TSUKUBA J. MATH.
Vol. 3, No. 1 (1979), 1–29

ON CONJUGATE LOCI AND CUT LOCI OF COMPACT SYMMETRIC SPACES II

By
Masaru TAKEUCHI

Introduction. This is a continuation of Part I, appeared in this journal, **2** (1977), 35–68.

In the present paper, we shall first study the topological structures of the cut locus and the conjugate locus of a compact symmetric space. We shall show that the cut locus C is the disjoint union of finite cell bundles E^{λ} over compact manifolds B^{λ}. Here each E^{λ} is the interior of a disk bundle $\bar{E}^{\lambda}$ over B^{λ}, and C is obtained by the successive attachments of E^{λ} at boundaries ∂E^{λ} of $\bar{E}^{\lambda}$. The conjugate locus can be also stratified in a similar way.

We shall next give another stratification of the cut locus of an irreducible symmetric R-space $M=G/U$ by means of orbits of a certain subgroup $\bar{U}$ of G. We shall prove the following facts: M has finite $\bar{U}$-orbits, say $V_0, V_1, \cdots, V_r$; There exists a unique open $\bar{U}$-orbit, say V_0, among them; Then the cut locus is the union of $V_1, \cdots, V_r$; Each V_l is described by means of generalized Schubert cells of M, and it has the structure of a vector bundle over a symmetric R-space B_l. These results include those on cut loci of Grassmann manifolds, $U(n)/O(n)$, $U(n)$, $SO(n)$ and $U(2n)/Sp(n)$ by Wong [12], [13], Sakai [5], [6].

We retain the definitions and notations in Part I.

§ 4. Topological structures

In this section, we shall study the topology of conjugate loci and cut loci of compact symmetric spaces. Compare § 1 in Part I for the notation.

Let Δ be an admissible subset of $\Pi^{\sharp}$. The boundary $\bar{S}^{\Delta}-S^{\Delta}$ of S^{Δ} will be denoted by ∂S^{Δ}. By Lemma 1.4, 3), we have decompositions:

$$\bar{S}^{\Delta}=\bigcup_{\Delta'\subset\Delta} S^{\Delta'}, \quad \partial S^{\Delta}=\bigcup_{\Delta'\subsetneqq\Delta} S^{\Delta'}.$$

Each element $\tau\in\bar{N}^{\Delta}$ leaves $\bar{S}^{\Delta}$ and ∂S^{Δ} invariant, and each element $\tau\in\bar{Z}^{\Delta}$ fixes any point of $\bar{S}^{\Delta}$. Therefore $\bar{W}^{\Delta}=\bar{N}^{\Delta}/\bar{Z}^{\Delta}$ acts on $\bar{S}^{\Delta}$ and ∂S^{Δ} continuously. Thus $\bar{W}^{\Delta}$ acts on $K/Z^{\Delta}\times\bar{S}^{\Delta}$ and $K/Z^{\Delta}\times\partial S^{\Delta}$ continuously in the same way as for $K/Z^{\Delta}\times S^{\Delta}$.

Received April 7, 1978

The quotient spaces relative to these actions are denoted by

$$\bar{E}^{\Delta}=K/Z^{\Delta}\times_{\overline{W}^{\Delta}}\bar{S}^{\Delta}\,,$$

$$\partial E^{\Delta}=K/Z^{\Delta}\times_{\overline{W}^{\Delta}}\partial S^{\Delta}\,,$$

respectively. We identify in the natural manner E^{Δ} and ∂E^{Δ} with an open subset and a closed subset of $\bar{E}^{\Delta}$ respectively.

Let

$$M_0=\mathfrak{c}_{\mathfrak{m}}/\Gamma_0\,.$$

It is a torus of dimension r_0. Let $\pi_0:\mathfrak{c}_{\mathfrak{m}}\rightarrow M_0$ denote the natural projection. We define a C^{∞} action of W^{Δ} on $M_0\times S'^{\Delta}$ as follows. Regarding Γ_0 as a subgroup of $\overline{W}^{\Delta}$ by ι^{Δ}, we may identify $M_0\times S'^{\Delta}$ with the quotient manifold $\Gamma_0\backslash S^{\Delta}$, since $S^{\Delta}=\mathfrak{c}_{\mathfrak{m}}\times S'^{\Delta}$ and since Γ_0 acts trivially on the second factor S'^{Δ}. Now $\overline{W}^{\Delta}$ acts on $\Gamma_0\backslash S^{\Delta}$ in the natural way, since Γ_0 is a normal subgroup of $\overline{W}^{\Delta}$. But Γ_0 acts trivially on $\Gamma_0\backslash S^{\Delta}$, and hence the quotient group $W^{\Delta}\cong\overline{W}^{\Delta}/\Gamma_0$ acts on $\Gamma_0\backslash S^{\Delta}$. Explicitly, the action of $[k]\in W^{\Delta}$ is given as follows. Choose an element $\tau\in\bar{N}^{\Delta}$ such that $\pi^{\Delta}[\tau]=[k]$. Decompose τ as

$$\tau=t(A'')\tau' \quad A''\in Z\,,\ \tau'\in\overline{W}^{*}_{S'} \ \text{ with } \ \tau'S'^{\Delta}=S'^{\Delta}\,. \tag{4.1}$$

Then

$$[k](\pi_0(H''),H')=(\pi_0(H''+A''),\tau'H') \qquad \text{for } H''\in\mathfrak{c}_{\mathfrak{m}},H'\in S'^{\Delta}\,.$$

Let

$$B^{\Delta}=K/N^{\Delta}\,.$$

It is a compact connected C^{∞} manifold. Then K/Z^{Δ} is a C^{∞} principal bundle over B^{Δ} with the group W^{Δ}. Let

$$\mathcal{E}^{\Delta}=K/Z^{\Delta}\times_{W^{\Delta}}(M_0\times S'^{\Delta})$$

be the C^{∞} fibre bundle over B^{Δ} associated to K/Z^{Δ} with the fibre $M_0\times S'^{\Delta}$, by means of the above defined action of W^{Δ} on $M_0\times S'^{\Delta}$. i.e., $\mathcal{E}^{\Delta}$ is the quotient manifold of $K/Z^{\Delta}\times(M_0\times S'^{\Delta})$ relative to the action:

$$(k'Z^{\Delta},h)\longmapsto((k'Z^{\Delta})[k],[k]^{-1}h) \qquad \text{for } k'\in K,h\in M_0\times S'^{\Delta}$$

of W^{Δ}. In the same way we define topological fibre bundles

$$\bar{\mathcal{E}}^{\Delta}=K/Z^{\Delta}\times_{W^{\Delta}}(M_0\times\bar{S}'^{\Delta})\,,$$

$$\partial\mathcal{E}^{\Delta}=K/Z^{\Delta}\times_{W^{\Delta}}(M_0\times\partial S'^{\Delta})$$

over B^{Δ} with fibres $M_0\times\bar{S}'^{\Delta}$ and $M_0\times\partial S'^{\Delta}$ respectively. Note that both $\bar{\mathcal{E}}^{\Delta}$ and $\partial\mathcal{E}^{\Delta}$ are compact since their fibres are compact. The equivalence class of (kZ^{Δ},h) will be denoted by $[kZ^{\Delta},h]$. The following lemma is an easy consequence of definitions.

LEMMA 4.1. *We define maps of $E^\Delta, \bar{E}^\Delta$ and ∂E^Δ into $\mathcal{E}^\Delta$, $\bar{\mathcal{E}}^\Delta$ and $\partial\mathcal{E}^\Delta$ respectively by*

$$[kZ^\Delta, H''+H'] \longmapsto [kZ^\Delta, (\pi_0(H''), H')] \qquad \text{for } k\in K, H''\in \mathfrak{c}_{\mathfrak{m}}, \; H'\in S'^\Delta, \bar{S}'^\Delta \text{ or } \partial S'^\Delta .$$

Then they are all homemorphisms.

LEMMA 4.2. *Let Δ be an admissible subset of $\Pi^\natural$. Then $\bar{E}^\Delta$ is a compact Hausdoff space and ∂E^Δ is a closed subspace of $\bar{E}^\Delta$. There exist a closed subset $\mathcal{A}$ and an open subset $\mathcal{U}$ of $\bar{E}^\Delta$ such that:*

i) $\mathrm{Int}\mathcal{A}\supset\bar{\mathcal{U}}$ *and* $\mathcal{U}\supset\partial E^\Delta$, *where* $\mathrm{Int}\mathcal{A}$ *and* $\bar{\mathcal{U}}$ *mean the interior of* $\mathcal{A}$ *and the closure of* $\mathcal{U}$ *respectively;*

ii) ∂E^Δ *is a strong deformation retract of* $\mathcal{A}$.

PROOF. The first assertion follows from Lemma 4.1. For the proof of the existence of $\mathcal{A}$ and $\mathcal{U}$, we prove first: There exist an $\bar{N}^\Delta$-invariant closed subset $\mathcal{B}$ of $\bar{S}^\Delta$ and an $\bar{N}^\Delta$-invariant open subset $\mathcal{V}$ of $\bar{S}^\Delta$ such that:

i) $\mathrm{Int}\mathcal{B}\supset\bar{\mathcal{V}}$ and $\mathcal{V}\supset\partial S^\Delta$;

ii) ∂S^Δ is an $\bar{N}^\Delta$-equivariant deformation retract of $\mathcal{B}$.

Case 1: $\mathfrak{g}$ is semi-simple and $(\mathfrak{g}, \mathfrak{k})$ is irreducible. Under the notation in the proof of Lemma 1.4, $\bar{S}^\Delta$ is given by

$$\bar{S}^\Delta=\{H=\sum_{\gamma\in\Delta} h_\gamma P_{\gamma^\natural};\ 0\leqslant h_\gamma\leqslant 1,\ \sum_{\gamma\in\Delta} h_\gamma=1\}.$$

Take a small $\varepsilon>0$ and put

$$\mathcal{B}=\{H\in\bar{S}^\Delta;\ \min_{\gamma\in\Delta} h_\gamma\leqslant\varepsilon\},$$

$$\mathcal{V}=\{H\in\bar{S}^\Delta;\ \min_{\gamma\in\Delta} h_\gamma<\varepsilon/2\}.$$

Then $\mathcal{B}$ is closed, $\mathcal{V}$ is open in $\bar{S}^\Delta$, and they satisfy $\mathrm{Int}\mathcal{B}\supset\bar{\mathcal{V}}\supset\mathcal{V}\supset\partial S^\Delta$. By Lemma 2.3, the action of $\tau\in\bar{N}^\Delta$ on $\bar{S}^\Delta$ is given by

$$\sum_{\gamma\in\Delta} h_\gamma P_\gamma \longmapsto \sum_{\gamma\in\Delta} h_\gamma P_{(\tau\cdot\gamma)^\natural}.$$

This shows the $\bar{N}^\Delta$-invariance of $\mathcal{B}$ and $\mathcal{V}$. For an element $H=\sum h_\gamma P_{\gamma^\natural}\in\mathcal{B}$, we put

$$\Delta_H=\{\gamma\in\Delta;\ h_\gamma=\min_{\gamma'\in\Delta} h_{\gamma'}\}.$$

For each $t\in I=[0,1]$ we define $f_t(H)\in\mathcal{B}$ by

$$f_t(H)=\sum{}' t h_\gamma P_{\gamma^\natural}+((1-t\sum{}' h_\gamma)/\sum{}'' h_\gamma)\sum{}'' h_\gamma P_{\gamma^\natural},$$

where $\sum'$ and $\sum''$ mean the summations over Δ_H and $\Delta-\Delta_H$ respectively. We choose $\varepsilon>0$ so small that $\Delta-\Delta_H$ is not empty for each $H\in\mathcal{B}$. Then f_t define a continuous map: $\mathcal{B}\times I\to\mathcal{B}$. We have

$$f_0(H)=(1/\textstyle\sum'' h_\gamma)\sum'' h_\gamma P_{\gamma^\natural}\in S^{\Delta'}\subset\partial S^\Delta\,,$$

where Δ' is an admissible subset of $\Pi^\sharp$ with $\Delta'\subsetneqq\Delta$ defined by $\Delta'=\Delta-\Delta_H$, and

$$f_1(H)=\textstyle\sum' h_\gamma P_{\gamma^\natural}+\sum'' h_\gamma P_{\gamma^\natural}=H\,.$$

If $H\in\partial S^\Delta$, then there exists an admissible subset Δ' of $\Pi^\sharp$ with $\Delta'\subsetneqq\Delta$ such that $H\in S^{\Delta'}$, and hence it has an expression:

$$H=\sum_{\gamma\in\Delta'} h_\gamma P_{\gamma^\natural}\quad 0<h_\gamma<1\,,\quad \sum_{\gamma\in\Delta'} h_\gamma=1\,.$$

Thus $\Delta_H=\Delta-\Delta'$, $\Delta-\Delta_H=\Delta'$ and hence

$$f_t(H)=\textstyle\sum'' h_\gamma P_{\gamma^\natural}=\displaystyle\sum_{\gamma\in\Delta'} h_\gamma P_{\gamma^\natural}=H\,.$$

These show that f_t is a strong deformation retraction of $\mathcal{B}$ into ∂S^Δ. Since we have $\tau\cdot\Delta_H=\Delta_{\tau H}$ for each $\tau\in\bar{N}^\Delta$, we get $f_t(\tau H)=\tau f_t(H)$ for each $H\in\mathcal{B}$. This means the $\bar{N}^\Delta$-equivariance of f_t.

