TSUKUBA J. MATH.
Vol. 28 No. 1 (2004), 223–243

PARALLEL CURVED SURFACES

By

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Abstract. A surface S in $\boldsymbol{R}^3$ is called *parallel curved* if there exists a plane such that at each point of S, there exists a principal direction parallel to this plane. In [2], we studied real-analytic, parallel curved surfaces and in particular, we showed that a connected, complete, real-analytic, embedded, parallel curved surface is homeomorphic to a sphere, a plane, a cylinder, or a torus. In the present paper, we shall show that a connected, complete, embedded, parallel curved surface such that any umbilical point is isolated is also homeomorphic to a sphere, a plane, a cylinder or a torus. However, we shall also show that for each non-negative integer $g \in \boldsymbol{N} \cup \{0\}$, there exists a connected, compact, orientable, embedded, parallel curved surface of genus g.

1. Introduction

A surface S in $\boldsymbol{R}^3$ is called *parallel curved* if there exists a plane P such that at each point of S, there exists a principal direction parallel to P; if S is parallel curved, then such a plane as P is called a *base plane* of S. For example, a surface of revolution is a parallel curved surface such that a plane normal to an axis of rotation is its base plane.

Let F be a smooth function defined on a connected neighborhood of $(0,0)$ in $\boldsymbol{R}^2$ satisfying

$$F(0,0) = \frac{\partial F}{\partial x}(0,0) = \frac{\partial F}{\partial y}(0,0) = 0$$

and the condition that the graph G_F of F is a parallel curved surface such that the xy-plane is its base plane. In [2], we studied real-analytic, parallel curved

Key Words and Phrases. parallel curved surface, base plane, surface of revolution, generating pair.
2000 *Mathematics Subject Classification.* Primary 53A05; Secondary 53A99, 53B25.
Received January 10, 2003.
Revised September 12, 2003.

surfaces. If F is real-analytic, then we proved the following: if the origin o of $\boldsymbol{R}^3$ is an isolated umbilical point of G_F, then G_F is part of a surface of revolution such that o lies on the axis of rotation; if o is not any isolated umbilical point of G_F, then one of the following (a) and (b) happens:

(a) G_F is part of a plane or a round sphere;
(b) There exist a neighborhood U_o of $(0,0)$ in $\boldsymbol{R}^2$ and a real-analytic curve C_0 in U_o satisfying
  (i) C_0 is the set of the zero points of F in U_o,
  (ii) The set of the umbilical points of the graph of $F|_{U_o}$ is empty or given by C_0,
  (iii) For any point $q \in C_0$ and the plane $P_q^\perp$ in $\boldsymbol{R}^3$ normal to C_0 at q, the intersection $C_q^\perp$ of $P_q^\perp$ with the graph of $F|_{U_o}$ is a curve such that at any point of $C_q^\perp$, the tangent line to $C_q^\perp$ is a principal direction of G_F.

In addition, we proved that a connected, complete, real-analytic, embedded and parallel curved surface is homeomorphic to a sphere, a plane, a cylinder or a torus.

The purpose of the present paper is to study parallel curved surfaces which are not always real-analytic. Suppose that F is not always real-analytic. We shall prove the following:

THEOREM 1.1. *If o is an isolated umbilical point of G_F, then G_F is part of a surface of revolution such that o lies on the axis of rotation.*

THEOREM 1.2. *Suppose the following: o is not any isolated umbilical point of G_F; not all the partial derivatives of F at $(0,0)$ are equal to zero. Then G_F is part of a surface of revolution such that o lies on the axis of rotation, or there exist a neighborhood U_o of $(0,0)$ in $\boldsymbol{R}^2$ and a curve C_0 in U_o satisfying such conditions as the above-mentioned (i)$\sim$(iii).*

THEOREM 1.3. *If all the partial derivatives of F at $(0,0)$ are equal to zero, then it is possible that F satisfies the following conditions: G_F is not part of any surface of revolution; there does not exist any curve in $\boldsymbol{R}^2$ through $(0,0)$ on which $F \equiv 0$.*

In addition, we shall prove the following:

THEOREM 1.4. *A connected, complete, embedded, parallel curved surface such that any umbilical point is isolated is homeomorphic to a sphere, a plane, a cylinder or a torus.*

