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SOME REMARKS ON ORDINARY ABELIAN VARIETIES

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Let k be an algebraically closed field of positive characteristic p. Let X be an abelian variety over k, of dimension g. If p^{σ} $(0\leqq\sigma\leqq g)$ is the separable degree of the isogeny p_X: $X\longrightarrow X$ defined by $x\longmapsto px$, then σ is called the p-rank of X. Suppose p-rank of X is equal to g; such an abelian variety is said to be *ordinary*. Let L be an ample invertible sheaf on X and P the Poincaré invertible sheaf on the product $X\times\hat{X}$ of X and the dual abelian variety $\hat{X}$ of X. For any closed point α of $\hat{X}$, we put $P|_{X\times\{\alpha\}}=P_{\alpha}$. In the present paper we have two main purposes, one of which is to prove Theorem 3.3, which asserts:

Assume $p>2$. *Then the canonical map*

$$\sum_{\alpha\in(\hat{X}_p)(k)}\Gamma(L\otimes P_{\alpha+\beta})\otimes\Gamma(L^{p-1}\otimes P_{-\alpha+\gamma})\longrightarrow\Gamma(L^{p}\otimes P_{\beta+\gamma})$$

is surjective for any closed points β *and* γ *of* $\hat{X}$, *where* $(\hat{X}_p)(k)$ *denotes the group of the closed points of* $\hat{X}_p=\ker(p_{\hat{X}})$.

The other is to give a simple proof of Theorem 4.1, which was first proved by T. Sekiguchi [12], using the lifting theory of abelian varieties. The theorem asserts:

Assume $p=2$. *Then for any closed points* β *and* γ, *there exists a non-empty open subset* U *of* $\hat{X}$ *such that the canonical map*

$$\Gamma(L^{2}\otimes P_{\alpha+\beta})\otimes\Gamma(L^{2}\otimes P_{-\alpha+\gamma})\longrightarrow\Gamma(L^{4}\otimes P_{\beta+\gamma})$$

is surjective for all closed points α *of* U.

By virtue of these theorems, we see that the following theorem ([11], Main Theorem) holds as long as X is ordinary, even if char $k=p$ is 2 or 3.

If Φ_{L^3}: $X\longrightarrow\boldsymbol{P}(\Gamma(L^{3}))$ *is the canonical embedding of* X *into the projective space, then the image variety is ideal-theoretically an intersection of cubics.*

(*)H. Morikawa [3] showed this to be the case for generic abelian varieties of any characteristic. In view of a result of P. Norman and F. Oort [7] to the effect that generic abelian varieties are ordinary, we get another proof of this statement.

In §1, we recall some fundamental facts from the theory of theta functions in

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* This is noticed by the referee.

abstract geometry, which were established by D. Mumford ([4], [5], [6]). In §2, we shall give a fundamental theta relation from which every theta relation results as its special case. The following two sections 3 and 4 are devouted to proving our main results.

The author had conversation with Dr. T. Sekiguchi during the preparation of this paper and obtained some useful suggestions from them, and thanks him for asking a question concerning abelian varieties in characteristic p which led the author to prove the results of this paper.

Notation and Terminology.

Throughout this paper we fix an algebraically closed field k and we are concerned with schemes over k. For a group scheme G over k and a scheme S over k (resp. k-algebra R), we denote by $G(S)$ (resp. $G(R)$) the group of the S-valued (resp. R-valued) points of G. Since no confusion occurs in this paper, we treat S-valued points of G as if they were k-valued points. For a finite group scheme F, F_{red} (resp. F_{loc}) denotes the maximal reduced subgroup scheme (resp. the identity component) of F and $\hat{F}$ is the dual of F. If X is an abelian variety, then $\hat{X}$ denotes the dual abelian variety of X. For an integer n, we denote by n_X the isogeny defined by $x \longmapsto nx$ and by X_n the kernel of n_X. Let L be an invertible sheaf on X. Then we define the homomorphism ϕ_L: $X \longrightarrow \hat{X}$ by $x \longmapsto T_x{}^*L \otimes L^{-1}$ and denote the kernel of ϕ_L by $K(L)$. The Euler-Poincaré characteristic $\chi(X, L)$ is the square root of the rank of $K(L)$. P denotes the Poincaré invertible sheaf on $X \times \hat{X}$ and $P_\alpha (\alpha \in \hat{X})$ is the pull-back of P via the inclusion $X \times \{\alpha\} \longrightarrow X \times \hat{X}$.

§1. Preliminaries.

Let X be an abelian variety over k and L an ample invertible sheaf on X. Then we denote by $G(L)$ the group scheme over k defined as follows:
For every k-scheme S, the S-valued points of $G(L)$ are functorially isomorphic to the group of the pairs (x, ϕ), where x is an S-valued point of X, and

$$\phi: \ L \otimes O_S \longrightarrow \mathrm{T}_x{}^*(L \otimes O_S)$$

is an isomorphism, where T_x: $X \times S \longrightarrow X \times S$ is the translation by x. It then fits into an exact sequence of group schemes:

$$1 \longrightarrow \boldsymbol{G}_m \longrightarrow G(L) \xrightarrow{j_L} \mathrm{K}(L) \longrightarrow 1\,,$$

where j_L is the canonical surjection. The canonical representation U of $G(L)$ on $\Gamma(X, L) = \Gamma(L)$ is given as follows:

$$U_{(x,\phi)}: \ \Gamma(L \otimes O_S) \xrightarrow{T_x{}^*} \Gamma(T_x{}^*(L \otimes O_S)) \xrightarrow{\Gamma(\phi)^{-1}} \Gamma(L \otimes O_S)$$

for an S-valued point (x, ϕ) of $G(L)$.

For any two S-valued points a, b of $G(L)$, $aba^{-1}b^{-1}$ is an S-valued point of $\boldsymbol{G}_m$ and it depends only on $j_L(a)$ and $j_L(b)$. Define $\boldsymbol{e}^L$: $K(L)\times K(L)\longrightarrow \boldsymbol{G}_m$ by $\boldsymbol{e}^L(j_L(a), j_L(b))=aba^{-1}b^{-1}$. Then $\boldsymbol{e}^L$ is skew-symmetric and bi-multiplicative. By the definition of $\boldsymbol{e}^L$, we see that a subgroup H of $K(L)$ satisfies the property $\boldsymbol{e}^L|_{H\times H}\equiv 1$ if and only if there exists a group section ρ: $H\longrightarrow G(L)$ of j_L. Such ρ is called a *lifter* of H and the image $\rho(H)$ is called a *level* subgroup of $G(L)$ over H.

Now we can state the fundamental theorem of Mumford's theory, which asserts:

THEOREM 1.1. (*Theta structure theorem*). *Let L be an ample invertible sheaf on an abelian variety X over k. Then $\Gamma(L)$ is an irreducible $G(L)$-module under the action U. Moreover, it is the unique irreducible representation of $G(L)$ for which $\boldsymbol{G}_m$ acts by its natural character. If H is a maximal subgroup of $K(L)$ such that $\boldsymbol{e}^L|_{H\times H}\equiv 1$, then*

$$\dim_k(\Gamma(L)^{H^*})=1$$

where H^* is a level subgroup of $G(L)$ over H.