Case 2: $\mathfrak{g}$ is semi-simple. Take the direct product of subsets $\mathcal{B}_k, \mathcal{V}_k$ and that of deformation retractions $f_t^{(k)}$ for irreducible factors $(\mathfrak{g}_k, \mathfrak{k}_k)$ $(1\leqslant k\leqslant s)$.

Case 3: General case. Let $\mathcal{B}', \mathcal{V}'$ and f_t' be the required ones for $\bar{S}'^\Delta$. We define subsets $\mathcal{B}$ and $\mathcal{V}$ of $\bar{S}^\Delta=\mathfrak{c}_{\mathfrak{m}}\times\bar{S}'^\Delta$ by

$$\mathcal{B}=\mathfrak{c}_{\mathfrak{m}}\times\mathcal{B}'\,,\quad \mathcal{V}=\mathfrak{c}_{\mathfrak{m}}\times\mathcal{V}'\,,$$

and then define a continuous map f_t: $\mathcal{B}\to\mathcal{B}$ for $t\in I$ by

$$f_t(H'', H')=(H'', f_t'(H'))\qquad \text{for } H''\in\mathfrak{c}_{\mathfrak{m}}, H'\in\mathcal{B}'\,.$$

It follows from the decomposition (4.1) for $\tau\in\bar{N}^\Delta$ that $\mathcal{B}$ and $\mathcal{V}$ are $\bar{N}^\Delta$-invariant and that f_t is $\bar{N}^\Delta$-equivariant. Thus $\mathcal{B}, \mathcal{V}$ and f_t are the required ones for $\bar{S}^\Delta$.

Now we shall prove the existence of subsets $\mathcal{A}$ and $\mathcal{U}$ of $\bar{E}^\Delta$ with i) and ii). We define $\bar{W}^\Delta$-invariant subsets $\tilde{\mathcal{A}}$ and $\tilde{\mathcal{U}}$ of $K/Z^\Delta\times\bar{S}^\Delta$ by

$$\tilde{\mathcal{A}}=K/Z^\Delta\times\mathcal{B}\,,\quad \tilde{\mathcal{U}}=K/Z^\Delta\times\mathcal{V}\,,$$

and then define a continuous map $\tilde{F}_t$: $\tilde{\mathcal{A}}\to\tilde{\mathcal{A}}$ for $t\in I$ by

$$\tilde{F}_t(kZ^\Delta, H)=(kZ^\Delta, f_t(H))\qquad \text{for } k\in K, H\in\mathcal{B}\,.$$

Let $[\tau]\in\bar{W}^\Delta$ with $\tau\in\bar{N}^\Delta$ and $\pi^\Delta[\tau]=[l]$ with $l\in N^\Delta$. Then

$$\begin{aligned}\tilde{F}_t((kZ^\Delta, H)[\tau]) &= \tilde{F}_t(klZ^\Delta, \tau^{-1}H) = (klZ^\Delta, f_t(\tau^{-1}H)) \\ &= (klZ^\Delta, \tau^{-1}f_t(H)) = (kZ^\Delta, f_t(H))[\tau] \\ &= \tilde{F}_t(kZ^\Delta, H)[\tau]\end{aligned}$$

for each $k \in K, H \in \bar{S}^\Delta$. Thus $\tilde{F}_t$ is a $\bar{W}^\Delta$-equivariant strong deformation retraction of $\tilde{\mathcal{A}}$ into $K/Z^\Delta \times \partial S^\Delta$. Now put

$$\mathcal{A} = \tilde{\mathcal{A}}/\bar{W}^\Delta = K/Z^\Delta \times_{\bar{W}^\Delta} \mathcal{B},$$

$$\mathcal{U} = \tilde{\mathcal{U}}/\bar{W}^\Delta = K/Z^\Delta \times_{\bar{W}^\Delta} \mathcal{V}.$$

Then $\tilde{F}_t$ induces a continuous map F_t: $\mathcal{A} \to \mathcal{A}$ giving a strong deformation retraction of $\mathcal{A}$ into ∂E^Δ. These $\mathcal{A}$ and $\mathcal{U}$ are the required ones. q.e.d.

For an integer k with $0 \leqslant k \leqslant r = \operatorname{rank} M$, we define subsets $\mathcal{Q}_k$ and $\mathcal{Q}^k$ of $\mathcal{Q}^*$ (cf. Theorem 2.2 in Part I) by

$$\mathcal{Q}_k = \{\Delta \in \mathcal{Q}^*;\ k_\Delta = k\},$$

$$\mathcal{Q}^k = \bigcup_{0 \leqslant l \leqslant k} \mathcal{Q}_l,$$

and then define subsets Q^k of M by

$$Q^k = \bigcup_{\Delta \in \mathcal{Q}^k} M^\Delta.$$

Then we have inclusions:

$$Q^0 \subset Q^1 \subset \cdots \subset Q^{r-1} = Q \subset Q^r = M$$

and decompositions:

$$Q^k - Q^{k-1} = \bigcup_{\Delta \in \mathcal{Q}_k} M^\Delta \quad (0 \leqslant k \leqslant r),$$

where $Q^{-1} = \phi$. With these definitions we have

THEOREM 4.1. 1) *Each* Q^k *is a closed subset of* M.

2) *For each* $\Delta \in \mathcal{Q}_k$, *there exists a continuous map* ψ^Δ : $(\bar{E}^\Delta, \partial E^\Delta) \to (Q^k, Q^{k-1})$, *which extends the previously defined diffeomorphism* ψ^Δ : $E^\Delta \to M^\Delta$ *and satisfies* $\psi^\Delta(\bar{E}^\Delta) = \bar{M}^\Delta$.

3) *There exists an exact sequence*:

$$\cdots \longrightarrow H^{p-1}(Q^{k-1}, A) \longrightarrow \sum_{\Delta \in \mathcal{Q}_k} \oplus H^p(\bar{E}^\Delta, \partial E^\Delta; A) \longrightarrow H^p(Q^k, A) \longrightarrow H^p(Q^{k-1}, A) \longrightarrow \cdots$$

for any coefficient group A.

4) *The pair* $(\bar{E}^\Delta, \partial E^\Delta)$ *is homemorphic with the pair* $(\bar{\mathcal{E}}^\Delta, \partial \mathcal{E}^\Delta)$ *of the compact fibre bundle* $\bar{\mathcal{E}}^\Delta$ *over the compact connected* C^∞ *manifold* B^Δ *with the fibre* $M_0 \times \bar{S}'^\Delta$,

and its boundary $\partial\mathcal{E}^{\Delta}$. *The group* $H^*(\bar{E}^{\Delta}, \partial E^{\Delta}; A)$ *can be obtained from* $H^*(B^{\Delta}, A)$ *in the following cases.*

Let A *be a principal ideal ring.*

(i) *If* $\pi_1(M)$ *is finite and* $\bar{W}^{\Delta}$ *acts trivially on* $H^{k_{\Delta}}(\bar{S}^{\Delta}/\partial S^{\Delta}, A)$, *then*

$$H^p(\bar{E}^{\Delta}, \partial E^{\Delta}; A) \cong H^{p-k_{\Delta}}(B^{\Delta}, A).$$

(ii) *If* $\bar{W}^{\Delta} = \{1\}$, *then*

$$H^p(\bar{E}^{\Delta}, \partial E^{\Delta}; A) \cong H^{p-k'_{\Delta}}(B^{\Delta} \times M_0, A),$$

where $k'_{\Delta} = \dim S'^{\Delta}$.

PROOF. 1) Note that the cellular decomposition:

$$\bar{S} = \bigcup_{\Delta} S^{\Delta}$$

is closed under the closure operation in the sense that for each S^{Δ} the closure $\bar{S}^{\Delta}$ is the union of S^{Δ} and some lower dimensional cells $S^{\Delta'}$. This implies the closedness of Q^k.

2) For an admissible subset Δ of $\Pi^{\natural}$, we define a continuous map $\psi^{\Delta}: \bar{E}^{\Delta} \to M$ by

$$\psi^{\Delta}([kZ^{\Delta}, H]_{\Delta}) = k \operatorname{Exp} H \qquad \text{for } k \in K, H \in \bar{S}^{\Delta},$$

where $[kZ^{\Delta}, H]_{\Delta}$ denotes the class in $\bar{E}^{\Delta}$ containing $(kZ^{\Delta}, H) \in K/Z^{\Delta} \times \bar{S}^{\Delta}$. On E^{Δ} it coincides with the previously defined diffeomorphism $\psi^{\Delta}: E^{\Delta} \to M^{\Delta}$. For an admissible subset $\Delta' \subset \Delta$, $H' \in S^{\Delta'}$ and $k \in K$, we have

$$\psi^{\Delta}([kZ^{\Delta}, H']_{\Delta}) = k \operatorname{Exp} H' = \psi^{\Delta'}([kZ^{\Delta'}, H']_{\Delta'}).$$

Thus, for each $\Delta \in Q_k$

$$\psi^{\Delta}(\bar{E}^{\Delta}) = \bigcup_{\Delta' \subset \Delta} M^{\Delta'} = \bar{M}^{\Delta} \subset Q^k$$

and

$$\psi^{\Delta}(\partial E^{\Delta}) = \bigcup_{\Delta' \subsetneqq \Delta} M^{\Delta'} \subset Q^{k-1}.$$

This proves the assertion 2).

3) Under the existence of subsets $\mathcal{A}$ and $\mathcal{U}$ in Lemma 4.2, the ordinary argument shows that

$$(\psi^{\Delta})^*: H^*(Q^k, Q^{k-1}; A) \longrightarrow H^*(\bar{E}^{\Delta}, \partial E^{\Delta}; A)$$

is surjective for each $\Delta \in Q_k$ and that

$$\sum_{\Delta \in Q_k} \oplus (\psi^{\Delta})^*: H^*(Q^k, Q^{k-1}; A) \longrightarrow \sum_{\Delta \in Q_k} \oplus H^*(\bar{E}^{\Delta}, \partial E^{\Delta}; A)$$

is an isomorphism. Together with the cohomology exact sequence:

$$\cdots \longrightarrow H^{p-1}(Q^{k-1}, A) \overset{\partial^*}{\longrightarrow} H^p(Q^k, Q^{k-1}; A) \longrightarrow H^p(Q^k, A)$$
$$\longrightarrow H^p(Q^{k-1}, A) \longrightarrow \cdots$$

for the pair (Q^k, Q^{k-1}), we get the required exact sequence.

4) The first assertion follows from Lemma 4.1.