THEOREM 1.5. *For each non-negative integer $g \in \boldsymbol{N} \cup \{0\}$, there exists a connected, compact, orientable, embedded, parallel curved surface of genus g.*

REMARK 1.6. We easily see that there exists a principal direction parallel to the xy-plane at a point of the graph of a smooth function of two variables if and only if its gradient vector field is in a principal direction at the same point. Therefore we see in particular that the gradient vector field of F is in a principal direction at each point of G_F. We found the class of parallel curved surfaces in studying the graph of a real-analytic function such that its gradient vector field is in a principal direction at each point. We often studied relations between the behavior of the principal distributions and the behavior of the gradient vector field. The gradient vector field of a nonzero homogeneous polynomial g of degree $k \geqq 2$ in two variables is in a principal direction of its graph at a point if and only if at the same point, one of the following happens: the gradient vector field is represented by the "position vector field" $x\partial/\partial x + y\partial/\partial y$ up to a constant; the Gaussian curvature of the graph is equal to zero ([1]). In particular, we see that if the gradient vector field of g is in a principal direction at each point of its graph, then g is represented as $g = \lambda_1(x^2 + y^2)^l$ or $g = \lambda_2(\alpha x + \beta y)^k$, where $\lambda_i \in \boldsymbol{R}\backslash\{0\}$, $(\alpha, \beta) \in \boldsymbol{R}^2\backslash\{(0,0)\}$ and $l \in \boldsymbol{N}$. The former (respectively, latter) type is the simplest one which appears in Theorem 1.1 (respectively, Theorem 1.2). In [1], we studied relations between the behavior of the principal distributions and the behavior of the gradient vector field of a homogeneous polynomial g on its graph in the case where g is of none of the above-mentioned two types. As we saw in [2], if F is real-analytic and nonzero, then the behavior of its gradient vector field around o is given by either the position vector field or the set of curves $\{C_q^{\perp}\}_{q \in C_0}$ as in the above-mentioned (iii). Theorem 1.1 says that if o is an isolated umbilical point of G_F, then the assumption that F is real-analytic is removable; Theorem 1.2 says that even if o is not any isolated umbilical point, if not all the partial derivatives of F at $(0,0)$ are equal to zero, then the assumption that F is real-analytic is also removable; on the other hand, Theorem 1.3 says that if all the partial derivatives of F at $(0,0)$ are equal to zero, then there exists a type which does not appear in the real-analytic case. In [3], we studied the behavior of the principal distributions around an isolated umbilical point on a real-analytic surface. We may grasp the behavior of the principal distributions in most cases in the way of studying the limit of each principal distribution toward the isolated umbilical point along the intersection of the surface with each normal plane at this point. However, there exist cases in which we may not grasp the behavior in only such a way. Then adding the way of studying the behavior of the principal distributions in relation to the behavior of the gradient vector field of a function the graph of which is a neighborhood of the isolated umbilical point in the surface, we were able to grasp the behavior of the principal distributions in some case (see [3]). In [4], we described a similar discussion on the graph of a smooth function with such coefficients as nonzero real-analytic functions have in Taylor's formula. Let f be a smooth function on a neighborhood of $(0,0)$ in $\boldsymbol{R}^2$ satisfying $f(0,0)=0$ and $f>0$ on a punctured neighborhood of $(0,0)$. Then $\exp(-1/f)$ is a smooth function defined on a punctured neighborhood of $(0,0)$ and smoothly extended to $(0,0)$ so that all the partial derivatives of $\exp(-1/f)$ at $(0,0)$ are equal to zero. If for each positive number $c>0$, there exists a punctured neighborhood of $(0,0)$ on which the norm of the gradient vector field of $\log f$ is bounded from below by the number c, then we showed in [5] that o is an isolated umbilical point on the graph of $\exp(-1/f)$ and that around o, a principal distribution is approximated by (the distribution defined by) the gradient vector field of $\exp(-1/f)$ on its graph. For example, if there exists a homogeneous polynomial g of degree k in two variables satisfying $g>0$ on $\boldsymbol{R}^2\backslash\{(0,0)\}$ and

$$f=g+o((x^2+y^2)^{k/2}),$$

then f satisfies the assumption. If the graph of f is locally strictly convex at any point, then f also satisfies the assumption. Hence in the set of the smooth functions such that the values and all the partial derivatives at $(0,0)$ are equal to zero, we may find many examples for each of which, o is an isolated umbilical point on its graph such that there exists a principal distribution approximated by the gradient vector field around o. On the other hand, it is exceptional that around an isolated umbilical point on the graph of a real-analytic function, a principal distribution is approximated by the gradient vector field.