(cf. [4], § 1 Th. 2 and [10], Appendix)

Such a subgroup H of $K(L)$ as in Theorem 1.1 is said to be *maximal isotropic*.

In terms of theta groups, the descent theory of invertible sheaves on abelian varieties is translated as follows:

THEOREM 1.2. *Let π: $X\longrightarrow Y$ be an isogeny of abelian varieties and L an ample invertible sheaf on X. Then there is a natural one-to-one correspondence between the sets of*

(a) *isomorphism classes of invertible sheaves M on Y such that $\pi^*M\simeq L$, and*

(b) *homomorphisms α: $\ker\pi\longrightarrow G(L)$ lifting the inclusion $\ker\pi\longrightarrow X$.*

(cf. [6] § 23 Th. 2)

For such M as in (a), the corresponding hommorphism α is called *the descent data on L for π associated with the descended sheaf M.*

THEOREM 1.3. *Let π: $X\longrightarrow Y$ be an isogeny of abelian varieties and L and M ample invertible sheaves on X and Y, respectively, such that $\pi^*M\simeq L$. Let α be the descent data on L associated with M. Then*

(1) $\pi^{-1}(K(M))\subset K(L)$,

(2) *the centralizer G^* of $\alpha(\ker\pi)$ in $G(L)$ is $j^{-1}[\pi^{-1}(K(L))]$, where j: $G(L)\longrightarrow K(L)$ is the natural projection,*

(3) $G(M)\simeq G^*/\alpha(\ker\pi)$, *canonically.*

(cf. [6], § 23)

As for the product of theta groups, we have the following which is proved in the same method as in [4, § 3 Lemma 1].

PROPOSITION 1.4. *Let L and M be ample invertible sheaves on abelian varieties X and Y, respectively. Let* $i_L: \boldsymbol{G}_m \longrightarrow G(L)$ *and* $i_M: \boldsymbol{G}_m \longrightarrow G(M)$ *be the canonical inclusions. Then we have an exact sequence:*

$$1 \longrightarrow \boldsymbol{G}_m \xrightarrow{(i_L,\ i_M^{-1})} G(L)\times G(M) \longrightarrow G(p_X^*L\otimes p_Y^*M) \longrightarrow 1$$

where p_X *and* p_Y *are projections from* $X\times Y$ *to X and Y, respectively.*

We here recall an addition formula of sheaves.

PROPOSITION 1.5 *Let L be a symmetric* (i.e., $(-1_X)^*L\simeq L$) *invertible sheaf on an abelian variety X. Let a and b be integers. If we define an isogeny* ξ: $X\times X\longrightarrow X\times X$ *by* $(x, y)\longmapsto(x-by, x+ay)$, *then we have*

$$\xi^*(p_1^*(L^a\otimes P_\alpha)\otimes p_2^*(L^b\otimes P_\beta))$$
$$\simeq p_1^*(L^{a+b}\otimes P_{\alpha+\beta})\otimes p_2^*(L^{ab(a+b)}\otimes P_{a\beta-b\alpha})$$

for any α, $\beta\in X(k)$, *where* p_i *is the i-th projection of* $X\times X$.

(cf. [9], Lemma 4.1)

Before bringing this section to an end, we give the following lemma, which is the positive characteristic version of Lemma 2.1 in [2].

LEMMA 1.6. *Let K be a field of positive characteristic p and let E(g) be an elementary abelian p-group of rank g. Suppose* $\{T(a)|a\in E(g)\}$ *is a set of independent variable over K. If we define a* $p^g\times p^g$ *-matrix M by*

$$M=[T(a-b)]_{(a,b)\in E(g)\times E(g)},$$

then we have

$$\det M=(\sum_{a\in E(g)} T(a))^{p^g}.$$

PROOF. If we regard M as the matrix with the coefficients in $Z[\zeta][\cdots, T(a), \cdots]$, where ζ is a primitive p-th root of unity in the field of the complex numbers, then we can easily see that

$$\det M=\prod_{\chi\in E(g)^*}(\sum_{a\in E(g)}\chi(a)T(a))$$

in which $E(g)^*$ is the dual group of $E(g)$ (cf. loc. cit.). Since we can specialize ζ to 1 over the canonical homomorphism $\boldsymbol{Z}\longrightarrow\boldsymbol{Z}/(p)$, we have

$$\det M \text{ (in the ring } K[\cdots, T(a), \cdots])=(\sum T(a))^{p^g}$$

Q.E.D.

§ 2. A fundamental formula of theta relations

We start this section with the following definition.

DEFINITION 2.1. Let L be an ample invertible sheaf on an abelian variety X. Let H_1 and H_2 be two subgroup schemes of $K(L)$ such that $K(L)=H_1 \oplus H_2$. This decomposition is called a Göpel decomposition of $K(L)$ with respect to $\boldsymbol{e}^L$ if $\boldsymbol{e}^L|H_i \times H_i \equiv 1$ for $i=1,2$ and $H_2 \longrightarrow K(L) \xrightarrow{\gamma} (K(L))^\wedge \xrightarrow{\text{res.}} (H_1)^\wedge$ is an isomorphism where γ is the canonical isomorphism associated to $\boldsymbol{e}^L$ and res. is the restriction map.

It is well-known that if $K(L)$ is reduced, then it has always a Göpel decomposition. But, unfortunately, $K(L)$ need not have a Göpel decomposition unless $K(L)$ is reduced. Take, for instance, an ample invertible sheaf L of degree p on a supersingular elliptic curve.

The following argument in this section is a slight generalization of § 3 in our paper [9].

Suppose we are in the following situation:

$$(*) \qquad \begin{array}{ccc} X & \xrightarrow{\pi} & Y \\ L & \xleftarrow[\phi]{} & M. \end{array}$$

Here π is an isogeny of abelian varieties, L and M are ample invertible sheaves and ϕ is an isomorphism $\pi^*M \simeq L$. Then (π, ϕ) induces a linear map:

$$\pi^*: \ \Gamma(Y, M) \longrightarrow \Gamma(X, L).$$

Let d: $\ker \pi \longrightarrow G(L)$ be the descent data on L for π associated with M. Then we have, by Theorem 1.3, the exact sequence:

$$0 \longrightarrow \ker \pi \xrightarrow{d} G(L)^* \xrightarrow{T} G(M) \longrightarrow 1,$$

where $G(L)^*$ is the centralizer of $d(\ker \pi)$ in $G(L)$ and T is the canonical surjection. Assume that there exists a Göpel decomposition $K(L)=H_1(L) \oplus H_2(L)$ satisfying the following properties:

(i) $H_1(L)$ is reduced.

(ii) If we put $K_1 = \ker \pi \cap H_1(L)$ and $K_2 = \ker \pi \cap H_2(L)$, then $\ker \pi = K_1 \oplus K_2$.

We then have the following:

(iii) If we put $(K_i)^\perp = \ker [K(L) \xrightarrow{\gamma} (K(L))^\wedge \xrightarrow{\text{res.}} (K_i)^\wedge] \cap H_j(L)$ for $i, j=1,2$ and $i \neq j$, then we have

$$K(M) = \pi((K_2)^\perp) \oplus \pi((K_1)^\perp)$$

and this is a Göpel decomposition. Put $\pi((K_2)^{\perp})=H_1(M)$ and $\pi((K_1)^{\perp})=H_2(M)$; hence $H_1(M)$ is reduced.