(i) Theorem 2.1 and (2.7) imply that $\pi_1(M)$ is finite if and only if $\mathfrak{c}_{\mathfrak{m}}=\{0\}$. Thus $\bar{\mathcal{E}}^{\Delta}$ and $\partial\mathcal{E}^{\Delta}$ are fibre bundles over B^{Δ} with the fibre the closed k_{Δ}-disk $\bar{S}^{\Delta}$ and the $(k_{\Delta}-1)$-sphere ∂S^{Δ} respectively. Moreover, the local system of the k_{Δ}-th cohomologies of fibres of the pair $(\bar{\mathcal{E}}^{\Delta}, \partial\mathcal{E}^{\Delta})$ is trivial, since $\bar{W}^{\Delta}$ acts trivially on $H^{k_{\Delta}}(\bar{S}^{\Delta}/\partial S^{\Delta}, A)$. Thus Thom isomorphism provides the required isomorphism.

(ii) In this case, both $\bar{\mathcal{E}}^{\Delta}$ and $\partial\mathcal{E}^{\Delta}$ are trivial:

$$\bar{\mathcal{E}}^{\Delta}=B^{\Delta}\times M_0\times\bar{S}'^{\Delta},$$
$$\partial\mathcal{E}^{\Delta}=B^{\Delta}\times M_0\times\partial S'^{\Delta}.$$

Thus the same argument as in (i) shows the assertion (ii). q.e.d.

REMARK. In the case (i), the triviality of the action of $\bar{W}^{\Delta}$ holds always for $A=\boldsymbol{Z}_2$ and it holds for $A=\boldsymbol{Z}$ if and only if the cell bundle $\mathcal{E}^{\Delta}$ is orientable.

For an integer k with $0\leqslant k\leqslant r-1$, we define (cf. Theorem 2.2 in Part I)

$$\mathcal{F}_k=\mathcal{F}\cap Q_k,$$
$$\mathcal{F}^k=\mathcal{F}\cap Q^k,$$
$$F^k=\bigcup_{\Delta\in\mathcal{F}^k} M^{\Delta}$$

to get inclusions:

$$F^0\subset F^1\subset\cdots\subset F^{r-1}=F$$

and decompositions:

$$F^k-F^{k-1}=\bigcup_{\Delta\in\mathcal{F}_k} M^{\Delta}\quad(0\leqslant k\leqslant r-1),$$

where $F^{-1}=\phi$. Note that the cellular decomposition:

$$\tilde{F}\cap\bar{S}=\bigcup_{\Delta\not\ni\Sigma^*} S^{\Delta}$$

is also closed under the closure operation. Thus we get also the following

THEOREM 4.2. 1) *Each F^k is a closed subset of M.*

2) *For each* $\Delta\in\mathfrak{F}_k$, *there exists a continuous map* $\psi^\Delta : (\bar{E}^\Delta, \partial E^\Delta)\rightarrow(F^k, F^{k-1})$, *which extends the diffeomorphism* $\psi^\Delta : E^\Delta\rightarrow M^\Delta$ *and satisfies* $\psi^\Delta(\bar{E}^\Delta)=\bar{M}^\Delta$.

3) *There exists an exact sequence*:

$$\cdots\longrightarrow H^{p-1}(F^{k-1}, A)\longrightarrow\sum_{\Delta\in\mathfrak{F}_k}\oplus H^p(\bar{E}^\Delta, \partial E^\Delta ; A)\longrightarrow H^p(F^k, A)$$
$$\longrightarrow H^p(F^{k-1}, A)\longrightarrow\cdots$$

for any coefficient group A.

For an integer k with $0\leqslant k\leqslant r-1$, we define subsets $\mathcal{C}_k$ and $\mathcal{C}^k$ of $\mathcal{C}$ (cf. §3 in Part I) by

$$\mathcal{C}_k=\{(\Delta, \Phi)\in\mathcal{C};\ k_{\Delta,\Phi}=k\},$$
$$\mathcal{C}^k=\bigcup_{0\leqslant l\leqslant k}\mathcal{C}_l,$$

and then define subsets C^k of M by

$$C^k=\bigcup_{(\Delta,\Phi)\in\mathcal{C}^k}M^{\Delta,\Phi},$$

to get inclusions:

$$C^0\subset C^1\subset\cdots\subset C^{r-1}=C,$$

and decompositions:

$$C^k-C^{k-1}=\bigcup_{(\Delta,\Phi)\in\mathcal{C}_k}M^{\Delta,\Phi}\quad(0\leqslant k\leqslant r-1),$$

where $C^{-1}=\phi$.

Let (Δ, Φ) be a c-pair (cf. §3 in Part I). Recall that $K/Z^{\Delta,\Phi}$ is a C^∞ principal bundle over a compact connected C^∞ manifold $B^{\Delta,\Phi}=K/N^{\Delta,\Phi}$ with the group $\bar{W}^{\Delta,\Phi}$. In the same way as for an admissible subset Δ of Π^*, we define compact topological fibre bundles $\bar{E}^{\Delta,\Phi}$ and $\partial E^{\Delta,\Phi}$ over $B^{\Delta,\Phi}$ by

$$\bar{E}^{\Delta,\Phi}=K/Z^{\Delta,\Phi}\times_{\bar{W}^{\Delta,\Phi}}\bar{S}^{\Delta,\Phi},$$
$$\partial E^{\Delta,\Phi}=K/Z^{\Delta,\Phi}\times_{\bar{W}^{\Delta,\Phi}}\partial S^{\Delta,\Phi},$$

where $\partial S^{\Delta,\Phi}=\bar{S}^{\Delta,\Phi}-S^{\Delta,\Phi}$. We can also prove Lemma 4.2 for our pair $(\bar{E}^{\Delta,\Phi}, \partial E^{\Delta,\Phi})$ making use of end points of the convex polyhedron $\bar{S}^{\Delta,\Phi}$ instead of vertices of the simplex $\bar{S}^\Delta$. Moreover, the cellular decomposition:

$$\tilde{C}\cap\bar{S}=\bigcup_{(\Delta,\Phi)}{}' S^{\Delta,\Phi}$$

is also closed under the closure operation. Thus we can prove the following theorem in the same way as for Theorem 4.1.

THEOREM 4.3. 1) *Each* C^k *is a closed subset of* M.

2) *For each* $(\Delta, \Phi)\in C_k$, *there exists a continuous map* $\psi^{\Delta,\Phi}: (\bar{E}^{\Delta,\Phi}, \partial E^{\Delta,\Phi})\to (C^k, C^{k-1})$, *which extends the previously defined diffeomorphism* $\psi^{\Delta,\Phi}: E^{\Delta,\Phi}\to M^{\Delta,\Phi}$ *and satisfies* $\psi^{\Delta,\Phi}(\bar{E}^{\Delta,\Phi})=\bar{M}^{\Delta,\Phi}$.

3) *There exists an exact sequence*:

$$\cdots \longrightarrow H^{p-1}(C^{k-1}, A) \longrightarrow \sum_{(\Delta,\Phi)\in C_k} \oplus H^p(\bar{E}^{\Delta,\Phi}, \partial E^{\Delta,\Phi}; A) \longrightarrow H^p(C^k, A) \longrightarrow H^p(C^{k-1}, A) \longrightarrow \cdots$$

for any coefficient group A.

4) $\bar{E}^{\Delta,\Phi}$ *is a compact fibre bundle over a compact connected* C^∞ *manifold* $B^{\Delta,\Phi}$ *with the fibre* $\bar{S}^{\Delta,\Phi}$, *and* $\partial E^{\Delta,\Phi}$ *is the boundary of* $\bar{E}^{\Delta}$.

5) *If* A *is a principal ideal ring and* $\bar{W}^{\Delta,\Phi}$ *acts trivially on* $H^{k_{\Delta,\Phi}}(\bar{S}^{\Delta,\Phi}/\partial S^{\Delta,\Phi}, A)$, *then*

$$H^p(\bar{E}^{\Delta,\Phi}, \partial E^{\Delta,\Phi}; A)\cong H^{p-k_{\Delta,\Phi}}(B^{\Delta,\Phi}, A).$$

REMARK 1. The triviality of the action of $\bar{W}^{\Delta,\Phi}$ on $H^{k_{\Delta,\Phi}}(\bar{S}^{\Delta,\Phi}/\partial S^{\Delta,\Phi}, A)$ holds always for $A=\boldsymbol{Z}_2$ and it holds for $A=\boldsymbol{Z}$ if and only if the cell bundle $E^{\Delta,\Phi}$ over $B^{\Delta,\Phi}$ is orientable.

REMARK 2. (cf. Warner [11]) Let $\dim M=n$. Considering the cohomology exact sequence for the pair (M, C), we get isomorphisms:

$$H^p(M, A)\cong H^p(C, A) \qquad (0\leqslant p\leqslant n-2)$$

and an exact sequence:

$$0 \longrightarrow H^{n-1}(M, A) \longrightarrow H^{n-1}(C, A) \longrightarrow H^n(M, C; A)=A \longrightarrow H^n(M, A) \longrightarrow 0.$$

Thus, if $A=\boldsymbol{Z}_2$ or if $A=\boldsymbol{Z}$ and M is orientable, then

$$H^{n-1}(M, A)\cong H^{n-1}(C, A).$$

§ 5. Cut loci of symmetric $\boldsymbol{R}$-spaces

In this section, we shall study cut loci of symmetric R-spaces, applying the results in the previous sections. Let us recall the notion of symmetric R-spaces. (cf. Takeuchi [7]) Let $G\subset GL(V)$ be a Zariski-connected reductive real algebraic group without compact simple factors, where V is a finite dimensional real vector space. Let $G^C\subset GL(V^C)$ denote the complexification of G, where V^C is the complexification of V. Take a maximal compact subgroup K of G and choose an automorphism θ' of $GL(V)$ with $\theta'(G)=G$ such that

$$K=\{x\in G; \theta'(x)=x\}.$$

Denote the extention of θ' to $GL(V^{C})$ by the same θ'. Then θ' leaves G^{C} invariant. Let $\sigma' : GL(V^{C}) \to GL(V^{C})$ be the complex conjugation, i.e.,

$$\sigma'(x)=\bar{x} \quad \text{where} \quad \bar{x}(v)=\overline{x(\bar{v})} \quad \text{for } v \in V^{C}.$$

Then σ' leaves G^{C} invariant and we have

$$G=\{x \in G^{C};\ \sigma'(x)=x\}.$$

Put $\tau'=\theta'\sigma'=\sigma'\theta'$. Then τ' leaves G^{C} and G invariant. Put

$$G_u=\{x \in G^{C};\ \tau'(x)=x\}.$$

It is a maximal compact subgroup of G^{C}, which is invariant by θ' and σ'. K is also given by

$$K=\{x \in G_u;\ \theta'(x)=x\}.$$

Thus (G_u, K) is a compact symmetric pair.

In what follows, for automorphisms of Lie groups, their differentials will be denoted by the same letters.

Let $\mathfrak{g}, \mathfrak{g}^{C}, \mathfrak{k}$ and $\mathfrak{g}_u$ be Lie algebras of G, G^{C}, K and G_u respectively. Then the complexifications of $\mathfrak{g}$ and $\mathfrak{g}_u$ are the same and equal to $\mathfrak{g}^{C}$. Take a G-invariant symmetric bilinear form B on $\mathfrak{g}^{C}$ such that it coincides on $[\mathfrak{g}^{C}, \mathfrak{g}^{C}]$ with the Killing form of $[\mathfrak{g}^{C}, \mathfrak{g}^{C}]$ and that it is negative definite on $\mathfrak{g}_u$. Such a B is always θ'-invariant. We define

$$(X, Y)=-B(X, Y) \qquad \text{for } X, Y \in \mathfrak{g}_u.$$

It is an inner product on $\mathfrak{g}_u$, which is invariant under θ' and the adjoint action of G_u.

Take a Cartan subalgebra $\mathfrak{A}'$ for $(\mathfrak{g}_u, \mathfrak{k})$ and a maximal abelian subalgebra $\mathfrak{T}'$ of $\mathfrak{g}_u$ containing $\mathfrak{A}'$. The various objects for $(\mathfrak{g}_u, \mathfrak{k})$ relative to $\mathfrak{A}'$ and $\mathfrak{T}'$ will be denoted by the same symbols as in §1 of Part I but with primes.