REMARK 1.7. Let C_b, C_g be simple curves in $\boldsymbol{R}^3$ with a unique intersection $p_{(C_b,C_g)}$ and contained in planes P_b, P_g, respectively. Then a pair (C_b, C_g) is called *generating* if we may choose as P_g the plane normal to C_b at $p_{(C_b,C_g)}$; if (C_b, C_g) is generating, then C_b and C_g are called the *base curve* and the *generating curve* of (C_b, C_g), respectively. For a generating pair (C_b, C_g), let $\Sigma_{(C_b,C_g)}$ be the set of the embedded, parallel curved surfaces such that each $S\in\Sigma_{(C_b,C_g)}$ satisfies the following:

(a) P_b is a base plane of S;
(b) A surface S contains a neighborhood O_b (respectively, O_g) of $p_{(C_b,C_g)}$ in C_b (respectively, C_g) so that the tangent line at each point is a principal direction of S.

In [2], we proved $\Sigma_{(C_b,C_g)} \neq \emptyset$ if (C_b, C_g) is a generating pair such that C_b and C_g are real-analytic. If for an embedded, parallel curved surface S and a point $p \in S$, there exists a generating pair (C_b, C_g) satisfying $p = p_{(C_b,C_g)}$ and $S \in \Sigma_{(C_b,C_g)}$, then S is called *generated* at p (by (C_b, C_g)). If S is an embedded surface of revolution which has the only one axis of rotation, then S is generated at a point which does not lie on the axis and S is not generated at any point on the axis. In [2], we showed that if S is a real-analytic, embedded and parallel curved surface and if S is not part of any surface of revolution, then S is generated at any point. In the present paper, we shall see that $\Sigma_{(C_b,C_g)} \neq \emptyset$ holds, even if (C_b, C_g) is a generating pair such that C_b and C_g are not always real-analytic and that if for a function F as in Theorem 1.2, G_F is not part of any surface of revolution, then G_F is generated at any point of a neighborhood of o in G_F. However, we shall also see that there exists an embedded, parallel curved surface which is neither part of any surface of revolution nor generated at some point (we shall see that the example we shall give implies Theorem 1.3).

REMARK 1.8. Such results as Theorems 1.3 and 1.5 do not hold in the real-analytic case. We shall see that not only the proof of Theorem 1.3 but also the proof of Theorem 1.5 depends on the existence of non-constant smooth functions such that all the partial derivatives at some point are equal to zero.

Acknowledgement

The author would like to express his cordial gratitude to the referee for his comments and suggestions. This work was supported by the Japan Society for the Promotion of Science.

2. Preliminaries

Let f be a smooth function of two variables x, y and G_f the graph of f. We set $p_f := \partial f/\partial x$, $q_f := \partial f/\partial y$ and

$$E_f := 1 + p_f^2, \quad F_f := p_f q_f, \quad G_f := 1 + q_f^2.$$

The *first fundamental form* of G_f is a symmetric tensor field I_f on G_f of type $(0,2)$ represented in terms of the coordinates (x, y) as

$$\mathrm{I}_f := E_f\, dx^2 + 2F_f\, dxdy + G_f\, dy^2,$$

where

$$dx^2 := dx \otimes dx, \quad dxdy := \frac{1}{2}(dx \otimes dy + dy \otimes dx), \quad dy^2 := dy \otimes dy.$$

We set $r_f := \partial^2 f/\partial x^2$, $s_f := \partial^2 f/\partial x \partial y$, $t_f := \partial^2 f/\partial y^2$ and

$$L_f := \frac{r_f}{\sqrt{\det(\mathrm{I}_f)}}, \quad M_f := \frac{s_f}{\sqrt{\det(\mathrm{I}_f)}}, \quad N_f := \frac{t_f}{\sqrt{\det(\mathrm{I}_f)}},$$

where $\det(\mathrm{I}_f) := E_f G_f - F_f^2$. The *Weingarten map* of G_f is a tensor field W_f on G_f of type $(1,1)$ satisfying

$$\left[\mathrm{W}_f\left(\frac{\partial}{\partial x}\right), \mathrm{W}_f\left(\frac{\partial}{\partial y}\right)\right] = \left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right] W_f,$$

where

$$W_f := \begin{pmatrix} E_f & F_f \\ F_f & G_f \end{pmatrix}^{-1} \begin{pmatrix} L_f & M_f \\ M_f & N_f \end{pmatrix}.$$

