, let $\alpha \in \mathrm{Hom}_{A^e}(K, Y)$ and consider the element $x \otimes f - f \otimes x \in K$ with $x \in fAf$. Then we get in the B-module Y

$$\alpha(x \otimes f - f \otimes x) = \alpha(f(x \otimes f - f \otimes x)) = f\alpha(x \otimes f - f \otimes x) \in JY = 0.$$

Hence $\mathrm{Hom}_{A^e}(K, Y) = 0$ and (b) is proved.

(c): If J is a projective A^e-module, then $\mathrm{Ext}^i_{A^e}(J,Y)=0$ for any A^e-module Y and $i\geq 1$. The result follows from (a). ■

Let $Z(A)$ denote the center of A. It is well-known (and trivial to show) that $H^0(A)=Z(A)$. More generally, $\mathrm{Hom}_{A^e}(A,J)=Z(A)\cap J$ for any two-sided ideal J of A.

Proposition 3.2. (a) *Assume that ${}_AJ$ is a projective A-module. Then there is an exact sequence*

$$0\to Z(A)\cap J\to H^0(A)\to H^0(B)\to \mathrm{Ext}^1_{A^e}(A,J)$$
$$\to H^1(A)\to H^1(B)\to \mathrm{Ext}^2_{A^e}(A,J)\to H^2(A).$$

(b) *Assume that J is a projective A^e-module. Then there exist a long exact sequence*

$$0\to Z(A)\cap J\to H^0(A)\to H^0(B)\to \mathrm{Ext}^1_{A^e}(A,J)\to H^1(A)$$
$$\to H^1(B)\to\cdots\to \mathrm{Ext}^i_{A^e}(A,J)\to H^i(A)\to H^i(B)\to\cdots.$$

(c) *Assume that J is a projective ideal in A generated by a primitive idempotent element f with $fAf\simeq k$. Then there exist a long exact sequence*

$$0\to Z(A)\cap J\to H^0(A)\to H^0(B)\to \mathrm{Ext}^1_{A}(D(fA),Af)\to H^1(A)$$
$$\to H^1(B)\to\cdots\to \mathrm{Ext}^i_{A}(D(fA),Af)\to H^i(A)\to H^i(B)\to\cdots,$$

where D is the usual duality $\mathrm{Hom}_k(-,k)$.

Proof. By applying $\mathrm{Hom}_{A^e}(A,-)$ to the exact sequence $0\to J\to A\to B\to 0$ we get a long exact sequence

$$0\to \mathrm{Hom}_{A^e}(A,J)\to \mathrm{Hom}_{A^e}(A,A)\to \mathrm{Hom}_{A^e}(A,B)\to \mathrm{Ext}^1_{A^e}(A,J)\to \mathrm{Ext}^1_{A^e}(A,A)$$
$$\to \mathrm{Ext}^1_{A^e}(A,B)\to\cdots\to \mathrm{Ext}^i_{A^e}(A,J)\to \mathrm{Ext}^i_{A^e}(A,A)\to \mathrm{Ext}^i_{A^e}(A,B)\to\cdots.$$

The first two statements follow from 3.1. To prove the statement (c), we notice that $J\simeq Af\otimes_k fA$ and that $D(Af\otimes_k fA)\simeq D(fA)\otimes_k D(Af)$. Thus, by [4, Chap. IX, Exercise 8, p. 181], we have $\mathrm{DExt}^i_{A^e}(A,J)=DH^i(A,J)\simeq H_i(A,DJ)=H_i(A,D(fA)\otimes_k D(Af))$. It follows further from [4, Chap. IX, Corollary 4.4, p. 170 and Chap. VI, Proposition 5.3, p. 120] that $H_i(A,D(fA)\otimes_k D(Af))\simeq \mathrm{Tor}^A_i(D(Af),D(fA))\simeq \mathrm{DExt}^i_A(D(fA),Af)$. Thus we have proved (c). ■

Let us remark that Theorem 5.3 in [10] now follows easily from 3.2(c): Suppose f is the idempotent of the one-point extension $A = B[M]$ such that $M = \mathrm{rad}(Af)$. Let S be the simple injective A-module $D(fA)$. Then $fAf \simeq k$. It follows from the exact sequence $0 \to M \to Af \to S \to 0$ that $\mathrm{Ext}^1_A(D(fA), Af) \simeq \mathrm{Hom}_A(M, Af)/k \simeq \mathrm{Hom}_A(M, M)/k$ and $\mathrm{Ext}^{i+1}_A(D(fA), Af) \simeq \mathrm{Ext}^i_A(M, Af) \simeq \mathrm{Ext}^i_A(M, M)$. Thus the long exact sequence in [10, Theorem 5.3] follows immediately from 3.2.