LEMMA 2.2. *There exist lifters* $\rho_i(L): H_i(L) \longrightarrow G(L)$ *and* $\rho_i(M): H_i(M) \longrightarrow G(M)$ *for* $i=1,2$ *satisfying the following properties*:

(i) $\rho_i(L)=d$ *on* K_i *for* $i=1,2$ *and*

(ii) *the following diagram is commutative*:

$$\begin{array}{ccc} (K_j) & \xrightarrow{\rho_i(L)} & G(L)^* \subset G(L) \\ {\scriptstyle \pi'=\pi|_{(K_j)^{\perp}}} \downarrow & & \downarrow {\scriptstyle T} \\ H_i(M) & \xrightarrow[\rho_i(M)]{} & G(M) \end{array}$$

for $i,\ j=1,2$ *and* $i \neq j$.

PROOF. Since $\rho_i(L) \cdot d^{-1}: K_i \longrightarrow G(L)$ factors through $\boldsymbol{G}_m$, it defines an element a' of $(K_i)^{\wedge}(k)$. Let a be an element of $[H_i(L)]^{\wedge}(k)$ which induces a' in $(K_i)^{\wedge}(k)$. If a corresponds to the homomorphism $\chi: H_i(L) \longrightarrow \boldsymbol{G}_m$, then $\chi^{-1} \cdot \rho_i(L)$ is equal to d on K_i and it is also a lifter. As for (ii), we have the equalities $K_i=\ker \pi \cap (K_j)^{\perp} =\ker (T \circ \rho_i(L)) \cap (K_j)^{\perp}$. Since π' is surjective, there exists a lifter $\rho_i(M)$ such that it fits into the above diagram.

Since $H_2(L)$ (resp. $H_2(M)$) is a maximal isotropic subgroup of $K(L)$ (resp. $K(M)$), it follows that there is a unique (up to constant multiple) non-zero element $\theta(L)[0]$ (resp. $\theta(M)[0]$) of $\Gamma(X, L)$ (resp. $\Gamma(Y, M)$) invariant under the action of $\rho_2(L)[H_2(L)]$ (resp. $\rho_2(M)[H_2(M)]$). Moreover, if we put $U_{\rho_1(L)(a)}\theta(L)[0]=\theta(L)[a]$ for $a \in H_1(L)(k)$ and $U_{\rho_1(M)(a)}\theta(M)[0]=\theta(M)[a]$ for $a \in H_1(M)(k)$, then we see that $\{\theta(L)[a] | a \in H_1(L)(k)\}$ and $\{\theta(M)[a] | a \in H_1(M)(k)\}$ form the bases of $\Gamma(X, L)$ and $\Gamma(Y, M)$, respectively. Such bases are called the canonical bases defined by ρ_i'*s*.

Now we can state our fundamental theta relation. It plays the same role in this paper as Koizumi's basic formula (cf. [1], Th. 1.3) and Mumford's general formula (cf. [4], §1, Th. 4).

THEOREM 2.3. *Under the notation as above, there is a scalar* $\lambda \in k^*$ *such that for all* $a \in (K_2)^{\perp}(k)$,

$$\pi^*(\theta(M)[\pi a])=\lambda \cdot \sum_{p \in K_1(k)} \theta(L)[a+p].$$

Remark. By suitably choosing our bases, we can always assume $\lambda=1$. In the sequel, we will always assume that this has been done.

PROOF. First of all, we note that π^* is injective and that its image is the subspace $\Gamma(X, L)^{d(\ker\pi)}$ consisting of the elements of $\Gamma(X, L)$ invariant under the action of $d(\ker\pi)$. Moreover, $\Gamma(X, L)^{d(\ker\pi)}$ is a module over $G'=G(L)^*/d(\ker\pi)$. In fact, $\Gamma(X, L)^{d(\ker\pi)}$ as a G'-module is isomorphic to $\Gamma(Y, M)$ as $G(M)$-module. This means the commutativity of the diagram:

$$\begin{array}{ccc} G(L)^*\times\Gamma(X, L)^{d(\ker\pi)} & \longrightarrow & \Gamma(X, L)^{d(\ker\pi)} \\ \downarrow T\times(\pi^*)^{-1} & & \downarrow (\pi^*)^{-1} \\ G(M)\times\Gamma(Y, M) & \longrightarrow & \Gamma(Y, M) \end{array}$$

Let us first prove the assertion for $a=0$. We may put

$$\pi^*(\theta(M)[0])=\sum_{p\in H_1(L)(k)} c_p\cdot\theta(L)[p]$$

with $c_p\in k$. Then, for all k-algebras R and all R-valued points x of $(K_1)^\perp$, we get the following:

$$\begin{aligned} U_{\rho_2(L)(x)}(\pi^*\theta(M)[0]) &= \pi^*(U_{T\circ\rho_2(L)(x)}\theta(M)[0]) \\ &= \pi^*(U_{\rho_2(M)(\pi x)}\theta(M)[0]) \\ &= \pi^*\theta(M)[0] \end{aligned}$$

by the commutativity of the above diagram, and

$$\begin{aligned} U_{\rho_2(L)(x)}\Big(\sum_{p\in H_1(L)(k)} c_p\cdot\theta(L)[p]\Big) &= \sum c_p U_{\rho_2(L)(x)}\theta(L)[p] \\ &= \sum c_p(U_{\rho_2(L)(x)}(U_{\rho_1(L)(p)}\theta(L)[0])) \\ &= \sum c_p \boldsymbol{e}^L(x, p)(U_{\rho_1(L)(p)}(U_{\rho_2(L)(x)}\theta(L)[0])) \\ &= \sum c_p\cdot\boldsymbol{e}^L(x, p)\cdot\theta(L)[p]\,. \end{aligned}$$

Therefore if $c_p\neq 0$, then $\boldsymbol{e}^L(x, p)=1$ for all R-valued points x of $(K_1)^\perp$. Hence p is contained in $K_1(k)$ by the definition of $(K_1)^\perp$. Thus we have

$$\pi^*(\theta(M)[0])=\sum_{p\in K_1(k)} c_p\cdot\theta(L)[p].$$

Since $\pi^*(\theta(M)[0])$ is $d(\ker\pi)$-invariant and $d=\rho_1(L)$ on K_1, it follows that

$$\begin{aligned} \pi^*(\theta(M)[0]) &= U_{d(q)}\Big(\sum_{p\in K_1(k)} c_p\cdot\theta(L)[p]\Big) \\ &= \sum c_p(U_{\rho_1(L)}\theta(L)[p]) \\ &= \sum c_p\cdot\theta(L)[p+q] \end{aligned}$$

for all $q\in K_1(k)$. Therefore there is $\lambda\in k^*$ such that

$$\pi^*(\theta(M)[0])=\lambda\cdot\sum_{p\in K_1(k)}\theta(L)[p].$$

As for the assertion in general, apply $U_{\rho_1(L)(a)}$ with $a\in(K_2)^{\perp}(k)$ to this equation. We see that

$$\begin{aligned} U_{\rho_1(L)(a)}(\pi^*(\theta(M)[0])) &= \pi^*(U_{T\circ\rho_1(L)(a)}\theta(M)[0]) \\ &= \pi^*(U_{\rho_1(M)(\pi a)}\theta(M)[0]) \\ &= \pi^*(\theta(M)[\pi a]) \end{aligned}$$

and that

$$U_{\rho_1(L)(a)}(\lambda\cdot\sum_{p\in K_1(k)}\theta(L)[p])=\lambda\cdot\sum_{p\in K_1(k)}\theta(L)[p+a]\,.$$

Thus we have obtained our formula. *Q.E.D.*

§ 3. Main theorem I.