A *principal direction* of G_f is a one-dimensional eigenspace of W_f. By the symmetry of W_f with respect to I_f, we see that at a point of G_f, a one-dimensional subspace of the tangent plane which is perpendicular to a principal direction with respect to I_f is also a principal direction.

Let PD_f be a symmetric tensor field on G_f of type $(0,2)$ represented in terms of the coordinates (x, y) as

$$\mathrm{PD}_f := \frac{1}{\sqrt{\det(\mathrm{I}_f)}}\{A_f\, dx^2 + 2B_f\, dxdy + C_f\, dy^2\},$$

where

$$A_f := E_f M_f - F_f L_f, \quad 2B_f := E_f N_f - G_f L_f, \quad C_f := F_f N_f - G_f M_f.$$

For vector fields V_1, V_2 on G_f,

$$\frac{1}{2}\sum_{\{i,j\}=\{1,2\}} V_i \wedge \mathrm{W}_f(V_j) = \frac{\mathrm{PD}_f(V_1, V_2)}{\sqrt{\det(\mathrm{I}_f)}}\left(\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}\right).$$

Therefore we see that at a point of G_f, a tangent vector v_0 is in a principal direction if and only if $\mathrm{PD}_f(v_0, v_0) = 0$. In particular, if we set

$$\varpi_f := s_f(p_f^2 - q_f^2) - (r_f - t_f)p_f q_f,$$

$$\varpi_f^{\perp} := s_f(1 + p_f^2) - p_f q_f r_f,$$

then we obtain the following:

PROPOSITION 2.1 ([2]). *At a point of* G_f, *there exists a principal direction parallel to the xy-plane if and only if* $\varpi_f = 0$.

PROPOSITION 2.2. *At a point of* G_f, *there exists a principal direction parallel to the xz-plane if and only if* $\varpi_f^{\perp} = 0$.

3. Parallel Curved Surfaces

Let S be an embedded, parallel curved surface and for a base plane P of S, let $\Xi_{S,P}$ be the subset of S such that for any $q \in \Xi_{S,P}$, the tangent plane $T_q(S)$ to S at q is not parallel to P. We see that $\Xi_{S,P}$ is an open set of S. A point of $S \backslash \Xi_{S,P}$ is called a *parallel point* of S with respect to a base plane P. If there exists a base plane P_0 of S satisfying $\Xi_{S,P_0} = \emptyset$, then each connected component of S is part of a plane in $\boldsymbol{R}^3$. In the following, suppose $\Xi_{S,P} \neq \emptyset$ for any base plane P.

For a base plane P_0 of S and a point $q \in \Xi_{S,P_0}$, let $P^{\perp}_{P_0,q}$ be the plane in $\boldsymbol{R}^3$ through q perpendicular to each of P_0 and $T_q(S)$, and $C^{\perp}_{P_0,q}$ the connected component of $P^{\perp}_{P_0,q} \cap \Xi_{S,P_0}$ containing q. We shall prove

PROPOSITION 3.1. *The plane* $P^{\perp}_{P_0,q}$ *is perpendicular to* $T_p(S)$ *for any* $p \in C^{\perp}_{P_0,q}$.

PROOF. For each $q \in \Xi_{S,P_0}$, let (x, y, z) be orthogonal coordinates on $\boldsymbol{R}^3$ satisfying the following:

(a) the point q corresponds to $(0,0,0)$;
(b) the xz-plane P_{xz} is parallel to P_0;
(c) the yz-plane P_{yz} is equal to $P^{\perp}_{P_0,q}$.