As a direct consequence of 3.2, we have the following corollary.

Corollary 3.3. *Assume that J is a projective ideal in A generated by a primitive idempotent element f with $fAf \simeq k$. If the injective dimension of the right A-module fA is at most m, then $H^i(A) = H^i(B)$ for all $i > m$. In particular, if $id(A_A) \leq 1$, then $H^i(A) = H^i(B)$ for all $i > 1$.* ■

Recall that an idempotent ideal I of A is a *heredity ideal* if I is a projective left module and $\mathrm{End}_A(I)$ is a semisimple ring. The following is probably well-known.

Lemma 3.4. *Assume that k is an algebraically closed field and that J is an indecomposable idempotent two-sided ideal of A. The following conditions are equivalent*:

(a) *J is a heredity ideal*;

(b) *J is a projective A^e-module with* $\mathrm{End}_{A^e}(J) = k$;

(c) *The multiplication map $m : J \otimes_A J \to J$ is an isomorphism and* $\mathrm{End}_{A^e}(J) = k$;

(d) $\mathrm{End}_{A^e}(J) = k$ *and* $\mathrm{Tor}^A_2(B, B) = 0$, *where* $B = A/J$.

Proof. (a) $\Rightarrow$ (b): Since J is an indecomposable two-sided ideal in A, it must be generated by a primitive idempotent element f in A. Thus $J \simeq Af \otimes_{fAf} fA$. Since fJf is isomorphic to k, we have that $\mathrm{End}_{A^e}(J) \simeq f \cdot A^e \cdot f \simeq fAf \otimes_k fAf \simeq k$.

(b) $\Rightarrow$ (a): This is clear.

(a) $\Rightarrow$ (c): Applying $J \otimes_A -$ to the exact sequence $0 \to J \to A \to B \to 0$, we get

$$0 \to \mathrm{Tor}^A_1(J, B) \to J \otimes_A J \xrightarrow{m} J \simeq J \otimes_A A \to J \otimes_A B = 0.$$

If J is a projective A-module, then $\mathrm{Tor}^A_1(J, B) = 0$.

(c) $\Leftrightarrow$ (d): Applying $- \otimes_A B$ to the canonical sequence, we get

$$0 = \mathrm{Tor}_2^A(A, B) \to \mathrm{Tor}_2^A(B, B) \to \mathrm{Tor}_1^A(J, B) \to \mathrm{Tor}_1^A(A, B) = 0.$$

Then $\mathrm{Tor}_1^A(J, B) = 0$ exactly when $\mathrm{Tor}_2^A(B, B) = 0$.

(c) $\Rightarrow$ (a): It is shown in [1, (5.3) and (1.4)]. ∎

Proposition 3.5. *Assume that k is algebraically closed and J is an indecomposable heredity ideal of A. Then*

(a) $\mathrm{Hom}_{A^e}(J, J) = k = \mathrm{Hom}_{A^e}(J, A)$.

(b) $H^i(B) = \mathrm{Ext}^i_{A^e}(A, B)$ *for all* $i \geq 0$.

(c) $H^i(A) = \mathrm{Ext}^i_{A^e}(B, A)$ *for all* $i \geq 1$.

(d) $\dim_k H^0(A) = \dim_k \mathrm{Hom}_{A^e}(B, A) + 1$.

Proof. We apply different functors to the canonical exact sequence $0 \to J \to A \to B \to 0$.

For (a) we apply $\mathrm{Hom}_{A^e}(J, -)$ and use 3.4 and 3.1(a).

(b) Follows from 3.4 and 3.1(c).