Now assume that there exists $\zeta \in \mathfrak{A}'$ such that

$$(5.1) \qquad (\alpha, \zeta)=0, 1 \ \text{ or } \ -1 \qquad \text{for } \alpha \in \tilde{\Sigma}'.$$

Put

$$\tilde{\Sigma}_1'=\{\alpha \in \tilde{\Sigma}';\ (\alpha, \zeta)=0\},$$

$$\Sigma_1'=\{\gamma \in \Sigma';\ (\gamma, \zeta)=0\}.$$

Choose a compatible order $>$ on $\mathfrak{T}'$ such that $(\alpha, \zeta) \geqq 0$ for each positive $\alpha \in \tilde{\Sigma}'$, and fix it once for all. Let $\tilde{\Pi}'$ and Π' denote the fundamental root systems for $\tilde{\Sigma}'$ and Σ' respectively with respect to this order $>$. Let Σ_+' and Σ_-' denote the set of positive roots and that of negative roots in Σ' respectively. Put

$$\tilde{\Pi}_1' = \tilde{\Pi}' \cap \tilde{\Sigma}_1' ,$$

$$\Pi_1' = \Pi' \cap \Sigma_1' ,$$

$$(\Sigma_1')_\pm = \Sigma_1' \cap \Sigma_\pm' .$$

For $\gamma \in \Sigma'$ we define a subspace $\mathfrak{g}_\gamma$ of $\mathfrak{g}$ by

$$\mathfrak{g}_\gamma = \{X \in \mathfrak{g} ; [H, X] = 2\pi\sqrt{-1}(\gamma, H)X \quad \text{for each } H \in \mathfrak{A}'\} .$$

We define some subalgebras of $\mathfrak{g}$ as follows.

$$\mathfrak{g}_0 = \{X \in \mathfrak{g} ; [X, \mathfrak{A}'] = \{0\}\} ,$$

$$\mathfrak{g}_1 = \{X \in \mathfrak{g} ; [\zeta, X] = 0\} ,$$

$$\mathfrak{n}^+ = \{X \in \mathfrak{g} ; [\zeta, X] = 2\pi\sqrt{-1}X\} ,$$

$$\mathfrak{u} = \mathfrak{g}_1 + \mathfrak{n}^+ .$$

Then we have

$$\mathfrak{g} = \mathfrak{g}_0 + \sum_{\gamma \in \Sigma'} \mathfrak{g}_\gamma ,$$

$$\mathfrak{g}_1 = \mathfrak{g}_0 + \sum_{\gamma \in \Sigma_1'} \mathfrak{g}_\gamma ,$$

$$\mathfrak{n}^+ = \sum_{\gamma \in \Sigma_+' - \Sigma_1'} \mathfrak{g}_\gamma ,$$

$$\mathfrak{u} = \mathfrak{g}_0 + \sum_{\gamma \in \Sigma_+' \cup \Sigma_1'} \mathfrak{g}_\gamma .$$

Now we define closed subgroups G_1, N^+ and U of G by

$$G_1 = \{x \in G ; \mathrm{Ad}x\, \zeta = \zeta\} ,$$

$$N^+ = \exp \mathfrak{n}^+ ,$$

$$U = \{x \in G ; \mathrm{Ad}x\, \mathfrak{n}^+ = \mathfrak{n}^+\} .$$

Then $\mathfrak{n}^+$ is abelian and $\exp : \mathfrak{n}^+ \to N^+$ is an isomorphism. G_1 is also a Zariski-connected reductive real algebraic group. U has a semi-direct decomposition:

$$U = G_1 \cdot N^+ .$$

The Lie algebras of G_1, N^+ and U are $\mathfrak{g}_1, \mathfrak{n}^+$ and $\mathfrak{u}$ respectively. The subgroup U is a so-called parabolic subgroup of G. The homogeneous space:

$$M = G/U$$

is called a *symmetric R-space*. It is known to be compact and connected. The origin U of M will be denoted by o. Put $K_1 = K \cap G_1$. Then the inclusion $K \subset G$ induces an identification: $K/K_1 = G/U$. We define an involutive automorphism θ of

G^C by

$$\theta(x)=\exp((1/2)\zeta)x\exp((1/2)\zeta)^{-1} \quad \text{for } x\in G^C.$$

Then θ leaves G_u and the identity component K^0 of K invariant. Put $K^*=K^0\cap K_1$ and denote the Lie algebra of K^* by $\mathfrak{k}^*$. Then the pair (K^0, K^*) is a compact symmetric pair with respect to θ, and the inclusion $K^0\subset G$ induces an identification:

$$M=K^0/K^*.$$

The inner product (,) on $\mathfrak{k}$ is invariant under θ and the adjoint action of K^0. Thus it defines a K^0-invariant Riemannian metric g on M. We shall study the cut locus of this symmetric space (M, g). The tangent space of M at o will be identified with

$$\mathfrak{m}=\{X\in\mathfrak{k};\ \theta X=-X\}.$$

We shall use the same notation for the pair (K^0, K^*) as in the previous sections.

Here we recall the notion of a symmetric pair of Dynkin diagrams. In general, a pair (Π', Π_1') of an irreducible Dynkin diagram Π' and a subdiagram Π_1', is said to be *irreducible symmetric* if $\Pi'-\Pi_1'$ consists of one element, say α_1, and if the highest root δ' of the reduced root system Σ' with the fundamental system Π' has an expression:

$$\delta'=\alpha_1+\sum_{\alpha_i\in\Pi_1'} m_i\alpha_i \quad m_i\in\boldsymbol{Z}.$$

For a general pair (Π', Π_1') of Dynkin diagrams, decompose Π' into the sum of irreducible components: $\Pi'=\Pi'^{(1)}\cup\cdots\cup\Pi'^{(s)}$ and put $\Pi_1'^{(k)}=\Pi_1'\cap\Pi'^{(k)}$ $(1\leqslant k\leqslant s)$. If each pair $(\Pi'^{(k)}, \Pi_1'^{(k)})$ is irreducible symmetric, the pair (Π', Π_1') is said to be *symmetric*. For each symmetric pair (Π', Π_1'), we can associate in a canonical way a hermitian symmetric space of compact type (cf. Takeuchi [10]). Then $\operatorname{rank}(\Pi', \Pi_1')$ is defined to be the rank of this symmetric space.

Now we come back to our symmetric R-space.

In what follows, we assume that the fundamental root system Π' for the pair $(\mathfrak{g}_u, \mathfrak{k})$ is irreducible. In this case, (M, g) is called an *irreducible symmetric R-space.*

LEMMA 5.1. *For an irreducible symmetric R-space (M, g), we have:*

1) *(Π', Π_1') is an irreducible symmetric pair;*
2) *Σ' is a reduced root system;*
3) *$r=\operatorname{rank} M$ is equal to $\operatorname{rank}(\Pi', \Pi_1')$.*

PROOF. 1) It follows from (5.1) that $(\tilde{\Pi}', \tilde{\Pi}_1')$ is a symmetric pair, which implies the assertion 1).

2) and 3) are checked by seeing the table of irreducible symmetric R-spaces (Takeuchi [7]). q.e.d.

Let $\{\beta_1, \cdots, \beta_r\} \subset \Sigma_+' - \Sigma_1'$ be the maximal system of strongly orthogonal roots of the same length with $\beta_1 = \delta'$, in the sense of Takeuchi [9]. For each i with $1 \leqslant i \leqslant r$, choose $X_i \in \mathfrak{g}_{\beta_i}$ in such a way that

$$-B(X_i, \tau' X_i) = 1/2\pi^2(\beta_i, \beta_i),$$

and put

$$X_{-i} = \tau' X_i \in \mathfrak{g}_{-\beta_i},$$

$$U_i = -\pi(X_i + X_{-i}) \in \mathfrak{m} \subset \mathfrak{k}.$$

We define a subspace $\mathfrak{a}$ of $\mathfrak{m}$ by

$$\mathfrak{a} = \{U_1, \cdots, U_r\}_{\boldsymbol{R}}.$$

It follows from Lemma 5.1, 3) that $\mathfrak{a}$ is a Cartan subalgebra for the symmetric pair $(\mathfrak{k}, \mathfrak{k}^*)$.

LEMMA 5.2. *Let P be a closed subgroup of $SL(2, \boldsymbol{R})$ defined by*

$$P = \left\{ \begin{pmatrix} a & b \\ 0 & 1/a \end{pmatrix};\ a, b \in \boldsymbol{R},\ a \neq 0 \right\},$$

and identify the 1-dimensional real projective space $P_1(\boldsymbol{R})$ with the quotient manifold $SL(2, \boldsymbol{R})/P$. Then there exists an $SL(2, \boldsymbol{R})^r$-equivariant imbedding $\varphi: P_1(\boldsymbol{R})^r \to M$ with the image $\hat{A} = (\exp \mathfrak{a})o$. Here $*^r$ means the r-fold direct product of $*$.

PROOF. Note first that the inclusion $SO(2) \subset SL(2, \boldsymbol{R})$ induces an identification: $SO(2)/\{\pm 1\} = SL(2, \boldsymbol{R})/P$.

We have $[X_i, X_{-i}] \in \sqrt{-1}\mathfrak{A}'$ and

$$\begin{aligned} B(H, [X_i, X_{-i}]) &= B([H, X_i], X_{-i}) = 2\pi\sqrt{-1}(\beta_i, H) B(X_i, \tau' X_i) \\ &= -2\pi\sqrt{-1}(\beta_i, H)/2\pi^2(\beta_i, \beta_i) = B(H, (\sqrt{-1}/\pi(\beta_i, \beta_i))\beta_i) \end{aligned}$$

for each $H \in \mathfrak{A}'$. Thus

$$[X_i, X_{-i}] = (\sqrt{-1}/\pi) A_{\beta_i} \quad (1 \leqslant i \leqslant r). \tag{5.2}$$

We define a basis $\{X_+, X_{-1}, H_0\}$ for $\mathfrak{sl}(2, \boldsymbol{C})$ by

$$X_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad X_- = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}, \quad H_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

It is also a basis for $\mathfrak{sl}(2, \boldsymbol{R})$. They have the bracket relations:

(5.3) $$[X_+, X_-] = -H_0\,, \quad [H_0, X_\pm] = \pm 2X_\pm\,.$$

Put

$$U_0 = -\pi(X_+ + X_-) = \begin{pmatrix} 0 & -\pi \\ \pi & 0 \end{pmatrix} \in \mathfrak{o}(2)\,.$$

Let $\phi_i : \mathfrak{sl}(2, \boldsymbol{C}) \to \mathfrak{g}^{\boldsymbol{C}}$ be an injective linear map defined by

$$\phi_i(X_\pm) = X_{\pm i}\,, \quad \phi_i(H_0) = -(\sqrt{-1}/\pi)A_{\beta_i}\,,$$

so that

(5.4) $$\phi_i(U_0) = U_i \quad (1 \leqslant i \leqslant r)\,.$$

It follows from (5.2) and (5.3) that ϕ_i is a homomorphism. Since $\phi_i(\bar{X}) = \sigma'\phi_i(X)$, $\phi_i(-{}^t\bar{X}) = \tau'\phi_i(X)$ for each $X \in \mathfrak{sl}(2, \boldsymbol{C})$, ϕ_i sends $\mathfrak{sl}(2, \boldsymbol{R})$, $\mathfrak{su}(2)$ and $\mathfrak{o}(2)$ into $\mathfrak{g}, \mathfrak{g}_u$ and $\mathfrak{k}$ respectively. Moreover, ϕ_i sends the Lie algebra $\mathfrak{p}$ of P into $\mathfrak{u}$. Next we define an injective homomorphism $\phi : \mathfrak{sl}(2, \boldsymbol{C})^r \to \mathfrak{g}^{\boldsymbol{C}}$ by

$$\phi(Y_1, \cdots, Y_r) = \sum_{i=1}^{r} \phi_i(Y_i) \qquad \text{for } Y_i \in \mathfrak{sl}(2, \boldsymbol{C})\,.$$

In virtue of (5.4), we have $\phi(\mathfrak{o}(2)^r) = \mathfrak{a}$. We extend ϕ to $SL(2, \boldsymbol{C})^r$ and denote it by the same ϕ: $SL(2, \boldsymbol{C})^r \to G^{\boldsymbol{C}}$. The above argument shows that ϕ sends $SL(2, \boldsymbol{R})^r$, $SO(2)^r$ and P^r into G, K^0 and U respectively. Thus ϕ induces an immersion φ: $P_1(\boldsymbol{R})^r = SL(2, \boldsymbol{R})^r/P^r \to G/U = M$. It is $SL(2, \boldsymbol{R})^r$-equivariant, i.e.,

$$\varphi(xp) = \phi(x)\varphi(p) \qquad \text{for each } x \in SL(2, \boldsymbol{R})^r\,,\ p \in P_1(\boldsymbol{R})^r\,.$$

It is verified that φ is an imbedding. The homomorphism ϕ induces also an $SO(2)^r$-equivariant immersion $\varphi' : SO(2)^r/\{\pm 1\}^r \to K^0/K^*$ which is compatible with φ relative to the identifications, in the sense that the diagram:

$$\begin{array}{ccc} SL(2, \boldsymbol{R})^r/P^r & \xrightarrow{\varphi} & G/U \\ \cong\uparrow & & \uparrow\cong \\ SO(2)^r/\{\pm 1\}^r & \xrightarrow[\varphi']{} & K^0/K^* \end{array}$$

is commutative. Now $\phi(\mathfrak{o}(2)^r) = \mathfrak{a}$ implies $\varphi(P_1(\boldsymbol{R})^r) = \hat{A}$. q.e.d.