Then the xy-plane P_{xy} is not perpendicular to $T_q(\Xi_{S,P_0})$. Let f be a smooth function on a neighborhood of q in P_{xy} such that G_f is a neighborhood of q in Ξ_{S,P_0}. The function f satisfies $f(0,0) = p_f(0,0) = 0$. Noticing that $\partial/\partial x$ is in a principal direction at each point of G_f, we see that a vector field

$$V_f := -F_f \frac{\partial}{\partial x} + E_f \frac{\partial}{\partial y},$$

which is perpendicular to $\partial/\partial x$ with respect to the first fundamental form I_f, is also in a principal direction at each point of G_f. In addition, by Proposition 2.2, we see that p_f is constant on each integral curve of V_f. Then by $p_f(0,0) = 0$ together with the definition of V_f, we see that the integral curve of V_f through q is contained in P_{yz} and that at any point p of this integral curve, P_{yz} is perpendicular to $T_p(S)$. Hence we obtain Proposition 3.1. □

REMARK 3.2. In [2], we presented another proof of Proposition 3.1 on condition that S is real-analytic.

COROLLARY 3.3. *The following hold*:

(a) $C^{\perp}_{P_0,q}$ *is a simple curve*;
(b) *A principal direction of* S *at each point of* $C^{\perp}_{P_0,q}$ *which is parallel to* P_0 *is perpendicular to* $P^{\perp}_{P_0,q}$;
(c) *The tangent line to* $C^{\perp}_{P_0,q}$ *at each point of* $C^{\perp}_{P_0,q}$ *is a principal direction of* S *and not parallel to* P_0.

For a base plane P_0 of S and a point $q \in \Xi_{S,P_0}$, let $P_{P_0,q}$ be the plane in $\boldsymbol{R}^3$ through q parallel to P_0 and $C_{P_0,q}$ the connected component of $P_{P_0,q} \cap S$ containing q. We shall prove

PROPOSITION 3.4. *The angle between* $T_p(S)$ *and* $P_{P_0,q}$ *does not depend on the choice of* $p \in C_{P_0,q}$.

PROOF. For each $q \in \Xi_{S,P_0}$, let (x, y, z) and f be as in the proof of Proposition 3.1. Let $\alpha_f(x, y)$ be the angle between $T_{(x,y)}(\mathrm{G}_f)$ and $P_{P_0,(x,y)}$. Then we see that $\alpha_f(x, y)$ is equal to the angle between $V_f(x, y)$ and $P_{P_0,(x,y)}$. Therefore we obtain

$$\cos^2 \alpha_f = \frac{q_f^2}{1 + p_f^2 + q_f^2}.$$

By Proposition 2.2, we obtain $\partial(\cos^2 \alpha_f)/\partial x \equiv 0$. This implies Proposition 3.4. □

COROLLARY 3.5. *The set* $C_{P_0,q}$ *is a simple curve in* Ξ_{S,P_0} *such that the tangent line to* $C_{P_0,q}$ *at each point of* $C_{P_0,q}$ *is a principal direction of* S.

4. Generating Pairs

Let S be an embedded, parallel curved surface and P_0 a base plane of S. Then from Corollary 3.3 and Corollary 3.5, we see that for any $q \in \Xi_{S,P_0}$, $(C_{P_0,q}, C^{\perp}_{P_0,q})$ is a generating pair such that $C_{P_0,q}$ and $C^{\perp}_{P_0,q}$ are the base curve and the generating curve of $(C_{P_0,q}, C^{\perp}_{P_0,q})$, respectively and that $(C_{P_0,q}, C^{\perp}_{P_0,q})$ satisfies $q = p_{(C_{P_0,q}, C^{\perp}_{P_0,q})}$ and $S \in \Sigma_{(C_{P_0,q}, C^{\perp}_{P_0,q})}$. Therefore we obtain

PROPOSITION 4.1. *Let S be an embedded, parallel curved surface and P_0 a base plane of S. Then S is generated at any point of Ξ_{S,P_0}.*

We shall prove

PROPOSITION 4.2. *Let (C_b, C_g) be a generating pair such that C_b and C_g are the base curve and the generating curve of (C_b, C_g), respectively. Then $\Sigma_{(C_b,C_g)} \neq \emptyset$.*

PROOF. For each $p \in C_b$, there exists an isometry Φ_p of $\boldsymbol{R}^3$ satisfying

(a) $\Phi_p(p_{(C_b,C_g)}) = p$;
(b) $\Phi_p(P_g)$ is normal to C_b at p;
(c) the angle between P_b and the tangent line to $\Phi_p(C_g)$ at p is equal to the angle between P_b and the tangent line to C_g at $p_{(C_b,C_g)}$;
(d) the map $\Phi : C_b \times \boldsymbol{R}^3 \to \boldsymbol{R}^3$ defined by $\Phi(p, X) := \Phi_p(X)$ is smooth;
(e) $\Phi_{p_{(C_b,C_g)}}$ is the identity map.