(c) and (d): By applying $\mathrm{Hom}_{A^e}(-, A)$ we get the following exact sequence

$$0 \to \mathrm{Hom}_{A^e}(B, A) \to \mathrm{Hom}_{A^e}(A, A) \xrightarrow{\alpha} \mathrm{Hom}_{A^e}(J, A)$$
$$\to \mathrm{Ext}^1_{A^e}(B, A) \to \mathrm{Ext}^1_{A^e}(A, A) \to 0$$

and $\mathrm{Ext}^i_{A^e}(B, A) \xrightarrow{\sim} \mathrm{Ext}^i_{A^e}(A, A)$ for $i \geq 2$. By (a), α is surjective since $\alpha(1) \neq 0$. Therefore (c) and (d) follow. ∎

The next lemma shows that the terms $\mathrm{Ext}^i_{A^e}(A, J)$ in the long exact sequence 3.2 may be replaced by $\mathrm{Ext}^i_{A^e}(B, J)$ in certain cases.

Lemma 3.6. *Under the assumptions of* 3.5, *we have* $\mathrm{Ext}^i_{A^e}(A, J) \xrightarrow{\sim} \mathrm{Ext}^i_{A^e}(B, J)$ *for* $i \neq 1$ *and* $\dim_k \mathrm{Ext}^1_{A^e}(A, J) = \dim_k \mathrm{Ext}^1_{A^e}(B, J) - 1$.

Proof. We apply $\mathrm{Hom}_{A^e}(-, J)$ to the canonical sequence to get

$$0 \to \mathrm{Hom}_{A^e}(B, J) \to \mathrm{Hom}_{A^e}(A, J) \to \mathrm{Hom}_{A^e}(J, J) \xrightarrow{\delta} \mathrm{Ext}^1_{A^e}(B, J)$$
$$\to \mathrm{Ext}^1_{A^e}(A, J) \to 0$$

and $\mathrm{Ext}^i_{A^e}(B, J) \xrightarrow{\sim} \mathrm{Ext}^i_{A^e}(A, J)$ for $i \geq 2$.

Since $\delta(1_J)$ corresponds to our canonical sequence, the map δ is a monomorphism and the result follows. ∎

By 3.2, to compare the Hochschild homologies of A and B, we need to calculate the groups $\mathrm{Ext}^n_{A^e}(A, J)$. In general, fAf may not be isomorphic to k, this implies that we cannot use the third exact sequence of 3.2. However, the following result will be helpful in some cases.

Proposition 3.7. *Suppose that J is an ideal in A generated by an idempotent element f such that ${}_AJ$ is projective. We denote by $\mathrm{Tr}_{fAf}(M)$ the transpose of an fAf-module M.*

If $\mathrm{Ext}^i_{fAf}(\mathrm{Tr}_{fAf}(fA), Af) = 0$ for all $i \geq 1$, then $\mathrm{Ext}^n_{A^e}(A, J) \simeq \mathrm{Ext}^n_{A\otimes_k(fAf)^{op}}(\mathrm{Hom}_{fAf}(fA, fAf), Af)$ for all $n \geq 1$.

Proof. It follows from [2, Proposition 3.2, p. 123] that $Af \otimes_{fAf} fA \simeq \mathrm{Hom}_{fAf}(\mathrm{Hom}_{fAf}(fA, fAf), Af)$ if $\mathrm{Ext}^1_{fAf}(\mathrm{Tr}_{fAf}(fA), Af) = 0 = \mathrm{Ext}^2_{fAf}(\mathrm{Tr}_{fAf}(fA), Af)$. By definition of the transpose, if $P_1 \to P_0 \to fA \to 0$ is a projective presentation of the fAf-module fA, then we have a presentation of right fAf-module $\mathrm{Tr}_{fAf}(fA)$:

$$0 \to \mathrm{Hom}_{fAf}(fA, fAf) \to \mathrm{Hom}_{fAf}(P_0, fAf) \to \mathrm{Hom}_{fAf}(P_1, fAf) \to \mathrm{Tr}_{fAf}(fA) \to 0.$$