The relations (5.4) imply the following

COROLLARY. *The lattice* $\Gamma = \{H \in \mathfrak{a}\,;\, \exp H \in K^*\}$ *is given by*

$$\Gamma = \sum_{i=1}^{r} \boldsymbol{Z} U_i\,.$$

Let $r'=\mathrm{rank}(\tilde{\Pi}', \tilde{\Pi}_1')$. We define $h_i \in \mathfrak{a}$ $(1\leqslant i\leqslant r)$ by

$$2(h_i, U_j)=\delta_{ij} \quad (1\leqslant i, j\leqslant r).$$

Now irreducible symmetric R-spaces are divided into the following five classes (cf. Takeuchi [7]).

(i) $2r=r'$, $\mathfrak{c}_{\mathfrak{m}}=\{0\}$, $\pi_1(M)=\{0\}$, $\tilde{\Pi}'$ is reducible.

$$\Sigma=\{\pm(h_i\pm h_j)\ (1\leqslant i<j\leqslant r),\ \pm h_i,\ \pm 2h_i\ (1\leqslant i\leqslant r)\} \quad \text{or}$$
$$\{\pm(h_i\pm h_j)\ (1\leqslant i<j\leqslant r),\ \pm 2h_i\ (1\leqslant i\leqslant r)\}.$$

(ii) $r=r'$, $\mathfrak{c}_{\mathfrak{m}}\neq\{0\}$, $\pi_1(M)=\boldsymbol{Z}$, $\tilde{\Pi}'$ is irreducible.

$$\Sigma=\{\pm(h_i-h_j)\ (1\leqslant i<j\leqslant r)\}.$$

(iii) $2r=r'$, $\mathfrak{c}_{\mathfrak{m}}=\{0\}$, $\pi_1(M)=\{0\}$, $\tilde{\Pi}'$ is irreducible.

Σ is the same as in (i).

(iv) $r=r'$, $\mathfrak{c}_{\mathfrak{m}}=\{0\}$, $\pi_1(M)=\boldsymbol{Z}_2$, $\tilde{\Pi}'$ is irreducible.

$$\Sigma=\{\pm(h_i\pm h_j)\ (1\leqslant i<j\leqslant r),\ \pm h_i\ (1\leqslant i\leqslant r)\},$$

(v) $r=r'\geqslant 2$, $\mathfrak{c}_{\mathfrak{m}}=\{0\}$, $\pi_1(M)=\boldsymbol{Z}_2$, $\tilde{\Pi}'$ is irreducible.

$$\Sigma=\{\pm(h_i\pm h_j)\ (1\leqslant i<j\leqslant r)\}.$$

The spaces in the class (i) are nothing but irreducible hermitian symmetric spaces of compact type. The classes (ii), (iii), (iv) and (v) include $U(r)$, $Sp(r)$, $SO(2r+1)$ and $SO(2r)$ respectively.

EXAMPLE. We shall determine the stratification:

$$C=\bigcup_{(\Delta,\Phi)\in C} M^{\Delta,\Phi}$$

of the cut locus C of an irreducible symmetric R-space (M, g) in the class (ii) with $r\geqslant 2$. Compare also §2 in Part I for the notation.

Put

$$\gamma_i=h_i-h_{i+1} \quad (1\leqslant i\leqslant r-1).$$

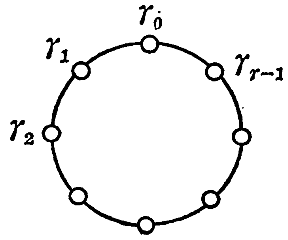
Choose an order $>$ such that Π is given by

$$\Pi=\{\gamma_1, \cdots, \gamma_{r-1}\}.$$

Then the highest root δ and $\Pi^\natural$ are given by

$$\delta=h_1-h_r=\gamma_1+\cdots+\gamma_{r-1}=-\gamma_0,$$
$$\Pi^\natural=\{\gamma_0, \gamma_1, \cdots, \gamma_{r-1}\}.$$

The diagram for $\Pi^\natural$ is given by



Any element $s \in W$ is of the form:

$$sh_i = h_{\sigma(i)} \quad (1 \leqslant i \leqslant r) \qquad \text{for some } \sigma \in \mathfrak{S}_r ,$$

where $\mathfrak{S}_r$ denotes the symmetric group acting on r letters $\{1, \cdots, r\}$. We identify W with $\mathfrak{S}_r$ by the correspondence $s \longmapsto \sigma$. $\mathfrak{a}$ will be identified with the space of r-row vectors by the correspondence $H = \sum x_i U_i \longmapsto (x_1, \cdots, x_r)$. Then

$$\mathfrak{a}_+ = \{H;\ x_1 \geqslant x_2 \geqslant \cdots \geqslant x_r\} ,$$

$$\bar{S} = \{H;\ x_1 \geqslant x_2 \geqslant \cdots \geqslant x_r,\ x_1 - x_r \leqslant 1\} ,$$

$$\mathfrak{a}' = \{H;\ \sum x_i = 0\} ,$$

$$\bar{S}' = \{H;\ x_1 \geqslant x_2 \geqslant \cdots \geqslant x_r,\ x_1 - x_r \leqslant 1,\ \sum x_i = 0\} ,$$

$$\mathfrak{c}_{\mathfrak{m}} = \{H;\ x_1 = \cdots = x_r\} .$$

Put

$$U_0 = (1/r, \cdots, 1/r) .$$

It is a basis for $\mathfrak{c}_{\mathfrak{m}}$. Projections $p_{\mathfrak{a}'}$ and $p_{\mathfrak{c}}$ are given by

$$p_{\mathfrak{a}'} :\ (x_1, \cdots, x_r) \longmapsto (x_1 - (\sum x_i)/r, \cdots, x_r - (\sum x_i)/r) ,$$

$$p_{\mathfrak{c}} :\ (x_1, \cdots, x_r) \longmapsto ((\sum x_i)/r, \cdots, (\sum x_i)/r) = (\sum x_i) U_0 .$$

Γ, Γ^* and Γ_0 are given by

$$\Gamma = \{H;\ x_i \in \boldsymbol{Z} \quad \text{for each } i\ (1 \leqslant i \leqslant r)\} ,$$

$$\Gamma^* = \{H;\ x_i - x_j \in \boldsymbol{Z} \quad \text{for each } i, j\ (1 \leqslant i, j \leqslant r)\} ,$$

$$\Gamma_0 = \{(n, \cdots, n);\ n \in \boldsymbol{Z}\} = \{nrU_0;\ n \in \boldsymbol{Z}\} .$$

For an integer p with $0 \leqslant p \leqslant r-1$, we define

$$s_p = \begin{pmatrix} 1 & \cdots & r-p & r-p+1 & \cdots & r \\ p+1 & \cdots & r & 1 & \cdots & p \end{pmatrix} \in W ,$$

$$A_p' = (1/r)(\underbrace{r-p, \cdots, r-p}_{p}, \underbrace{-p, \cdots, -p}_{r-p}) \in \bar{S}' \cap \Gamma^* ,$$

$$\tau_p' = t(A_p') s_p \in \bar{W}^*_{S'} .$$

Then

$$\pi_{\Gamma^*}(\tau_p')=A_p', \quad \pi_W(\tau_p')=s_p,$$

$$s_p\gamma_i=\gamma_{i+p} \quad \text{for } i\in \boldsymbol{Z}_r. \tag{5.5}$$

For an integer n with

$$n=mr+p \qquad m\in \boldsymbol{Z}, p\in \boldsymbol{Z} \text{ with } 0\leqslant p\leqslant r-1,$$

we define

$$A_n=(\underbrace{m+1,\cdots,m+1}_{p},\underbrace{m,\cdots,m}_{r-p})=mrU_0+(\underbrace{1,\cdots,1}_{p},\underbrace{0,\cdots 0}_{r-p})\in\bar{S}\cap\Gamma,$$

$$\tau_n=t(A_n)s_p\in\overline{W}_S.$$

Note that the sum of components of A_n is equal to n. We have

$$p_{\mathfrak{a}'}(A_n)=A_p', \quad p_{\mathfrak{c}}(A_n)=nU_0,$$

$$\pi_{\mathfrak{c}}(\tau_n)=nU_0, \quad \pi'(\tau_n)=\tau_p',$$

$$\pi_W(\tau_n)=s_p, \quad \pi_{\Gamma^*}(\tau_n)=A_n.$$

Easy computations show

$$\bar{S}\cap\Gamma=\{A_n;\, n\in\boldsymbol{Z}\},$$

$$\bar{S}'\cap\Gamma^*=\{A_p';\, 0\leqslant p\leqslant r-1\},$$

$$\bar{S}'\cap\Gamma=\{0\}.$$

Thus, by Lemma 2.4 we have

$$\overline{W}_S=\{\tau_n;\, n\in\boldsymbol{Z}\}\cong\boldsymbol{Z},$$

$$\overline{W}^*_{S'}=\{\tau_p';\, 0\leqslant p\leqslant r-1\}\cong\boldsymbol{Z}_r,$$

$$(\overline{W}_S)_*=\{1\}.$$

From $p_{\mathfrak{a}'}(\bar{S}\cap\Gamma)=\bar{S}'\cap\Gamma^*$ and Lemma 2.4 it follows

$$\boldsymbol{F}_*=\{1\},$$

$$\boldsymbol{F}=\boldsymbol{F}^*=\{s_p;\, 0\leqslant p\leqslant r-1\}\cong\boldsymbol{Z}_r.$$

As a subgroup of Aut($\Pi^\natural$), $\boldsymbol{F}$ is nothing but the group of rotations of the "ring" $\Pi^\natural$. More precisely, by (5.5), the action of $s_p\in\boldsymbol{F}$ on $\Pi^\natural$ is the p/r-times rotation of $\Pi^\natural$. $Z=p_{\mathfrak{c}}(\bar{S}\cap\Gamma)$ is given by

$$Z=\{nU_0;\, n\in\boldsymbol{Z}\}\cong\boldsymbol{Z}.$$

The injective homomorphism $\pi_{\mathfrak{c}}\times\pi_W:\, \boldsymbol{Z}=\overline{W}_S\rightarrow Z\times\boldsymbol{F}=\boldsymbol{Z}\times\boldsymbol{Z}_r$ is given by

$$(\pi_c \times \pi_W)(n) = (n, n \bmod r) \qquad \text{for } n \in \boldsymbol{Z}.$$

The action of $\bar{W}_S$ on $\bar{S} \cap \Gamma$ is given by

$$\tau_n A_{n'} = A_{n+n'} \quad \text{for } n, n' \in \boldsymbol{Z}. \tag{5.6}$$

The subsets $\mathcal{K}$ and $\mathcal{L}$ of $\bar{S}$ are given by

$$\mathcal{K} = \{H;\ 1/2 > x_1 \geqslant \cdots \geqslant x_r > -1/2\},$$

$$\mathcal{L} = \{H;\ 1/2 \geqslant x_1 \geqslant \cdots \geqslant x_r \geqslant -1/2,\ \ x_1 = 1/2 \text{ or } x_r = -1/2\}.$$