In addition, there exist neighborhoods O_b, O_g of $p_{(C_b,C_g)}$ in C_b, C_g, respectively such that

$$S_{O_b,O_g} := \bigcup_{p \in O_b} \Phi_p(O_g) \tag{1}$$

is an embedded surface. Let q be a point of S_{O_b,O_g} such that $T_q(S_{O_b,O_g})$ is not parallel to P_b and (x, y, z) orthogonal coordinates on $\boldsymbol{R}^3$ satisfying the following:

(a) the point q corresponds to $(0,0,0)$;
(b) P_{xz} is parallel to P_b;
(c) P_{yz} is perpendicular to each of P_b and $T_q(S_{O_b,O_g})$.

Let f be a smooth function defined on a neighborhood U of q in P_{xy} such that G_f is a neighborhood of q in S_{O_b,O_g}. Then we obtain $p_f(0, y) = s_f(0, y) = 0$ for any $y \in \boldsymbol{R}$ satisfying $(0, y) \in U$. Therefore we obtain

$$\mathrm{PD}_f\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial x}\right) = \mathrm{PD}_f\left(\frac{\partial}{\partial y}, \frac{\partial}{\partial y}\right) = 0$$

at $(0, y)$, i.e., we see that each of $\partial/\partial x$ and $\partial/\partial y$ is in a principal direction at $(0, y)$. In particular, we see that at q, there exists a principal direction parallel to P_b. If q is a point of S_{O_b,O_g} such that $T_q(S_{O_b,O_g})$ is parallel to P_b, then any principal direction at q is parallel to P_b. Therefore we see that S_{O_b,O_g} is a parallel curved surface such that P_b is a base plane of S_{O_b,O_g}. If q is a point of $O_b \cap$

$\Xi_{S_{O_b,O_g},P_b}$ or $O_g \cap \Xi_{S_{O_b,O_g},P_b}$, then we see from Corollary 3.3 that the tangent line at q is a principal direction of S_{O_b,O_g}. If q is a point of O_b or O_g and if q is an umbilical point of S_{O_b,O_g}, then the tangent line at q is a principal direction of S_{O_b,O_g}. Suppose that q is a point of O_b and that q is a non-umbilical point and a parallel point with respect to P_b. Then there exists a point of $\Xi_{S_{O_b,O_g},P_b}$ in any neighborhood of q in $\Phi_q(C_g)$. Therefore by the continuity of a principal distribution, we see that the tangent line to O_b at q is a principal direction of S_{O_b,O_g}. If q is a point of O_g and if q is a non-umbilical point and a parallel point with respect to P_b, then we similarly obtain the same result. Hence we obtain Proposition 4.2. □

REMARK 4.3. In the following, such a surface as S_{O_b,O_g} constructed in (1) is called a *canonical* parallel curved surface generated by a generating pair (C_b, C_g). We see that a canonical parallel curved surface is generated at any point.

REMARK 4.4. In Proposition 4.1 and Proposition 4.2, we may find relations between parallel curved surfaces and generating pairs. We take notice of the following question:

> For a given generating pair (C_b, C_g), does the non-empty set $\Sigma_{(C_b,C_g)}$ determine the only one germ of parallel curved surface? In other words, does any element S of $\Sigma_{(C_b,C_g)}$ contain a canonical parallel curved surface generated by (C_b, C_g)?