Now it follows from this sequence that $\mathrm{Ext}^i_{fAf}(\mathrm{Hom}_{fAf}(fA, fAf), Af) \simeq \mathrm{Ext}^{i+2}_{fAf}(\mathrm{Tr}_{fAf}(fA), Af) = 0$ for $i \geq 1$. By [4, Chap. IX, Theorem 2.8a, p. 167], $\mathrm{Ext}^i_{A\otimes A^{op}}(A, \mathrm{Hom}_{fAf}(\mathrm{Hom}_{fAf}(fA, fAf), Af)) \simeq \mathrm{Ext}^i_{A^{op}\otimes fAf}(A \otimes_A \mathrm{Hom}_{fAf}(fA, fAf), Af) = \mathrm{Ext}^i_{A^{op}\otimes fAf}(\mathrm{Hom}_{fAf}(fA, fAf), Af)$. If we understand each A-fAf-bimodule as left $A \otimes (fAf)^{op}$-module, then the last cohomology group is just what we want. This finishes the proof. ∎

The condition in the above proposition can be satisfied if Af is an injective right fAf-module, or fA is a projective fAf-module. The following is an example in which Af is an injective right fAf-module.

Let A be the algebra given by the following quiver with the relation:

$$\alpha \circlearrowright \underset{1}{\circ} \xleftarrow{\ \beta\ } \underset{2}{\circ} \qquad \alpha^2 = 0.$$

We consider the ideal J of A generated by the primitive idempotent element f corresponding to the vertex 1. Then J is a projective left ideal in A, fAf is isomorphic to $k[x]/(x^2)$ and $Af = fAf$.

In fact, we have the following more general result.

Corollary 3.8. *Let f be an idempotent element in A such that ${}_AJ = AfA$ is* *projective and that* $(1-f)AfA(1-f) = (1-f)A(1-f)$. *If* fAf *is self-injective, then* $\mathrm{Ext}^n_{A^e}(A, J) \simeq \mathrm{Ext}^n_{A\otimes_k (fAf)^{op}}(\mathrm{Hom}_{fAf}(fA, fAf), Af)$ *for* $n \geq 1$.

Proof. Let $e = 1 - f$. Then we have a matrix presentation of A:

$$A = \begin{pmatrix} eAe & eAf \\ fAe & fAf \end{pmatrix}.$$

Since $eAfAe = eAe$, both the right fAf-module eAf and the left fAf-module fAe are projective by [3, Theorem II.3.4]. Since fAf is self-injective , the right fAf-module $Af = fAf \oplus eAf$ is injective. Thus the corollary follows from 3.7 immediately. ∎

A very special case is that fAf is a symmetric algebra, that is, as an fAf-fAf-bimodules we have $fAf \simeq D(fAf)$. In this case we have

Corollary 3.9. *Under the assumption of* 3.7 *the following statement is true*: *If* fAf *is symmetric and* $\mathrm{Ext}^n_{fAf}(D(Af), \Omega^2(fA)) = 0$ *for all* $n \geq 1$, *then* $\mathrm{Ext}^n_{A^e}(A, J) \simeq \mathrm{Ext}^n_{A\otimes_k (fAf)^{op}}(D(fA), Af)$ *for all* $n \geq 1$, *where* Ω^2 *is the second syzygy operator.*

Proof. Since fAf is symmetric, we have that $\mathrm{DTr}_{fAf} = \Omega^2$ and $D \simeq \mathrm{Hom}_{fAf}(-, fAf)$. Thus the result follows. ∎

Now let us consider the Hochschild cohomology of a quasi-hereditary algebra. As is known, quasi-hereditary algebras were introduced in [6] and are a special kind of algebras of finite global dimension. They include many important algebras such as Temperley-Lieb algebras (see [16]) and Birman-Wenzl algebras (see [17]), and so on.

Recall that an algebra A is called quasi-hereditary if there is decomposition of $1 = e_1 + e_2 + \cdots + e_m$ into primitive orthogonal idempotents e_j such that each ideal $A(e_i + e_{i+1} + \cdots + e_m)A/A(e_{i+1} + \cdots + e_m)A$ is a heredity ideal in $A/A(e_{i+1} + \cdots + e_m)A$ for all i. Set $A_i = A/J_i$ with $J_i = A(e_i + e_{i+1} + \cdots + e_m)A$ and $J_{m+1} = 0$. So each A_i is an A-A-bimodule and the injective A_i-module $D(e_{i-1} + J_i)A_i$ is isomorphic with $D(e_{i-1}A_i)$. With the these notations we have the following result which is a direct consequence of 3.2.