In this section, the characteristic of k is not equal to 2. For a section s of an invertible sheaf M on an abelian variety X, we denote by $s(x)$ ($x\in X(k)$) the image of s in the fiber $M(x)$. If we choose an isomorphism $M(x)\simeq k$, then we can consider $s(x)$ to be a scalar. In what follows, we will always assume that this has been done.

PROPOSITION 3.1. *Let X be an abelian variety over k, L a principal* (i.e., *ample and* $\chi(X, L)=1$) *invertible sheaf on X and $K(L^{(p-1)(p-2)p^3})=H_1\oplus H_2$, with H_1 reduced, a Göpel decomposition. Let ρ_i be a lifter of H_i to $G(L^{(p-1)(p-2)p^3})$ $(i=1,2)$ and $\{\theta(L^{(p-1)(p-2)p^3})[a]\mid a\in H_1(k)\}$ the basis of $\Gamma(L^{(p-1)(p-2)p^3})$ defined by ρ_i's. Then*

$$\sum_{a\in H_1(p)(k)}\theta(L^{(p-1)(p-2)p^3})[a+b](0)\neq 0$$

for some $b\in H_1(p-1)(k)$. Here $H_i(n)=((p-1)(p-2)p^3/n)H_i$ when $n\mid(p-1)(p-2)p^3$.

PROOF. For simplicity, we put $(p-1)(p-2)p^3=q$. Let L_0 be a symmetric ample invertible sheaf such that $L_0\otimes P_\alpha\simeq L$ for some $\alpha\in\hat{X}(k)$. Let $\zeta: X\times X\longrightarrow X\times X$ be the isogeny defined by $(x, y)\longmapsto(x-py, x+p(p-2)y)$. If we put $\beta=(p-1)p^2\alpha$, then we have

$$\zeta^*[p_1^*(L_0^{(p-2)p})\otimes p_2^*(L_0^p\otimes P_\beta)]\simeq p_1^*(L_0^{(p-1)p}\otimes P_\beta)\otimes p_2^*(L^q).$$

As in § 2 if we put

$$\ker\zeta\cap\{H_i((p-1)p)\times H_i\}=K_i\quad(i=1,2)\,,$$

then we have

$$\zeta[(K_j)^{\perp}]=H_i((p-2)p)\times H_i(p)\quad(i\neq j)\,.$$

Moreover, it follows that

$$X_{(p-2)p}\times X_p=\{H_1((p-2)p)\times H_1(p)\}\oplus\{H_2((p-2)p)\times H_2(p)\}$$

is a Göpel decomposition with respect to e^M where $M=p_1^*(L_0^{(p-2)p})\otimes p_2^*(L_0^p\otimes P_\beta)$. We put $N=p_1^*(L_0^{(p-1)p}\otimes P_\beta)\otimes p_2^*L^q$. Let d: $\ker\zeta\longrightarrow G(N)$ be the descent data for ζ on N associated with M, and G^* the centralizer of $d(\ker\zeta)$ in $G(N)$. Then there exist lifters

$$\begin{aligned}
\rho_i' \quad &: H_i \longrightarrow G(L^q)\,,\\
\rho_i((p-1)p) &: H_i((p-1)p) \longrightarrow G(L_0^{(p-1)p}\otimes P_\beta)\,,\\
\rho_i(p) \quad &: H_i(p) \longrightarrow G(L_0^p\otimes P_\beta)
\end{aligned}$$

and

$$\rho_i((p-2)p): H_i((p-2)p) \longrightarrow G(L_0^{(p-2)p})$$

such that the following diagram is commutative:

$$\begin{array}{ccc}
\{H_i((p-1)p)\times H_i\}\cap\zeta^{-1}\{H_i((p-2)p)\times H_i(p)\} & \xrightarrow{\text{res. of } \phi\circ(\rho_i((p-1)p)\times\rho_i')} & G^* \\
\downarrow & & \downarrow \\
H_i((p-2)p)\times H_i(p) \xrightarrow[\rho_i((p-2)p)\times\rho_i(p)]{} G(L_0^{(p-2)p})\times G(L_0^p\otimes P_\beta) & \xrightarrow[\text{canonical}]{} & G(M)
\end{array}$$

where ϕ: $G(L_0^{(p-1)p}\otimes P_\beta)\times G(L^q)\longrightarrow G^*$ is the canonical homomorphism and that $\phi\circ\{\rho_i((p-1)p)\times\rho_i\}=d$ on K_i. Since $\ker\zeta\subset\{X_{p-1}\times X_{(p-1)p}\}$, we may assume that $\rho_i'=\rho_i$ on $H_i(p-2)$. Then we get $b_i\in H_i(p-1)$ $(i=1,2)$ and $a_1\in H_1(p^3)$ such that $\rho_1'=e^{L^q}(b_1,\ -)\cdot\rho_1$ and $\rho_2'=e^{L^q}(a_1+b_1,\ -)\cdot\rho_2$. Let $\{\theta(L^q)[a]\}$, $\{\theta(L_0^{(p-1)p}\otimes P_\beta)[a]\}$, $\{\theta(L_0^{(p-2)p})[a]\}$ and $\{\theta(L_0^p\otimes P_\beta)[a]\}$ be the bases defined by ρ_i', $\rho_i((p-1)p$, $\rho_i((p-2)p)$ and $\rho_i(p)$, respectively. Then we get, by Theorem 2.3,

$$\begin{aligned}
&\zeta^*\{\theta(L_0^{(p-1)p})[c-p(p-1)^2b]\otimes\theta(L_0^p\otimes P_\beta)[c-p(p-1)(p-2)b]\}\\
&=\sum_{a\in H_1((p-1)p)(k)}\theta(L_0^{(p-1)p}\otimes P_\beta)[c-(p-2)p^2b+pa]\otimes\theta'(L^q)[b+a]\\
&=\sum_{d\in H_1(p-1)(k)}\left(\sum_{a\in H_1(p)(K)}\theta'(L^q)[b+a+d]\right)\otimes\theta(L_0^{(p-1)p}\otimes P_\beta)[c-(p-2)p^2b+pd]
\end{aligned}$$

for any $(c-(p-2)p^2b,\ b)\in(K_2)^\perp(k)=\{(c-(p-2)P^2b,\ b)|c\in H_1(p)(k),\ b\in H_1(k)\}$. Let ι: $X\longrightarrow X\times X$ be the inclusion defined by $x\longmapsto(x,\ 0)$ and Δ: $X\longrightarrow X\times X$ be the diagonal morphism. Then $\zeta\circ\iota=\Delta$. Hence we have

$$\begin{aligned}
&\Delta^*\{\theta(L_0^{(p-1)p})[c-p(p-1)^2b]\otimes\theta(L_0^p\otimes P_\beta)[c-p(p-1)(p-2)b]\}\\
&=\sum_{d\in H_1(p-1)(k)}\left(\sum_{a\in H_1(p)(k)}\theta'(L^q)[b+a+d](0)\right)\times\theta(L_0^{(p-1)p}\otimes P_\beta)[c-(p-2)^2b+pd]\,.
\end{aligned}$$