By Proposition 3.1 together with Proposition 3.4, we see that if for a generating pair (C_b, C_g), C_g is not tangent to P_b at $p_{(C_b,C_g)}$, then the set $\Sigma_{(C_b,C_g)}$ determines the only one germ. Suppose that C_g is tangent to P_b at $p_{(C_b,C_g)}$ and let a_0 be a smooth function on a neighborhood of $p_{(C_b,C_g)}$ in $P_b \cap P_g$ such that the graph of a_0 in P_g is a neighborhood of $p_{(C_b,C_g)}$ in C_g. In Section 6, we shall show that if not all the derivatives of a_0 at $p_{(C_b,C_g)}$ are equal to zero, then the set $\Sigma_{(C_b,C_g)}$ determines the only one germ (Remark 6.5). However, we shall also show in the present section that if all the derivatives of a_0 at $p_{(C_b,C_g)}$ are equal to zero, then $\Sigma_{(C_b,C_g)}$ always gives plural germs (Example 4.5). We also take notice of the following question:

> Let S be a parallel curved surface and P_0 a base plane of S. Then is S generated at a parallel point with respect to P_0? In addition, if S is generated at some parallel point p_0 with respect to P_0, then is S *uniquely generated* at p_0? In other words, for two generating pairs $(C_b^{(1)}, C_g^{(1)})$,

$(C_b^{(2)}, C_g^{(2)})$ such that S is generated at p_0 by each of these two pairs, does there exist a neighborhood V of p_0 in S satisfying

$$(C_b^{(1)} \cup C_g^{(1)}) \cap V = (C_b^{(2)} \cup C_g^{(2)}) \cap V?$$

By Corollary 3.3 together with Corollary 3.5, we see that a parallel curved surface S is uniquely generated at any non-parallel point q with respect to a base plane if there exists no totally umbilical neighborhood of q in S. In addition, Theorem 1.2, which we shall prove in Section 6, implies that if for a smooth function F as in Theorem 1.2, G_F is not part of any surface of revolution, then G_F is uniquely generated at o. Even if all the partial derivatives of a smooth function F as in the beginning of the second paragraph in Section 1 are equal to zero at $(0,0)$, it is possible that G_F is uniquely generated at o: if we set

$$F(x, y) := \begin{cases} 0, & \text{if } x = 0, \\ \exp(-1/x^2), & \text{if } x \neq 0, \end{cases}$$

then the graph of F is a suitable example. However, in the present section, we shall construct an example of a parallel curved surface which is generated but not uniquely generated at some parallel point p with respect to a base plane and in which there exists no totally umbilical neighborhood of p (Example 4.5). We already know Theorem 1.1 in the real-analytic case. Therefore we already have an example of a parallel curved surface which is not generated at some point. In Section 5, we shall prove Theorem 1.1 in the general case. In addition, in the present section, we shall construct an example of a parallel curved surface which is neither part of any surface of revolution nor generated at some parallel point with respect to a base plane (Example 4.6).

We shall present types of parallel curved surface which never appear in the real-analytic case.

EXAMPLE 4.5. For a positive number $\delta_0 > 0$, let a_0 be a smooth function on an open interval $I_0 := (-\delta_0, \delta_0)$ satisfying $a_0(0) = 0$ and the condition that all the derivatives of a_0 at 0 are equal to zero, and C_b, C_g two curves in P_{xy} and P_{xz}, respectively defined by

$$C_b := \{(0, y, 0); y \in \boldsymbol{R}\},$$

$$C_g := \{(x, 0, a_0(x)); x \in I_0\}.$$

Then we see that (C_b, C_g) is a generating pair. We shall prove that there exists an element S_0 of $\Sigma_{(C_b, C_g)}$ satisfying the following:

(a) S_0 contains no canonical parallel curved surface generated by (C_b, C_g);
(b) S_0 is generated but not uniquely generated at o.

From (a), we see that $\Sigma_{(C_b,C_g)}$ gives plural germs and that there exists no totally umbilical neighborhood of o in S_0. Suppose that a_0 is not constant on any neighborhood of 0 in I_0. Let n be a positive integer and for $\delta \in (0, \delta_0)$, let $C_{b,+}^{(n)}, C_{b,-}^{(n)}$ be simple curves in the planes $\{z = a_0(\delta)\}, \{z = a_0(-\delta)\}$, respectively defined by

$$C_{b,+}^{(n)} := \{(\delta + y^{2n}, y, a_0(\delta)); y \in \boldsymbol{R}\},$$

$$C_{b,-}^{(n)} := \{(-\delta - y^{2n}, y, a_0(-\delta)); y \in \boldsymbol{R}\}.$$