Proposition 3.10. *Let* A *be a quasi-hereditary algebra. Then*

$$\sum_{i\geq 0}(-1)^i \dim_k H^i(A) = \sum_{i\geq 0}(-1)^i \dim_k H^i(A_j) + \sum_{j\leq s\leq m}\sum_{i\geq 0}(-1)^i \dim_k \operatorname{Ext}_A^i(D(e_sA_{s+1}), A_{s+1}e_s)$$

for all $1 \leq j \leq m$. ∎

As is well-known, a quasi-hereditary algebra A can be defined by standard modules $\Delta(i)$ and costandard modules $\nabla(j)$. Given an order of the idempotent elements e_j as above, the standard module $\Delta(i)$ is the maximal factor module of Ae_i with composition factors $S(j)$ such that $j \leq i$. Similarly, the costandard module $\nabla(i)$ is the maximal submodule of $D(e_iA)$ with composition factors $S(j)$ such that $j \leq i$. Thus the projective A_j-module A_je_{j-1} is just the standard module $\Delta(j)$ and the injective A_j-module $D(e_{j-1}A_j)$ is just the costandard module $\nabla(j)$. Note that $\Delta(m)$ is the projective module Ae_m and $\nabla(m)$ is the injective module $D(e_mA)$. The above proposition can be reformulated as follows:

Proposition 3.11. *Let A be a quasi-hereditary algebra. Then*

$$\sum_{i\geq 0}(-1)^i \dim_k H^i(A) = \sum_{i\geq 0}(-1)^i \dim_k H^i(A_j) + \sum_{j\leq s\leq m}\sum_{i\geq 0}(-1)^i \dim_k \operatorname{Ext}_A^i(\nabla(s), \Delta(s))$$

for all $1 \leq j \leq m$. ∎

As a corollary we have

Corollary 3.12. *Let A be a algebra (over k) given by a connected quiver with relations. If there is no oriented cycles in the quiver, then*

$$\sum_{i\geq 0}(-1)^i \dim_k H^i(A) = 1 + \sum_{j=1}^{m}\sum_{i\geq 1}(-1)^i \dim_k \operatorname{Ext}_A^i(E(S(j)), S(j)),$$

where $S(j)$ stands for the simple module corresponding to the vertex j and $E(S(j))$ is the injective envelope of $S(j)$.

Proof. Since the quiver of the algebra has no oriented cycles, we can have an order on the simple modules such that all standard modules are just the simple modules. In this case the costandard modules are just the indecomposable injective modules. Thus the result follows from the previous proposition. ■

The cohomology groups $\mathrm{Ext}_A^i(\nabla(j), \Delta(j))$ of the costandard modules and standard modules play an important role in the calculation of the Hochschild cohomology of a quasi-hereditary algebra. In this direction we have the following result.

PROPOSITION 3.13. *Let A be a quasi-hereditary algebra with standard modules $\Delta(j)$ and costandard modules $\nabla(j)$, $1 \leq j \leq m$. Suppose that there is a duality on the the module category mod-A which fixes simple modules. If* proj.dim $\Delta(j) \leq 1$ *for all j, then* $\mathrm{Ext}_A^i(\nabla(j), \Delta(j)) = 0$ *for all $i \geq 3$.*

PROOF. Since proj.dim $\Delta(j) \leq 1$ and the duality fixes each simple module but interchanges $\Delta(j)$ and $\nabla(j)$, we know that proj.dim $T \leq 1$, inj.dim $\nabla(j) \leq 1$ and inj.dim $T \leq 1$, where T is the characteristic tilting module of A. Furthermore, by a result of Ringel in [15], there is an exact sequence

$$0 \to \Delta(j) \to T(j) \to X(j) \to 0$$

with $T = \bigoplus_{j=1}^m T(j)$. Hence $\mathrm{Ext}_A^1(T, X(j)) = 0$ for all j. This implies that $X(j) \in \mathrm{add}(T)$. Now applying $\mathrm{Hom}_A(\nabla(j), -)$ to the above exact sequence, we obtain a new exact sequence

$$\cdots \to \mathrm{Ext}_A^2(\nabla(j), X(j)) \to \mathrm{Ext}_A^3(\nabla(j), \Delta(j)) \to \mathrm{Ext}_A^3(\nabla(j), T(j)) \to \cdots.$$

This new exact sequence implies that $\mathrm{Ext}_A^i(\nabla(j), \Delta(j)) = 0$ for $i \geq 3$ and $1 \leq j \leq m$. ■

As an example of quasi-hereditary algebras satisfying all conditions in the proposition we mention the dual extension of a finite dimensional hereditary algebra (for the definition of dual extensions we refer to [18]).