It follows

$$\Lambda = \{A_{-r}, A_{-r+1}, \cdots, A_{-1}, A_1, \cdots, A_r\},$$

$$\Theta = \phi.$$

Thus, (Δ, Φ) is a c-pair if and only if (i) (Δ, Φ) is admissible; (ii) $0 \in \Phi^c$; (iii) $\Phi^c - \{0\} \subset \{A_{-r}, \cdots, A_{-1}, A_1, \cdots, A_r\}$; (iv) $\Phi^c \neq \{0\}$. A subset $\Phi \subsetneqq \bar{S} \cap \Gamma$ satisfies (ii), (iii), (iv) and $T^{\Phi} \neq \phi$ if and only if Φ is of the form:

$$\Phi = \bar{S} \cap \Gamma - \{A_{-m}, \cdots, A_{-1}, 0, A_1, \cdots, A_l\}, \qquad \text{with } 0 \leqslant l, m \leqslant r,\ 1 \leqslant l+m \leqslant r.$$

In virtue of (5.6), we may assume that Φ is of the form:

$$\Phi_l = \bar{S} \cap \Gamma - \{0, A_1, \cdots, A_l\} \qquad \text{with } 1 \leqslant l \leqslant r.$$

In this case, T^{Φ_l} is given by

$$T^{\Phi_l} = \{H;\ 1/2 = x_1 = \cdots = x_l > x_{l+1} \geqslant \cdots \geqslant x_r > -1/2\}.$$

An admissible subset Δ of Π^* such that $S^{\Delta, \Phi_l} \neq \phi$ is of the form:

$$\Delta = \{\gamma_{i_1}, \cdots, \gamma_{i_a}, \delta\} \quad \text{with } l = i_1 < i_2 < \cdots < i_a \leqslant r-1, a \geqslant 1, \qquad \text{if } 1 \leqslant l \leqslant r-1,$$
$$\text{or } \Delta = \{\delta\}, \quad \text{if } l = r.$$

These c-pairs (Δ, Φ_l) are mutually inequivalent in virtue of (5.6). Thus the set C of all such pairs gives a set of complete representatives of equivalence classes of c-pairs. For such a pair (Δ, Φ_l), the cell S^{Δ, Φ_l} is given by

$$S^{\Delta, \Phi_l} = \{H;\ 1/2 = x_1 = \cdots = x_l > x_{l+1} = \cdots$$
$$\cdots = x_{i_2} > x_{i_2+1} = \cdots = x_{i_a} > x_{i_a+1} = \cdots = x_r > -1/2\} \quad \text{if } 1 \leqslant l \leqslant r-1,$$
$$\dim S^{\Delta, \Phi_l} = a,$$

$$\text{or } S^{\Delta, \Phi_l} = \{H;\ x_1 = \cdots = x_r = 1/2\} \quad \text{if } l = r,$$
$$\dim S^{\Delta, \Phi_l} = 0.$$

By Lemma 3.7, we have $\bar{N}^{\Delta, \Phi_l} = \{1\}$, and hence $\bar{W}^{\Delta, \Phi_l} = \{1\}$. Therefore

$$M^{\Delta,\Phi_l}\approx K^*/Z^{\Delta,\Phi_l}\times S^{\Delta,\Phi_l}\,.$$

The all cells $S^{\Delta,\Phi}$ with $(\Delta,\Phi)\in C$ may be described in a unified manner as follows:

$$S^{(i_1,\cdots,i_a)}=\{H;\ 1/2=x_1=\cdots=x_{i_1}>x_{i_1+1}=\cdots\quad\cdots=x_{i_a}>x_{i_a+1}=\cdots=x_r>-1/2\}\,,$$

where

$$1\leqslant i_1<\cdots<i_a\leqslant r-1\,,\quad 0\leqslant a\leqslant r-1\,,$$
$$\dim S^{(i_1,\cdots,i_a)}=a\,.$$

The subset $\bar{S}^0$ of $\bar{S}$ defined in Remark following Theorem 3.3 is given by

$$\bar{S}^0=\{H;\ 1/2>x_1\geqslant\cdots\geqslant x_r>-1/2\}\,.$$

These results also hold for the case $r=1$, as is easily verified.

The cells $S^{\Delta,\Phi}$ with $(\Delta,\Phi)\in C$ and the subset $\bar{S}^0$ of $\bar{S}$ for the spaces in the other classes are given as follows.

Classes (i), (iii) and (iv).

$$S^{(i_1,\cdots,i_a)}=\{H;\ 1/2=x_1=\cdots=x_{i_1}>x_{i_1+1}=\cdots\quad\cdots=x_{i_a}>x_{i_a+1}=\cdots=x_r=0\}\,,$$

where

$$1\leqslant i_1<\cdots<i_a\leqslant r\,,\quad 1\leqslant a\leqslant r\,.$$
$$\dim S^{(i_1,\cdots,i_a)}=a-1\,.$$
$$\bar{S}^0=\{H;\ 1/2>x_1\geqslant\cdots\geqslant x_r\geqslant 0\}\,.$$

Class (v).

$$1\leqslant i_1<\cdots<i_a\leqslant r\,,\quad 1\leqslant a\leqslant r\,,$$

$$S^{(i_1,\cdots,i_a)}=\begin{cases}\{H;\ 1/2=x_1=\cdots=x_{i_1}>x_{i_1+1}=\cdots=x_{i_a}>x_{i_a+1}=\cdots=x_r=0\}\,, & \text{if } 1\leqslant i_a\leqslant r-2\,,\\ \{H;\ 1/2=x_1=\cdots=x_{i_1}>x_{i_1+1}=\cdots=x_{i_{a-1}}>x_{i_{a-1}+1}=\cdots=x_r>0\}\,, & \text{if } i_a=r-1\,,\\ \{H;\ 1/2=x_1=\cdots=x_{i_1}>x_{i_1+1}=\cdots=x_{i_{a-1}}>x_{i_{a-1}+1}=\cdots=x_{r-1}>|x_r|\}\,, & \text{if } i_a=r\,.\end{cases}$$

$$\dim S^{(i_1,\cdots,i_a)}=a-1\,.$$
$$\bar{S}^0=\{H;\ 1/2>x_1\geqslant\cdots\geqslant x_{r-1}\geqslant|x_r|\}\,.$$

Seeing these tables, we have the other decomposition of the cut locus as follows.

LEMMA 5.3. *For an integer l with $0\leqslant l\leqslant r$, we define subsets $\bar{S}^l$ of $\bar{S}$ as follows.*

Classes (i), (iii) *and* (iv).

$$\bar{S}^l=\{H;\ 1/2=x_1=\cdots=x_l>x_{l+1}\geqslant\cdots\geqslant x_r\geqslant 0\}\,.$$

Class (ii).

$$\bar{S}^l=\{H;\ 1/2=x_1=\cdots=x_l>x_{l+1}\geqslant\cdots\geqslant x_r>-1/2\}\,.$$

Class (v).

$$\bar{S}^l=\{H;\ 1/2=x_1=\cdots=x_l>x_{l+1}\geqslant\cdots\geqslant x_{r-1}\geqslant|x_r|\}\,.$$

Then

$$C=\bigcup_{l=1}^{r} K^*\operatorname{Exp}\bar{S}^l \quad \text{(disjoint union).}$$

The interior M^0 is given by

$$M^0=K^*\operatorname{Exp}\bar{S}^0\,.$$

REMARK. Lemma 5.3 implies the results of Naitoh [4] on the structure of $\tilde{C}\cap\mathfrak{a}$ for an irreducible symmetric R-space.

We define a closed subgroup $\bar{U}$ of G by

$$\bar{U}=\tau' U\,.$$

We shall investigate the structures of $\bar{U}$-orbits in M.

LEMMA 5.4. *Let W_1' be the subgroup of W' generated by symmetries s_γ with $\gamma\in\Sigma_1'$. Let $\{\beta_1,\cdots,\beta_r\}\subset\Sigma_+'-\Sigma_1'$ be the maximal system of strongly orthogonal roots. Put*

$$\mathfrak{A}'^-=\{\beta_1,\cdots,\beta_r\}_{\boldsymbol{R}}\,,$$

and denote by $\varpi:\mathfrak{A}'\to\mathfrak{A}'^-$ the orthogonal projection. Then:

1) *We have $|W_1'\backslash W'/W_1'|=r+1$. More precisely, if we define*

$$s_l=s_{\beta_1}\cdots s_{\beta_l}\quad(0\leqslant l\leqslant r)\,,$$

then $\{s_0,s_1,\cdots,s_r\}$ is a set of complete representatives of $W_1'\backslash W'/W_1'$;

2) *For each l with $0\leqslant l\leqslant r$, we have*

$$s_l\zeta=\zeta-\zeta_l\,,$$

where

$$\zeta_l=\sum_{i=1}^{l}(2/(\beta_i,\beta_i))\beta_i=2\sum_{i=1}^{l}A_{\beta_i}\,;$$

3) $\varpi(\Sigma_+'-\Sigma_1')$ *and* $\varpi((\Sigma_1')_+)-\{0\}$ *are given by*

$$\begin{cases}\varpi(\Sigma_+'-\Sigma_1')=\{(1/2)(\beta_i+\beta_j)\ (1\leqslant i\leqslant j\leqslant r)\}, \\ \varpi((\Sigma_1')_+)-\{0\}=\{(1/2)(\beta_i-\beta_j)\ (1\leqslant i<j\leqslant r)\}\end{cases}$$

or

$$\begin{cases}\varpi(\Sigma_+'-\Sigma_1')=\{(1/2)(\beta_i+\beta_j)\ (1\leqslant i\leqslant j\leqslant r),\ (1/2)\beta_i\ (1\leqslant i\leqslant r)\}, \\ \varpi((\Sigma_1')_+)-\{0\}=\{(1/2)(\beta_i-\beta_j)\ (1\leqslant i<j\leqslant r),\ (1/2)\beta_i\ (1\leqslant i\leqslant r)\}\end{cases}$$

PROOF. 1) and 2). See Takeuchi [8].

3) See Takeuchi [9]. q.e.d.

COROLLARY. *We have*

$$(\gamma,\zeta_l)=\begin{cases}0 \text{ or } 1 & \text{if } \gamma\in(\Sigma_1')_+, \\ 0,1 \text{ or } 2 & \text{if } \gamma\in\Sigma_+'-\Sigma_1'.\end{cases}$$

LEMMA 5.5. 1) *For each* $s\in W'$, *there exist uniquely* $s'\in W_1'$ *and* $s''\in W'$ *with* $s''^{-1}(\Sigma_1')_+\subset\Sigma_+'$, *such that* $s=s's''$.

2) *For each* $s\in W'$, *there exist uniquely* $s'\in W_1'$ *and* $s''\in W'$ *with* $s''(\Sigma_1')_+\subset\Sigma_+'$, *such that* $s=s''s'$.

PROOF. 1) Put

$$W'^1=\{s\in W';\ s^{-1}(\Sigma_1')_+\subset\Sigma_+'\}.$$

Then the map $W_1'\times W'^1\longrightarrow W'$ defined by $(s',s'')\longmapsto s's''$ is a bijection (Kostant [3]). This implies the assertion 1).

2) We have

$$(W'^1)^{-1}=\{s\in W';\ s(\Sigma_1')_+\subset\Sigma_+'\}.$$

On the other hand, $W'=(W'^1)^{-1}W_1'$ by 1). These imply the assertion 2). q.e.d.

Let $s\in W'$ and let A and B be subgroups of G, one of which contains the centralizer $Z_K(\mathfrak{A}')$ of $\mathfrak{A}'$ in K. If we take an element k in the normalizer $N_K(\mathfrak{A}')$ of $\mathfrak{A}'$ in K such that $Adk|\mathfrak{A}'=s$, then the double coset AkB is determined by s and independent of the choice of k. So this double coset will be denoted by AsB. Note that both U and $\bar{U}$ contain $Z_K(\mathfrak{A}')$.