The left hand side of this equation is not the zero section. Hence, for any $b\in H_1(k)$, there exists $d\in H_1(p-1)(k)$ such that

$$\sum_{a\in H_1(p)(k)} \theta'(L^q)[b+a+d](0)\neq 0 .$$

On the other hand, the relations between ρ_i' and ρ_i give the following theta relation:

$$\theta'(L^q)[b]=e^{L^q}(b_2, b)\cdot\theta(L^q)[b+a_1+b_1]$$

for any $b\in H_1(k)$. Therefore we have the following theta relation:

$$\sum_{a\in H_1(p)(k)} \theta(L^q)[a+b]=\sum_{a\in H_1(p)(k)} e^{L^q}(b_2, a_1+b_1-a-b)\theta'(L^q)[a+b-a_1-b_1]$$

$$=e^{L^{p-1}}(b_2, p^3(p-2)(b+b_1))\cdot\sum_{a\in H_1(p)(k)} \theta'(L^q)[a+b-a_1-b_1]$$

Hence the left hand side of this equation is not zero at the origin. *Q.E.D.*

COROLLARY 3.2. *Let X and L be as in Proposition* 3.1. *Let* $K(L^{(p-1)p})=H_1((p-1)p)\oplus H_2((p-1)p)$, *with* $H_1((p-1)p)$ *reduced, be a Göpel decomposition,* $\rho_i((p-1)p)$ *a lifter of* $H_i((p-1)p)$ *to* $G(L^{(p-1)p})$ $(i=1,2)$ *and* $\{\theta(L^{(p-1)p})[a]|a\in H_1((p-1)p)(k)\}$ *the basis of* $\Gamma(L^{(p-1)p})$ *defined by* $\rho_i((p-1)p)'s$. *Then*

$$\sum_{a\in H_1(p)(k)} \theta(L^{(p-1)p})[a+b](0)\neq 0$$

for some $b\in H_1(p-1)(k)$, *where* $H_1(p)=(p-1)H_1((p-1)p)$ *and* $H_1(p-1)=pH_1((p-1)p)$.

PROOF. Let $(p-1)(p-2)p^3=q$ and $K(L^q)=H_1\oplus H_2$ be a Göpel decomposition with H_1 reduced and $(p-2)p^2H_1=H_i((p-1)p)$. Let π: $\hat{X}\longrightarrow\hat{X}/\phi_L[H_1((p-2)p^2)]=\hat{Y}$ be the canonical surjection. Then we have an exact sequence:

$$0\longrightarrow K\longrightarrow X\overset{\pi}{\longrightarrow} Y\longrightarrow 0 ,$$

where K is a subgroup of Y isomorphic to $H_2((p-2)p^2)$, and we have an invertible sheaf M on Y such that $\chi(Y, M)=1$ and $\pi^*L\simeq M^{(p-2)p^2}$. Then there exists a Göpel decomposition $K(M^q)=H_1'\oplus H_2'$ with H_1' reduced, $\ker\pi=K=H_2((p-2)p^2)$ and $\pi[H_i'(p-1)]=H_i(p-1)$. Here $H_2'((p-2)p^2)$ denote $(p-1)pH_2'$ and so on. Moreover, we have $(K)^\perp=H_1'((p-1)p)$, $\pi[H_1'((p-1)p)]=H_1((p-1)p)$ and $\pi(H_2')=H_2((p-1)p)$. Let d': $K\longrightarrow G(M^q)$ be the descent data on M^q for π associated with $L^{(p-1)p}$ and G'^* the centralizer of $d'(K)$. Then there exist lifters ρ_i': $H_i'\longrightarrow G(M^q)$ and $\bar{\rho}_i((p-1)p)$: $H_i((p-1)p)\longrightarrow G(L^{(p-1)p})$ such that

$$\begin{array}{ccc} H_i' \cap \pi^{-1}[H_i((p-1)p] & \xrightarrow{\text{rest. of } \rho_i'} & G'^* \\ \downarrow & & \downarrow \\ H_i((p-1)p) & \xrightarrow[\bar{\rho}_i((p-1)p)]{} & G(L^{(p-1)p}) \end{array}$$

commutes $(i=1,2)$ and that $\rho_2'=d'$ on K. We may then assume that $\bar{\rho}_1((p-1)p)=\rho_1((p-1)p)$ and $\bar{\rho}_2((p-1)p)=e^{L^{(p-1)p}}(a_1, -)\cdot\rho_2((p-1)p)$ for some $a_1\in H_1((p-1)p)(k)$. As usual if we denote by $\{\bar{\theta}(L^{(p-1)p})[a]|a\in H_1((p-1)p)(k)\}$ the basis of $\Gamma(L^{(p-1)p})$ defined by $\bar{\rho}_i's$, then the relations between $\rho_i((p-1)p)$ and $\bar{\rho}_i((p-1)p)$ give the theta relations:

$$\bar{\theta}(L^{(p-1)p})[a]=\theta(L^{(p-1)p})[a+a_1]$$

for all $a\in H_1((p-1)p)(k)$. Let $\{\theta(M^q)[a]|a\in H_1'(k)\}$ be the basis of $\Gamma(M^q)$ defined by $\rho_i' s$. Then, by Theorem 2.3, we have

$$\pi^*(\bar{\theta}(L^{(p-1)p})[\pi a])|=\theta(M^q)[a]$$

for all $a\in(K)^{\perp}(k)=H_1'((p-1)p)(k)$. Hence we have

$$\pi^*(\sum_{a\in H_1(p)(k)}\theta(L^{(p-1)p})[a+\pi b])=\sum_{a\in H_1'(p)(k)}\theta(M^q)[a+b]$$

for all $b\in H_1'(p-1)(k)$. Thus we have our conclusion by Proposition 3.1. *Q.E.D.*

Now we shall prove our first main theorem:

THEOREM 3.3. *Let X be an ordinary abelian variety over an algebraically closed field k of characteristic $p>2$. Let L be an ample invertible sheaf on X. Then the canonical map*

$$\sum_{\gamma\in(\hat{X})_p(k)}\Gamma(L\otimes P_{\alpha+\gamma})\otimes\Gamma(L^{p-1}\otimes P_{\beta-\gamma})\longrightarrow\Gamma(L^p\otimes P_{\alpha+\beta})$$

is surjective for all α, $\beta\in\hat{X}(k)$.