Let us end this section by an example which illustrates how we can use the results of this section to compute the Hochschild cohomology. Before we do this, let us recall the following result in [10] which is needed sometimes for particular computation.

Let k be a perfect field and A an algebra over k. If $e_1, e_2, \ldots, e_m$ form a complete set of primitive orthogonal idempotents in A, then $e_i \otimes e_j^{op}$, $1 \leq i$, $j \leq m$, are a compete set of primitive orthogonal idempotents in A^e. Let $S(i)$ denote the simple top of $P(i) := Ae_i$ and $S(i,j)$ denote the simple top of A^e-module $P(i,j) := A^e \cdot (e_i \otimes e_j^{op})$. Observe that $S(i,j) \simeq \mathrm{Hom}_k(S(i), S(j))$.

Lemma 3.14 [10]. *Let* $\cdots \to R_n \to R_{n-1} \to \cdots R_1 \to R_0 \to A \to 0$ *be a minimal projective resolution of* A *over* A^e. *Then*

$$R_n = \bigoplus_{i,j} P(i,j)^{\dim_k \mathrm{Ext}_A^n(S(i),S(j))}.$$

In particular, the projective dimension of A^e*-module* A *equals the global dimension of* A.

Note that Lemma 3.14 is valid for algebras given by quivers with relations over any field.

Example 4. Suppose A is an algebra (over a field k) given by the quiver with relations:

$$\underset{1}{\circ} \underset{\alpha}{\overset{\alpha'}{\rightleftarrows}} \underset{2}{\circ} \underset{\beta}{\overset{\beta'}{\rightleftarrows}} \underset{3}{\circ}, \quad \alpha\alpha' = \beta\beta' = 0.$$

This algebra is the dual extension of the hereditary algebra of the linear quiver of $\mathbf{A}_3$ and has global dimension 2. We denote by J the ideal of A generated by the idempotent e_3 corresponding to the vertex 3. An easy calculation shows that the center $Z(A)$ of A is of dimension 3, $Z(A) \cap J$ is 1-dimensional and that $Z(B)$ is of dimension 2. Thus it follows from 3.2(c) that the sequence

$$\begin{aligned} 0 &\to \mathrm{Ext}_A^1(\nabla(3), \Delta(3)) \to H^1(A) \to H^1(B) \to \mathrm{Ext}_A^2(\nabla(3), \Delta(3)) \\ &\to H^2(A) \to H^2(B) \to 0 \end{aligned}$$

is exact.

Note that $\dim_k H^0(B) = \dim_k \mathrm{Ext}_A^i(\nabla(3), \Delta(3)) = 2$ and $\dim_k H^i(B) = 1$ for $i = 1, 2$. By 3.11, we see that $\dim_k H^1(A) = \dim_k H^2(A)$. On the other hand, a simple calculation using the sequence (3.14), shows that $\dim_k H^1(A) = 3$.

Similarly, we can calculate the Hochschild cohomology of the following quasi-hereditary algebra A given by

$$\underset{1}{\circ} \underset{\alpha}{\overset{\alpha'}{\rightleftarrows}} \underset{2}{\circ} \underset{\beta}{\overset{\beta'}{\rightleftarrows}} \underset{3}{\circ}, \quad \begin{array}{l} \alpha\alpha' = \beta'\beta,\ \beta\beta' = 0, \\ \beta\alpha = \alpha'\beta' = 0. \end{array}$$

Here we have that $\dim_k H^0(A) = 3$ and $\dim_k H^i(A) = 1$ for $1 \le i \le 4$. (Note that the global dimension of this algebra is 4.)

4. Applications

In this section we apply our results to calculate the Hochschild cohomology of the Temperley-Lieb algebras. Our method can be used to determine the Hochschild cohomology of the partition algebras in [13].