LEMMA 5.6. *We have*

$$G=\bigcup_{l=0}^{r}\bar{U}s_lU \quad \text{(disjoint union)}.$$

PROOF. Put

$$\mathfrak{n}=\sum_{\gamma\in\Sigma_+'}\mathfrak{g}_\gamma , \quad N=\exp\mathfrak{n} ,$$

$$B=\{x\in G ; \operatorname{Ad}x\mathfrak{n}=\mathfrak{n}\} .$$

Then we have the Bruhat decomposition (Harish-Chandra [2]):

$$G=\bigcup_{s\in W'}BsB=\bigcup_{s\in W'}NsB \quad \text{(disjoint).}$$

Take $k_0\in N_K(\mathfrak{A}')$ such that $\operatorname{Ad}k_0|\mathfrak{A}'=w_0$ satisfies $w_0\Sigma_+'=\Sigma_-'$. Putting

$$\bar{N}=\tau'N, \quad \bar{B}=\tau'B ,$$

we have

$$\bigcup_{s\in W'}k_0NsB=\bigcup_{s\in W'}\bar{N}w_0sB=\bigcup_{s\in W'}\bar{N}sB ,$$

and hence

$$G=\bigcup_{s\in W'}\bar{N}sB=\bigcup_{s\in W'}\bar{B}sB \quad \text{(disjoint).} \tag{5.7}$$

In the same way, put

$$\mathfrak{n}_1=\sum_{\gamma\in(\Sigma_1')_+}\mathfrak{g}_\gamma , \quad N_1=\exp\mathfrak{n}_1 ,$$

$$B_1=\{x\in G_1 ; \operatorname{Ad}x\mathfrak{n}_1=\mathfrak{n}_1\} ,$$

$$\bar{N}_1=\tau'N_1 , \quad \bar{B}_1=\tau'B_1 .$$

Then we have also decompositions:

$$G_1=\bigcup_{w\in W_1'}\bar{N}_1wB_1=\bigcup_{w\in W_1'}\bar{B}_1wB_1 \quad \text{(disjoint).}$$

Thus

$$U=G_1N^+=\bigcup_{w\in W_1'}\bar{N}_1wB_1N^+=\bigcup_{w\in W_1'}\bar{N}_1wB=\bigcup_{w\in W_1'}\bar{B}_1wB , \tag{5.8}$$

and hence

$$\bar{U}=\bigcup_{w\in W_1'}\bar{B}wN_1 \quad \text{(disjoint).} \tag{5.9}$$

Now, for each $s\in W'$ we have

$$\bigcup_{w\in W_1'sW_1'}\bar{B}wB\subset\bar{U}sU .$$

Thus it suffices to show

$$\bar{U}sU\subset\bigcup_{w\in W_1'sW_1'}\bar{B}wB .$$

In fact, then, from (5.7) and Lemma 5.4, 1) we get the required decomposition.

Decompose s by Lemma 5.5, 2) as

$$s=s'w_1 \qquad w_1\in W_1', \quad s'(\Sigma_1')_+\subset\Sigma_+'.$$

Then, by (5.8)

$$\bar{U}sU=\bar{U}s'U=\bigcup_{w\in W_1'}\bar{U}s'\bar{N}_1wB=\bigcup_{w\in W_1'}\bar{U}s'wB.$$

Decompose $s'w$ by Lemma 5.5, 1) as

$$s'w=w_2s'' \qquad w_2\in W_1', \quad s''^{-1}(\Sigma_1')_+\subset\Sigma_+'.$$

Then, by (5.9)

$$\bar{U}s'wB=\bar{U}s''B=\bigcup_{w'\in W_1'}\bar{B}w'N_1s''B=\bigcup_{w'\in W_1'}\bar{B}w's''B,$$

where

$$w's''=w'w_2^{-1}s'w=w'w_2^{-1}w_1^{-1}sw\in W_1'sW_1'.$$

Thus we get the required inclusion. q.e.d.

For each l with $0\leqslant l\leqslant r$, we define

$$\Sigma'^{(l)}=\{\gamma\in\Sigma'\,;\,(\gamma,\zeta_l)=0\},$$

$$\Sigma_\pm'^{(l)}=\Sigma'^{(l)}\cap\Sigma_\pm',$$

$$\Xi_l'=\{\gamma\in\Sigma_-'-\Sigma_1'\,;\,(\gamma,\zeta_l)<0\},$$

$$\Sigma_1'^{(l)}=\Sigma'^{(l)}\cap\Sigma_1'.$$

Then we have

$$\Sigma_1'=\Sigma_1'^{(l)}\cup((\Sigma_1')_--\Sigma'^{(l)})\cup((\Sigma_1')_+-\Sigma'^{(l)}).$$

Note also that by Corollary of Lemma 5.4 we have

$$\Sigma_-'-\Sigma_1'=(\Sigma_-'^{(l)}-\Sigma_1'^{(l)})\cup\Xi_l'. \tag{5.10}$$

We define subalgebras of $\mathfrak{g}$ and closed subgroups of G by

$$\bar{\mathfrak{n}}^+=\sum_{\gamma\in\Sigma_-'-\Sigma_1'}\mathfrak{g}_\gamma, \qquad \bar{N}^+=\exp\bar{\mathfrak{n}}^+,$$

$$\bar{\mathfrak{n}}_l'=\sum_{\gamma\in\Sigma_-'^{(l)}-\Sigma_1'^{(l)}}\mathfrak{g}_\gamma, \qquad \bar{N}_l'=\exp\bar{\mathfrak{n}}_l',$$

$$\bar{\mathfrak{n}}_l''=\sum_{\gamma\in\Xi_l'}\mathfrak{g}_\gamma, \qquad \bar{N}_l''=\exp\bar{\mathfrak{n}}_l'',$$

$$\bar{\mathfrak{n}}_1^{(l)}=\sum_{\gamma\in(\Sigma_1')_--\Sigma_1'^{(l)}}\mathfrak{g}_\gamma, \qquad \bar{N}_1^{(l)}=\exp\bar{\mathfrak{n}}_1^{(l)},$$

$$\bar{\mathfrak{u}}_1^{(l)}=\mathfrak{g}_0+\sum_{\gamma\in\Sigma_1'^{(l)}\cup(\Sigma_1')_-}\mathfrak{g}_\gamma,$$

$$\bar{U}_1^{(l)}=\{x\in G_1\,;\,\mathrm{Ad}x\bar{\mathfrak{n}}_1^{(l)}=\bar{\mathfrak{n}}_1^{(l)}\}.$$

These subgroups have Lie algebras denoted by corresponding German small letters.

It follows from (5.10) that $\bar{\mathfrak{n}}^+ = \bar{\mathfrak{n}}_l' + \bar{\mathfrak{n}}_l''$, and hence

$$\bar{N}^+ = \bar{N}_l' \cdot \bar{N}_l'' .$$

$\bar{U}_1^{(l)}$ leaves $\bar{\mathfrak{n}}^+$ and $\bar{\mathfrak{n}}_l''$ invariant under the adjoint action, and hence it acts linearly on $\bar{\mathfrak{n}}_l' \cong \bar{\mathfrak{n}}^+/\bar{\mathfrak{n}}_l''$ in the natural way. The action of $u \in \bar{U}_1^{(l)}$ will be denoted by

$$X \longmapsto u \cdot X \qquad \text{for } X \in \bar{\mathfrak{n}}_l' .$$

Moreover, we put

$$G_1^{(l)} = \{x \in G_1 ; \operatorname{Ad} x\, \zeta_l = \zeta_l\} ,$$

$$K_1^{(l)} = K_1 \cap G_1^{(l)} .$$

Then we have a semi-direct decomposition:

$$\bar{U}_1^{(l)} = G_1^{(l)} \cdot \bar{N}_1^{(l)} .$$

From Corollary of Lemma 5.4, $G_1/\bar{U}_1^{(l)}$ is also a symmetric R-space and identified with $K_1/K_1^{(l)}$. Let $G_1 \times_{\bar{U}_1^{(l)}} \bar{\mathfrak{n}}_l'$ denote the vector bundle over $G_1/\bar{U}_1^{(l)}$ associated to the principal bundle $\bar{U}_1^{(l)} \to G_1 \to G_1/\bar{U}_1^{(l)}$ by the above action of $\bar{U}_1^{(l)}$ on $\bar{\mathfrak{n}}_l'$. It is also identified with $K_1 \times_{K_1^{(l)}} \bar{\mathfrak{n}}_l'$ in the natural way.

We put

$$a_l = \exp (1/2) \sum_{i=1}^{l} U_i \quad (0 \leqslant l \leqslant r) .$$

Under the notation in the proof of Lemma 5.2, we have $\operatorname{Ad}(\exp(1/2)U_0)H_0 = -H_0$. Moreover, if we denote by $\mathfrak{A}'^+$ the orthogonal complement of $\mathfrak{A}'^-$ in $\mathfrak{A}'$, then $[U_i, \mathfrak{A}'^+] = \{0\}$ for each i. It follows from these and (5.4) that $a_l \in N_K(\mathfrak{A}')$ and $\operatorname{Ad} a_l | \mathfrak{A}' = s_l$ for each l. We define submanifolds B_l and V_l of M by

$$B_l = G_1 a_l o , \quad V_l = \bar{U} a_l o \quad (0 \leqslant l \leqslant r) .$$

LEMMA 5.7. 1) *B_l is diffeomorphic with the symmetric R-space $G_1/\bar{U}_1^{(l)}$.*

2) *A C^∞ map Ψ^l: $G_1 \times \bar{\mathfrak{n}}_l' \to M$ defined by*

$$\Psi^l(x, X) = x \exp X a_l o \qquad \textit{for } x \in G_1 ,\ X \in \bar{\mathfrak{n}}_l'$$

induces a diffeomorphism ψ^l: $G_1 \times_{\bar{U}_1^{(l)}} \bar{\mathfrak{n}}_l' \to V_l$.

Thus, V_l is diffeomorphic with a C^∞ vector bundle over B_l.

PROOF. Let

$$E_l = \{x \in \bar{U} ; x a_l o = a_l o\} = \bar{U} \cap a_l U a_l^{-1}$$

be the stabilizer in $\bar{U}$ at the point $a_l o$. Then the Lie algebra of E_l is given by

$$\mathfrak{e}_l=\bar{\mathfrak{u}}\cap \mathrm{Ad}a_l\mathfrak{u}=\mathfrak{g}_0+\sum_{\gamma}\mathfrak{g}_\gamma\,,$$

where in the summation γ ranges over all roots $\gamma\in\Sigma'$ such that

$$(5.11)\qquad (\gamma,\zeta)\leqslant 0 \quad\text{and}\quad (\gamma, s_l\zeta)\geqslant 0\,.$$

If $\gamma\in\Sigma_1'$, i.e., $(\gamma,\zeta)=0$, then by Lemma 5.4, 2) the condition (5.11) becomes

$$(\gamma, s_l\zeta)=(\gamma,\zeta-\zeta_l)=-(\gamma,\zeta_l)\geqslant 0\,,$$

or equivalently, $\gamma\in\Sigma_1'^{(l)}\cup(\Sigma_1')_-$. If $\gamma\in\Sigma_-'-\Sigma_1'$, i.e., $(\gamma,\zeta)=-1$, then (5.11) becomes

$$(\gamma, s_l\zeta)=(\gamma,\zeta-\zeta_l)=-1-(\gamma,\zeta_l)\geqslant 0\,,$$

or equivalently, $\gamma\in\Xi_l'$. Thus we get

$$(5.12)\qquad \mathfrak{e}_l=\bar{\mathfrak{u}}_1^{(l)}+\bar{\mathfrak{n}}_l''\,,$$

where

$$\bar{\mathfrak{u}}_1^{(l)}=\mathfrak{e}_l\cap\mathfrak{g}_1\,,\quad \bar{\mathfrak{n}}_l''=\mathfrak{e}_l\cap\bar{\mathfrak{n}}^+\,.$$

Now we proceed to the proof of the lemma.