PROOF. First of all, we may assume $\chi(X, L)=1$. Indeed let H be a maximal isotropic subgroup of $K(L)$ containing $K(L)_{\text{loc}}$. Since X is ordinary, it follows that $H\cap(X_p)_{\text{red}}=\{0\}$ and $\hat{H}$ is reduced. Let $\pi: X\longrightarrow X/H=Y$ be the canonical projection. Then there exists an ample invertible sheaf M on Y such that $\pi^*M\simeq L$ and $\chi(Y, M)=1$. We denote by H^* the kernel of $\hat{\pi}$. Since H^* is isomorphic to $\hat{H}$, H^* is reduced; hence we have, for any positive integer n and any $\alpha'\in(\hat{Y})(k)$,

$$\Gamma(L^n\otimes P_{\hat{\pi}(\alpha')})\simeq\sum_{\delta'\in H^*(k)}\Gamma(M^n\otimes P'_{\alpha'+\delta'})$$

where P' is the Poincaré invertible sheaf on $Y\times\hat{Y}$. Moreover, we have the following commutative diagram:

$$\begin{array}{ccc}
\sum_{\gamma\in(\hat{X})_p(k)} \Gamma(L\otimes P_{\alpha+\gamma})\otimes\Gamma(L^{p-1}\otimes P_{\beta-\gamma}) & \longrightarrow & \Gamma(L^p\otimes P_{\alpha+\beta}) \\
\uparrow & & \uparrow\wr \\
\sum_{\delta'\in H^*(k)}\left(\sum_{\gamma'\in(\hat{Y})_p(k)} \Gamma(M\otimes P'_{\alpha'+\gamma'+\delta'})\otimes\Gamma(M^{p-1}\otimes P'_{\beta'-\gamma'})\right) & \longrightarrow & \sum_{\delta'\in H^*(k)} \Gamma(M^p\otimes P'_{\alpha'+\beta'+\delta'})
\end{array}$$

where α', $\beta'\in\hat{Y}(k)$ and $\hat{\pi}(\alpha')=\alpha$, $\hat{\pi}(\beta')=\beta$. This diagram reduces the proof to the case $\chi(X, L)=1$.

Since there exists $\alpha\in\hat{X}(k)$ such that $L\otimes P_\alpha$ is symmetric, we may assume L to be symmetric. For simplicity, we prove the theorem when $\alpha=\beta=0$. If α or β is not zero, the same proof works. Now we shall prove the theorem when L is symmetric and $\chi(X, L)=1$ and $\alpha=\beta=0$. If we define an isogeny $\xi: X\times X\longrightarrow X\times X$ by $(x, y)\longmapsto(x-(p-1)y, x+y)$, then we have

$$\xi^*(p_1{}^*(L\otimes P_\alpha)\otimes p_2{}^*(L^{p-1}\otimes P_\beta)\simeq p_1{}^*(L^p\otimes P_{\alpha+\beta})\otimes p_2{}^*(L^{(p-1)p}\otimes P_{\beta-(p-1)\alpha})$$

for all α, $\beta\in\hat{X}(k)$. The kernel of ξ is the image of X_p via the homomorphism $X\longrightarrow X\times X$ define by $x\longmapsto((p-1)x, x)$. Since X is ordinary, $K(L^p)_{\mathrm{red}}$ and $K(L^p)_{\mathrm{loc}}$ are maximal isotropic subgroups of $K(L^p)$. If we denote $K(L^p)_{\mathrm{red}}$ and $K(L^p)_{\mathrm{loc}}$ by $H_1(p)$ and $H_2(p)$, respectively, then we see that $K(L^p)=H_1(p)\oplus H_2(p)$ is a Göpel decomposition. Let $K(L^{p-1})=H_1(p-1)\oplus H_2(p-1)$ be an arbitrary Göpel decomposition. Then we see that

$$K(M)=\{H_1(p)\times H_1((p-1)p)\}\oplus\{H_2(p)\times H_2((p-1)p\}$$

is a Göpel decomposition, where $M=p_1{}^*L^p\otimes p_2{}^*L^{(p-1)p}$ and $H_i((p-1)p)=H_i(p)\oplus H_i(p-1)$ $(i=1, 2)$. For $\gamma\in(\hat{X})_p(k)$, let d_γ: $\ker\xi\longrightarrow G(M)$ be the descent data on M for ξ associated with the descended sheaf $N_\gamma=p_1{}^*(L\otimes P_\gamma)\otimes p_2{}^*(L^{p-1}\otimes P_{-\gamma})$. Among the d_γ's, there exist the following relations.

LEMMA 3.4. (i) *If we put* $c=\phi_L{}^{-1}(\gamma)$, *then* $d_\gamma=e^M(-, (c, 0))\cdot d$ *where* $d=d_0$.
(ii) *The centralizer* G^* *of* $d_\gamma(\ker\xi)$ *in* $G(M)$ *does not depend on* γ.

PROOF. The second assertion immediately follows from the first. Let us now prove (i). If we abbreviate $p_1{}^*L\otimes p_2{}^*L^{p-1}=N$, then we have

$$\begin{aligned}
\xi^*(p_1{}^*P_\gamma\otimes p_2{}^*P_{-\gamma})&\simeq\xi^*[p_1{}^*(T_c{}^*L\otimes L^{-1})\otimes p_2{}^*(T_c{}^*L^{p-1}\otimes L^{1-p})]\\
&\simeq\xi^*[T^*_{(c,c)}(p_1{}^*L\otimes p_2{}^*L^{p-1})\otimes(p_1{}^*L\otimes p_2{}^*L^{p-1})^{-1}]\\
&\simeq\xi^*[T^*_{(c,c)}N\otimes N^{-1}]\\
&\simeq T^*_{(c,0)}(\xi^*N)\otimes\xi^*N^{-1}\\
&\simeq\mathrm{Hom}(\xi^*N,\ T^*_{(c,0)}\xi^*N)\,.
\end{aligned}$$

The last sheaf is trivial, so there exists a nowhere vanishing section which is unique up to scalars. This section naturally corresponds to the isomorphism ϕ: $\xi^*N \longrightarrow \xi^*N$ covering the translation $T_{(c,0)}$. For a k-algebra R, let x be an R-valued point of $\ker\xi$. Then $d(x)$: $\xi^*N \longrightarrow \xi^*N$ is the natural isomorphism covering T_x and the natural action of x on the sheaf $\mathrm{Hom}(\xi^*N, T^*_{(c,0)}\xi^*N)$ is defined by $\phi \longmapsto d(x)\circ\phi\circ d(x)^{-1}$. Since $d(x)\circ\phi\circ d(x)^{-1}\circ\phi^{-1} = \boldsymbol{e}^{\xi^*N}(x, (c, 0))$, the ratio of $d_\gamma(x)$ to $d(x)$ is equal to $\boldsymbol{e}^{\xi^*N}(x, (c, 0))$, and thus we have Lemma 3.4.