Let k be a field and n an integer. Recall that the Temperley-Lieb algebra $A_n(\delta)$ for $\delta \in k$ is defined to be a k-algebra with identity generated by $t_1, t_2, \dots, t_{n-1}$ subject to the relations:

$$(1) \quad t_i t_j t_i = t_i \quad \textit{if} \;\; |j-i| = 1,$$

$$(2) \quad t_i t_j = t_j t_i \quad \textit{if} \;\; |j-i| > 1,$$

$$(3) \quad t_i^2 = \delta t_i \quad \textit{for} \;\; 1 \le i \le n-1.$$

It was proved in [16] (see also [12]) that a block of a non-semisimple Temperley-Lieb algebra is Morita equivalent to the algebra A_m given by the following quiver with relations:

$$\begin{tikzcd} \underset{1}{\circ} \arrow[r, shift left, "\beta_1"] & \underset{2}{\circ} \arrow[l, shift left, "\alpha_1"] \arrow[r, shift left, "\beta_2"] & \underset{3}{\circ} \arrow[l, shift left, "\alpha_2"] \cdots \underset{m-1}{\circ} \arrow[r, shift left, "\beta_{m-1}"] & \underset{m}{\circ} \arrow[l, shift left, "\alpha_{m-1}"] \end{tikzcd} \qquad \begin{aligned} &\alpha_{i+1}\alpha_i = \beta_{i-1}\beta_i = 0, \\ &\beta_{i+1}\alpha_{i+1} = \alpha_i\beta_i, \\ &\alpha_{m-1}\beta_{m-1} = 0. \end{aligned}$$

As was proved in [19], the non-trivial block of the representation-finite q-Schur algebra $S_q(m,r)$ with $m \ge r$ is Morita equivalent to an algebra of the form A_n. For the definition of q-Schur algebras we refer to [7]. Hence, to get the Hochschild cohomology of these algebras, it is sufficient to calculate the Hochschild cohomology for the algebra A_n, and this will be done in the following.

Proposition 4.1. *Let k be any field and A_n the k-algebra defined as above. Then*

$$\dim_k H^i(A_n) = \begin{cases} n & i = 0; \\ 1 & 1 \le i \le 2n-2; \\ 0 & i \ge 2n-1. \end{cases}$$

Proof. We show this by induction on n. For $n = 1$, the algebra A_1 is a simple algebra and the proposition is trivially true. For $n = 2$ or 3, the proposition follows from Example 3 and Example 4 in the previous sections. Suppose now that the proposition holds for $n-1$ with $n \ge 4$. Let J be the ideal in A_n generated by the idempotent e_n corresponding to the vertex n. Then J is a heredity ideal such that $A_n/J \simeq A_{n-1}$. We may use the following minimal projective resolution of $\nabla(n)$ to compute $\mathrm{Ext}^j_A(\nabla(n), \Delta(n))$:

$$0 \longrightarrow Q_{2n-2} \xrightarrow{d_{2n-2}} Q_{2n-3} \xrightarrow{d_{2n-3}} \cdots \longrightarrow Q_{n-1} \longrightarrow Q_{n-2}$$

$$\longrightarrow \cdots \longrightarrow Q_1 \xrightarrow{d_1} Q_0 \xrightarrow{d_0} \nabla(n) \longrightarrow 0,$$

where $Q_i = P(n-i-1)$ for $0 \le i \le n-1$, and $Q_j = P(j-n+2)$ for $n-1 \le j \le 2n-2$. (Here $P(j)$ stands for the indecomposable projective A_n-module corresponding to the vertex j.) In fact, the kernel of d_i is $\nabla(n-i)$, if $1 \le i \le n-1$; and $\Delta(i-n+2)$, if $n-1 \le i \le 2n-2$. Thus we have for $i \ge 1$

$$\dim_k \mathrm{Ext}^i_{A_n}(\nabla(n), \Delta(n)) = \begin{cases} 1 & i \in \{2n-3, 2n-2\}; \\ 0 & \text{otherwise.} \end{cases}$$

Note that gl.dim $A_n = 2n-2$. Thus the proposition follows directly from 3.11 and induction. ■

Since Hochschild cohomology of algebras is Morita-invariant by [10, Theorem 4.2], the above proposition describes also the Hochschild cohomology of both the Temperley-Lieb algebras and the representation-finite q-Schur algebras $S_q(n,r)$ for $n \ge r$ and $r < 2p$, where p is the characteristic of the field.

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