1) It suffices to show $E_l\cap G_1=\bar{U}_1^{(l)}$. For each $x\in G_1^{(l)}$, we have

$$\begin{aligned}(\mathrm{Ad}x)(\mathrm{Ad}a_l\zeta)&=\mathrm{Ad}x(s_l\zeta)=\mathrm{Ad}x(\zeta-\zeta_l)\\&=\zeta-\mathrm{Ad}x\zeta_l=\zeta-\zeta_l=\mathrm{Ad}a_l\zeta\,,\end{aligned}$$

and hence $x\in a_lUa_l^{-1}$. Thus $G_1^{(l)}\subset E_l\cap G_1$. It is clear that $\bar{N}_1^{(l)}\subset E_l\cap G_1$. Hence $\bar{U}_1^{(l)}=G_1^{(l)}\bar{N}_1^{(l)}\subset E_l\cap G_1$. Since a parabolic subgroup of a Zariski-connected real reductive algebraic group coincides with the normalizer of its identity component, we get $U_1^{(l)}=E_l\cap G_1$.

2) Note that $E_l\subset\bar{U}=G_1\cdot\bar{N}^+$. Let $\pi_1\colon \bar{U}\to G_1$ denote the projection to the first factor. It induces a homomorphism $\pi\colon E_l\to G_1$. The image $\pi(E_l)$ contains $\bar{U}_1^{(l)}=E_l\cap G_1$ and has the Lie algebra $\bar{\mathfrak{u}}_1^{(l)}$ by (5.12). By the same argument as above, we get $\pi(E_l)=\bar{U}_1^{(l)}$. Therefore we have

$$E_l=\bar{U}_1^{(l)}(E_l\cap\bar{N}^+)\,.$$

This and the decomposition: $\bar{N}^+=\bar{N}_l'\cdot\bar{N}_l''$, where $\bar{N}_l'\cap a_lUa_l^{-1}=\{1\}$ and $\bar{N}_l''\subset E_l$, imply

$$(5.13)\qquad E_l=\bar{U}_1^{(l)}\cdot\bar{N}_l''\quad\text{(semi-direct product).}$$

Now it is easily seen that the C^∞ map Ψ^l induces a C^∞ map ψ^l in virtue of $\bar{N}_l''\subset E_l$. The decomposition: $\bar{U}=G_1\cdot\bar{N}_l'\cdot\bar{N}_l''$ implies that ψ^l is surjective. We shall show that ψ^l is injective. Assume that $x\exp Xa_lo=x'\exp X'a_lo$ for $x, x'\in G_1$ and $X, X'\in\bar{\mathfrak{n}}_l'$. Put

$$y=(\exp X')^{-1}x'^{-1}x\exp X\in E_l\,,$$
$$u=x'^{-1}x\in G_1\,.$$

Then

$$y=u\exp(-\mathrm{Ad}\,u^{-1}X')\exp X=u\exp(X-\mathrm{Ad}\,u^{-1}X')\,.$$

Hence, by (5.13) we have $u\in\bar{U}_1^{(l)}$ and $X-\mathrm{Ad}\,u^{-1}X'\in\bar{\mathfrak{n}}_l''$. Thus we get $x'=xu^{-1}$ and $X'=u\cdot X$. This means that the class of (x,X) and (x',X') in $G_1\times_{\bar{U}_1^{(l)}}\bar{\mathfrak{n}}_l'$ are the same. This shows the injectivity of ψ^l. q.e.d.

REMARK. The following facts are known (Takeuchi [7]).

1) Connected components of the set of fixed points of the symmetry of (M,g) at o are $B_0, B_1, \cdots, B_r$. Hence each B_l is a totally geodesic submanifold of (M,g).

2) We can define a C^∞ function f on M by

$$f(xo)=(\mathrm{Ad}x\zeta,\zeta)\quad\text{for } x\in K\,.$$

Then $B_0, B_1, \cdots, B_r$ coincide with the totality of non-degenerate critical submanifolds of f in the sense of Bott [1]. The dimension of $\bar{\mathfrak{n}}_l$ is the same as the rank of the negative normal bundle for B_l.

We recall now cellular decompositions of symmetric R-space by generalized Schubert cells. For $s\in W'$, the coset sW_1' in W'/W_1' will be denoted by $[s]$. For an element $[s]\in W'/W_1'$, choosing $a\in N_K(\mathfrak{A}')$ such that $\mathrm{Ad}a|\mathfrak{A}'=s$, we define a *generalized Schubert cell* $V_{[s]}$ of M by

$$V_{[s]}=\bar{N}ao\,.$$

It is determined by $[s]$ and does not depend on the choice of $a\in N_K(\mathfrak{A}')$. With these definitions, we have the following

LEMMA 5.8. (Takeuchi [7])

1) $M=\bigcup_{[s]\in W'/W_1'}V_{[s]}$ (disjoint union).

2) *Let* $\Pi'=\{\alpha_1,\cdots,\alpha_\nu\}$. *Then,* $\bar{V}_{[s]}\supset V_{[s']}$ *if and only if*

$$s\zeta-s'\zeta=\sum_{i=1}^{\nu}m_i\alpha_i\quad\textit{with some}\quad m_i\geqslant 0\,.$$

LEMMA 5.9. 1) *For each l with $0\leqslant l\leqslant r$, we have a decomposition*:

$$V_l=\bigcup_{[s]\in W_1'[s_l]}V_{[s]}\quad\text{(disjoint union)}.\tag{5.14}$$

2) *The closure relations are given by*

$$\bar{V}_l\supset V_{l+1}\quad(0\leqslant l\leqslant r-1)\,.\tag{5.15}$$

PROOF. 1) In the proof of Lemma 5.6, we have shown

$$\bar{U}s_l U = \bigcup_{w \in W_1' s_l W_1'} \bar{B}wB .$$

Thus by (5.8) we get

$$\bar{U}s_l U = \bigcup_{w \in W_1' s_l} \bar{N}wU .$$

This implies (5.14).

2) Let $0 \leqslant l \leqslant l' \leqslant r$. By Lemma 5.4, 2) we have

$$s_l\zeta - s_{l'}\zeta = \zeta_{l'} - \zeta_l = \sum_{i=l+1}^{l'} (2/(\beta_i, \beta_i))\beta_i ,$$

and hence $\bar{V}_{[s_l]} \supset V_{[s_{l'}]}$ by Lemma 5.8, 2). Since both V_l and $V_{l'}$ are $\bar{U}$-orbits and since $V_l \supset V_{[s_l]}$ and $V_{l'} \supset V_{[s_{l'}]}$, we get $\bar{V}_l \supset V_{l'}$. q.e.d.

THEOREM 5.1. *Let (M, g) be an irreducible symmetric R-space of rank r. Then the number of $\bar{U}$-orbits in M is $r+1$. Let $V_0, V_1, \cdots, V_l$ be the totality of $\bar{U}$-orbits with* $\dim V_l \geqslant \dim V_{l+1}$ $(0 \leqslant l \leqslant r-1)$. *Then*:

1) *Each V_l is described by* (5.14) *in terms of generalized Schubert cells. V_0 is the unique open $\bar{U}$-orbit.*

2) *The closure relations between the $\bar{U}$-orbits V_l are given by* (5.15).

3) *Each V_l is diffeomorphic with a C^∞ vector bundle over a symmetric R-space B_l.*

4) *The cut locus C of (M, g) with respect to the origin o is given by*

$$C = \bigcup_{l=1}^{r} V_l .$$

5) *The interior M^0 of (M, g) with respect to o is given by*

$$M^0 = V_0 .$$

PROOF. By Lemma 5.6, our $V_l = \bar{U}a_l o$ $(0 \leqslant l \leqslant r)$ are the totality of $\bar{U}$-orbits in M. The numbering of the V_l is also the same as that in the theorem, in virtue of (5.15). Thus the assertions 1), 2) and 3) are the consequences of the previous lemmas. It remains to show 4) and 5).

We define a closed subgroup $\bar{P}$ of $SL(2, \boldsymbol{R})$ by

$$\bar{P} = \left\{ \begin{pmatrix} a & 0 \\ c & 1/a \end{pmatrix} ; a, c \in \boldsymbol{R}, a \neq 0 \right\} .$$

Then, under the notation in the proof of Lemma 5.2, the Lie algebra of $\bar{P}$ is spanned by H_0 and X_-, and hence $\phi(\bar{P}^r) \subset \bar{U}$. If we denote a point of $P_1(\boldsymbol{R}) =$

$SL(2, \boldsymbol{R})/P$ with homogeneous coordinate $(\xi_1 : \xi_2)$ by $\begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix}$, then

$$(\exp xU_0)P = \begin{bmatrix} 1 \\ \tan \pi x \end{bmatrix} \quad \text{for each } x \in \boldsymbol{R} \quad \text{with} \quad |x| < 1/2 .$$

Thus we have

$$\{(\exp xU_0)P;\ |x| < 1/2\} = \left\{ \begin{bmatrix} 1 \\ t \end{bmatrix};\ t \in \boldsymbol{R} \right\} .$$

On the other hand, the subgroup $\bar{P}$ acts transitively on the set of right hand side. Thus, for each $x \in \boldsymbol{R}$ with $|x| < 1/2$, there exists $g \in \bar{P}$ such that $g(\exp xU_0)P = P$. Recalling the map φ in Lemma 5.2 is $SL(2, \boldsymbol{R})^r$-equivariant, we know that for each $x_{l+1}, \cdots, x_r \in \boldsymbol{R}$ with $|x_{l+1}|, \cdots, |x_r| < 1/2$, there exists $u \in \phi(\bar{P}^r)$ such that

$$u \operatorname{Exp}(1/2, \cdots, 1/2, x_{l+1}, \cdots, x_r) = \operatorname{Exp}(1/2, \cdots, 1/2, 0 \cdots 0) = a_l o .$$

Seeing the definitions of $\bar{S}^l$ $(0 \leqslant l \leqslant r)$ in Lemma 5.3, we know that for each $p \in \operatorname{Exp}(\bar{S}^l)$ there exists $u \in \bar{U}$ such that $p = ua_l o$ $(0 \leqslant l \leqslant r)$. Considering the inclusions $K^* \subset K_1 \subset G_1 \subset \bar{U}$, we get

$$K^* \operatorname{Exp} \bar{S}^l \subset \bar{U} a_l o = V_l \qquad \text{for each} \quad l .$$

It follows from Lemma 5.3 that

$$K^* \operatorname{Exp} \bar{S}^l = V_l \qquad \text{for each } l ,$$

and

$$C = \bigcup_{l=1}^{r} V_l , \quad M^0 = V_0 . \qquad \text{q.e.d.}$$

REMARK. Theorem 5.1 includes the results on cut loci of Grassmann manifolds by Wong [12], [13] and those on cut loci of $U(r)/O(r)$, $U(r)$, $SO(n)$ and $U(2r)/Sp(r)$ by Sakai [5], [6].

References

[1] Bott, R., The stable homotopy of the classical groups, Ann. Math. **70** (1959), 313-337.
[2] Chandra-Harish: On a lemma of Bruhat, J. Math. Pur. Appl. **35** (1956), 203-210.
[3] Kostant, B., Lie algebra cohomology and the generalized Borel-Weil theorem, Ann. Math. **74** (1961), 329-387.
[4] Naitoh, H., On cut loci and first conjugate loci of the irreducible symmetric R-spaces and the irreducible compact hermitian symmetric spaces, to appear.
[5] Sakai, T., On the geometry of manifold of Lagrangean subspaces of a symplectic vector space, to appear.
[6] ———: On cut loci of compact symmetric spaces, to appear.

[7] Takeuchi, M., Cell decompositions and Morse equalities on certain symmetric spaces, J. Fac. Sci. Univ. Tokyo I, **12** (1965), Part 1, 81-192.
[8] ———: On orbits in a compact hermitian symmetric space, Amer. J. Math. **90** (1968), 657-680.
[9] ———: Polynomial representations associated with symmetric bounded domains, Osaka J. Math. **10** (1973), 441-475.
[10] ———: Homogeneous Kähler submanifolds in complex projective spaces, to appear.
[11] Warner, F.W., Conjugate loci of constant order, Ann. Math. **86** (1967), 192-212.
[12] Wong, Y.C., Differential geometry of Grassmann manifolds, Proc. Nat. Acad. Sci. U.S.A. **57** (1967), 589-594.
[13] ———: A class of Schubert varieties, J. Diff. Geom. **4** (1970), 37-51.

Osaka University
Machikaneyama, Toyonaka 560
Japan