Now we put, for i, $j=1,2$ and $i\neq j$,

$$K_i = \ker\xi \cap \{H_i(p)\times H_i((p-1)p)\}$$

and

$$(K_i)^\perp = \ker[K(M) \xrightarrow{f} K(M)^\wedge \xrightarrow{g} (K_i)^\wedge] \cap \{H_j(p)\times H_j((p-1)p)\},$$

where f is the isomorphism induced by $\boldsymbol{e}^M$, and g is the restriction. Then we see that $\ker\xi = K_1 \oplus K_2$ and $\xi[(K_j)^\perp] = \{0\}\times H_i(p-1)$. Moreover, we have lifters ρ_i: $H_i(p)\times H_i((p-1)p) \longrightarrow G(M)$ for $i=1,2$ such that $\rho_i = d$ on K_i. If we put $\rho_{i,\gamma} = \boldsymbol{e}^M(-, (c_\gamma, 0))\cdot\rho_i$ with $c_\gamma = \phi_L{}^{-1}(\gamma)$, then we have, by the above lemma and the definition of d_γ, that $\rho_{i,\gamma} = d_\gamma$ on K_i. Since $\phi_L{}^{-1}[(\hat{X})_{p,\mathrm{red}}] = X_{p,\mathrm{red}} = H_1(p)$, it follows that $\rho_{1,\gamma} = \rho_1$ and $\rho_{2,\gamma}(a, b) = \boldsymbol{e}^{L^p}(a, c_\gamma)\cdot\rho_2(a, b)$ for all R-valued points (a, b) of $H_2(p)\times H_2((p-1)p)$. Therefore we have lifters $\rho_i(n)$ of $H_i(n)$ to $G(L^n)$ for $n=p$ and $(p-1)p$ $(i=1,2)$ which fit into the commutative diagram:

$$\begin{array}{ccc}
H_i(p)\times(H_i((p-1)p) & \xrightarrow{\rho_{i,\gamma}(p)\times\rho_i((p-1)p)} & G(L^p)\times G(L^{(p-1)p}) \\
& \searrow^{\rho_{i,\gamma}} & \downarrow \\
& & G(p_1{}^*L^p\otimes p_2{}^*L^{(p-1)p})
\end{array}$$

where $\rho_{1,\gamma}(p) = \rho_1(p)$ and $\rho_{2,\gamma}(p) = \boldsymbol{e}^{L^p}(-, c_\gamma)\cdot\rho_2(p)$. For $\gamma\in(\hat{X})_p(k)$, let $\rho_{i,\gamma}(p-1)$ be a lifter of $H_i(p-1)$ to $G(L^{p-1}\otimes P_{-\gamma})$ $(i=1,2)$ which fits into the following commutative diagram:

$$\begin{array}{ccc}
\{H_i(p)\times H_i((p-1)p)\}\cap\xi^{-1}[\{0\}\times H_i(p-1)] & \xrightarrow{\qquad\rho_{i,\gamma}\qquad} & G^* \\
\downarrow & & \downarrow \\
\{0\}\times H_i(p-1) \xrightarrow[\{0\}\times\rho_{i,\gamma}(p-1)]{} G(L\otimes P_\gamma)\times G(L^{p-1}\otimes P_{-\gamma}) & \xrightarrow[\text{canonical}]{} & G(N_\gamma).
\end{array}$$

Let $\{\theta(L^{(p-1)p})[a] | a\in H_1((p-1)p)(k)\}$ (resp. $\{\theta_\gamma(L^p)[a] | a\in H_1(p)(k)\}$, $\{\theta_\gamma(L^{p-1})[a] | a\in H_1(p-1)(k)\}$) be the basis of $\Gamma(L^{(p-1)p})$ (resp. $\Gamma(L^p)$, $\Gamma(L^{p-1}\otimes P_{-\gamma})$) defined by $\rho_i((p-1)p)$ (resp. $\rho_{i,\gamma}(p)$, $\rho_{i,\gamma}(p-1)$) and let $\theta_\gamma(L)$ be a non-zero element of $\Gamma(L\otimes P_\gamma)$, which is uniquely determined up to scalars. By the relations between $\rho_i(p)$ and $\rho_{i,\gamma}(p)$, we easily have the following theta relations:

$$\theta_\gamma(L^p)[a]=\theta(L^p)[a-\phi_L{}^{-1}(\gamma)]$$

for all $a\in H_1(p)(k)$ and $\gamma\in(\hat{X})_p(k)$, where $\{\theta(L^p)[a]|a\in H_1(p)(k)\}$ is the basis of $\Gamma(L^p)$ defined by $\rho_i(p)'s$. Therefore we have, by Theorem 2.3 and the above relations,

$$\begin{aligned}\xi^*(\theta_\gamma(L)\otimes\theta_\gamma(L^{p-1})[b]) &= \sum_{a\in H_1(p)(k)}\theta_\gamma(L^p)[a]\otimes\theta(L^{(p-1)p})[a+b]\\ &=\sum_{a\in H_1(p)(k)}\theta(L^p)[a-\phi_L{}^{-1}(\gamma)]\otimes\theta(L^{(p-1)p})[a+b]\end{aligned}$$

for all $b\in H_1(p-1)(k)$. Here note that $(K_2)^{\perp}(k)=\{(a, a+b)|a\in H_1(p)(k), b\in H_2(p-1)(k)\}$. If $\iota: X\longrightarrow X\times X$ is the inclusion defined defined by $x\longmapsto(x, 0)$, then $\xi\circ\iota=$the diagonal morphism $\varDelta$. Hence we have

$$\begin{aligned}\varDelta^*(\theta_\gamma(L)\otimes\theta_\gamma(L^{p-1})[b]) &=\iota^*\xi^*(\theta_\gamma(L)\otimes\theta_\gamma(L^{p-1})[b])\\ &=\sum_{a\in H_1(p)(k)}\theta(L^{(p-1)p})[a+b](0)\cdot\theta(L^p)[a-\phi_L{}^{-1}(\gamma)]\\ &=\sum\theta(L^{(p-1)p})[a+\phi_L{}^{-1}(\gamma)+b](0)\cdot\theta(L^p)[a].\end{aligned}$$

Thus we have the following relation:

$$\varDelta^*\begin{bmatrix}\vdots\\ \theta_\gamma(L)\otimes\theta_\gamma(L^{p-1})[b]\\ \vdots\end{bmatrix}_{\gamma\in(\hat{X})_p(k)}=\begin{bmatrix}\theta(L^{(p-1)p})[-a+\phi_L{}^{-1}(\gamma)+b](0)\end{bmatrix}_{(\gamma,a)\in[(\hat{X})_p\times H_1(p)](k)}\times\begin{bmatrix}\vdots\\ \theta(L^p)[-a]\\ \vdots\end{bmatrix}_{a\in H_1(p)(k)}.$$

By Lemma 1.6, we have

$$\det\left(\begin{bmatrix}\theta(L^{(p-1)p})[-a+a'+b](0)\end{bmatrix}_{(a,a')\in[H_1(p)\times H_1(p)](k)}\right)=\Big(\sum_{a\in H_1(p)(k)}\theta(L^{(p-1)p})[a+b](0)\Big)^{p^g},$$

where g is the dimension of X. This is not zero for some $b\in H_1(p-1)(k)$ by Corollary 3.2. Therefore we see that the canonical map in the theorem is surjective.

Q.E.D.

§ 4. Main theorem II.

Our task in this section is to prove the following theorem.

THEOREM 4.1. *Assume char $k=2$. Let X be an ordinary abelian variety k and let L be an ample invertible sheaf on X. Then for any α, $\beta\in\hat{X}(k)$ there exists a non-empty open subset U of X such that the canonical map*

$$\Gamma(L^2\otimes P_{\alpha+\gamma})\otimes\Gamma(L^2\otimes P_{\beta-\gamma}) \longrightarrow \Gamma(L^4\otimes P_{\alpha+\beta})$$

is surjective for all $\gamma\in U(k)$.

PROOF. As in the proof of Theorem 3.5, we may assume that L is symmetric and $\chi(X, L)=1$. Moreover, it is enough to prove the theorem in the case $\alpha=\beta=0$. If we define the isogeny $\xi: X\times X\longrightarrow X\times X$ by $(x, y)\longmapsto(x-y, x+y)$, then we have

$$\xi^*(p_1{}^*L^2\otimes p_2{}^*L^2)\simeq p_1{}^*L^4\otimes p_2{}^*L^4$$

and the kernel of ξ is the image of X_2 via the diagonal morphism $\Delta: X\longrightarrow X\times X$. If we denote $K(L)_{\mathrm{red}}$ (resp. $K(L^4)_{\mathrm{loc}}$) by $H_1(4)$ (resp. $H_2(4)$), then we see that $K(L^4)=X_4=H_1(4)\oplus H_2(4)$ is a Göpel decomposition. Let $H_i(2)$ be the image of $H_i(4)$ by 2_X. Then $H_1(2)=K(L^2)_{\mathrm{red}}$ and $H_2(2)=K(L^2)_{\mathrm{loc}}$. For simplicity, we put $p_1{}^*L\otimes p_2{}^*L=N$. Then one can easily see that $K(N^n)=\{H_1(n)\times H_2(n)\}\oplus\{H_1(n)\times H_2(n)\}$ $(n=2, 4)$ is a Göpel decomposition. If we put $\ker\xi\cap\{H_i(4)\times H_i(4)\}=K_i$ $(i=1, 2)$ and define $(K_i)^\perp$ as in §1, then we have $\xi[(K_j)^\perp]=H_i(2)\times H_i(2)$ $(i, j=1, 2$ and $i\neq j)$. By Lemma 2.2, we get lifters $\rho_i{}^{(j)}(n): H_i(n)\longrightarrow G(L^n)$ $(i, j=1, 2$ and $n=2, 4)$ satisfying the following properties:

(i) $\phi(4)\circ\{\rho_i{}^{(1)}(4)\times\rho_i{}^{(2)}(4)\}=d$ on K_i $(i=1, 2)$,

(ii) the following diagram is commutative:

$$\begin{array}{ccccc}
\{H_i(4)\times H_i(4)\}\cap\xi^{-1}[H_i(2)\times H_i(2)] & & \xrightarrow{\ \text{res.}\ } & & G^* \\
\downarrow & & & & \downarrow \\
H_i(2)\times H_i(2) & \xrightarrow[\rho_i{}^{(1)}(2)\times\rho_i{}^{(2)}(2)]{} & G(L^2)\times G(L^2) & \xrightarrow[\phi(2)]{} & G(N^2)
\end{array}$$

Here $\phi(n): G(L^n)\times G(L^n)\longrightarrow G(N^n)$ $(n=2, 4)$ is the canonical homomorphism, d: $\ker\xi\longrightarrow G(N^4)$ is the descent data on N^4 for ξ associated with N^2, G^* is the centralizer of $d(\ker\xi)$ in $G(N^4)$ and res. is the restriction of $\phi(4)_\circ\{\rho_i{}^{(1)}(4)\times\rho_i{}^{(2)}(4)\}$. Since no confusion occurs, we identify $\rho_i{}^{(1)}(n)$ with $\rho_i{}^{(2)}(n)$ and denote them by $\rho_i(n)$. For $n=2, 4$, let $\{\theta_n[a]|a\in H_1(n)(k)\}$ be the basis of $\Gamma(L^n)$ defined by $\rho_i(n)$. Then, by Theorem 2.3, we have

$$\xi^*(\theta_2[b]\otimes\theta_2[2a+b])=\sum_{q\in H_1(2)(k)}\theta_4[a+q]\otimes\theta_4[a+b+q]$$

for all $(a, a+b)\in(K_2)^\perp(k)=\{(a, a+b)|a\in H_1(4)(k),\ b\in H_1(2)(k)\}$. Let $x_0\in X(k)$. Then we get the following commutative diagram:

$$\begin{array}{ccc}
X\simeq\{x_0\}\times X & \xrightarrow{\ \iota\ } & X\times X \\
 & \underset{\delta}{\searrow} & \downarrow\xi \\
 & & X\times X
\end{array}$$

where $\delta(x)=(x_0-x,\ x_0+x)$ and ι is the inclusion. Hence we have

$$\iota^*\xi^*(\theta_2[b]\otimes\theta_2[2a+b])=\sum_{q\in H_1(2)(k)}\theta_4[a+q](x_0)\cdot\theta_4[a+b+q]\,.$$

Since $\delta=((-1_x)\times 1_x)\circ(T_{-x_0}\times T_{x_0})\circ\Delta$, we get

$$\delta^*(\theta_2[b]\otimes\theta_2[2a+b])=\Delta^*[(T^*_{-x_0}\theta_2'[b])\otimes(T^*_{x_0}\theta_2[2a+b])]$$

where " ′ " is the image of the automorphism ε of $\Gamma(L^2)$ induced by -1_X. If we put $\alpha=\phi_L(x_0)$, then we have the following commutative diagram:

$$\begin{array}{ccc}
\Gamma(L^2\otimes P_{-\alpha})\otimes\Gamma(L^2\otimes P_\alpha) & \longrightarrow & \Gamma(L^4) \\
{\scriptstyle T^*_{-x_0}\times T^*_{x_0}}\downarrow\wr & & \uparrow \\
\Gamma(L^2)\otimes\Gamma(L^2) & & \Big| \\
{\scriptstyle \varepsilon\times id}\downarrow\wr & & \Big| \\
\Gamma(L^2)\otimes\Gamma(L^2) & \underset{\xi^*}{\longrightarrow} & \Gamma(L^4)\otimes\Gamma(L^4)
\end{array}$$

For each $a\in H_1(4)(k)$, we have

$$\iota^*\xi^*\left(\begin{bmatrix}\vdots\\ \theta_2[2]\otimes\theta_2[2a+b]\\ \vdots\end{bmatrix}_{b\in H_1(2)(k)}\right)$$

$$=\begin{bmatrix}\theta_4[a+b+q](x_0)\end{bmatrix}_{(b,\ q)\in[H_1(2)\times H_1(2)](k)}\times\begin{bmatrix}\vdots\\ \theta_4[a+q]\\ \vdots\end{bmatrix}_{q\in H_1(2)(k)}$$

By Lemma 1.6, we get

$$\det\left(\begin{bmatrix}\theta_4[a+b+q](x_0)\end{bmatrix}_{(b,q)}\right)=[\sum_{q\in H_1(2)(k)}\theta_4[a+q](x_0)]^{2^g}\,.$$

Since $\sum_q\theta_4[a+q]$ is a non-zero section for all $a\in H_1(4)(k)$, there exists a non-empty subset V of X such that for all $a\in H_1(4)(k)$ and all $x\in V(k)$,

$$\sum_{q\in H_1(2)(k)}\theta_4[a+q](x)\neq 0\,.$$

Then we see that $\phi_L(V)$ is a required open subset of X. *Q.E.D.*

REMARK 4.2. It is easily seen that Theorem 4.1 holds without the assumption "ordinary" in the case of elliptic curves.

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