We set

$$C_{g,+} := C_g \cap \{x > 0\}, \quad C_{g,-} := C_g \cap \{x < 0\}.$$

Then we may choose $\delta \in (0, \delta_0)$ so that for any $\varepsilon \in \{+,-\}$, $(C_{b,\varepsilon}^{(n)}, C_{g,\varepsilon})$ is a generating pair such that

$$S_{C_{b,\varepsilon}^{(n)}, C_{g,\varepsilon}} := \bigcup_{p \in C_{b,\varepsilon}^{(n)}} \Phi_p(C_{g,\varepsilon})$$

is a canonical parallel curved surface generated by $(C_{b,\varepsilon}^{(n)}, C_{g,\varepsilon})$. We set

$$X := P_{xy} \cap (\overline{S_{C_{b,+}^{(n)}, C_{g,+}}} \cup \overline{S_{C_{b,-}^{(n)}, C_{g,-}}}),$$

where $\overline{S_{C_{b,\varepsilon}^{(n)}, C_{g,\varepsilon}}}$ is the closure of $S_{C_{b,\varepsilon}^{(n)}, C_{g,\varepsilon}}$ in $\boldsymbol{R}^3$. Let A_+, A_- be two connected components of $P_{xy} \backslash X$ which contain points $(0,1,0), (0,-1,0)$, respectively. Then

$$S_0^{(n)} := S_{C_{b,+}^{(n)}, C_{g,+}} \cup \overline{A_+} \cup S_{C_{b,-}^{(n)}, C_{g,-}} \cup \overline{A_-}$$

is an element of $\Sigma_{(C_b,C_g)}$. We see that $S_0^{(n)}$ contains no canonical parallel curved surface generated by (C_b, C_g) and that $S_0^{(n)}$ is generated but not uniquely generated at o. Suppose that a_0 is constant on I_0. Let n be a positive integer and D_n an open disc in P_{xy} defined by

$$D_n := \left\{ \left(x - \frac{1}{2^n}\right)^2 + \left(y - \frac{1}{2^n}\right)^2 < \frac{1}{2^{2n+3}} \right\}.$$

Then for arbitrary distinct two positive integers $n_1, n_2 \in \boldsymbol{N}$, $D_{n_1} \cap D_{n_2} = \varnothing$. We set

$$Y := P_{xy} \backslash \bigcup_{n \in \boldsymbol{N}} D_n.$$

For each $n \in \boldsymbol{N}$, let F_n be a smooth function on D_n defined by

$$F_n(x, y) := \exp\left(-2^n - \frac{1}{1 - 2^{2n+3}\{(x - 1/2^n)^2 + (y - 1/2^n)^2\}}\right).$$

Then

$$S_0 := Y \cup \bigcup_{n \in \boldsymbol{N}} \mathrm{G}_{F_n}$$

is an element of $\Sigma_{(C_b, C_g)}$. We see that S_0 contains no canonical parallel curved surface generated by (C_b, C_g) and that S_0 is generated but not uniquely generated at o. We may prove that $\Sigma_{(C_b, C_g)}$ gives plural germs as long as C_b is a curve in P_{xy} through o tangent to the y-axis at o.

Example 4.6. Let k be a smooth, positive-valued function on an open interval $(-2\pi/3, 2\pi/3)$ satisfying the following:

(a) $k' > 0$ on $(-2\pi/3, -\pi/3)$;
(b) $k \equiv 1$ on $[-\pi/3, \pi/3]$;
(c) $k' < 0$ on $(\pi/3, 2\pi/3)$.

Let $\lambda : (-2\pi/3, 2\pi/3) \to \{z = 1\}$ be a smooth map from $(-2\pi/3, 2\pi/3)$ into the plane $\{z = 1\}$ in $\boldsymbol{R}^3$ satisfying $|\lambda'| \equiv 1$, $|\lambda''| \equiv k$ and

$$\lambda([-\pi/3, \pi/3]) = \{(\cos\theta, \sin\theta, 1); \theta \in [-\pi/3, \pi/3]\}.$$

We set $C_b := \lambda((-2\pi/3, 2\pi/3))$. In addition, we set

$$C_g := \{(r, 0, z) \in \boldsymbol{R}^3; z = e \cdot e^{-1/r}, r > 0\}.$$

Then (C_b, C_g) is a generating pair. We see that

$$S_{C_b, C_g} := \bigcup_{p \in C_b} \Phi_p(C_g)$$

is a canonical parallel curved surface generated by (C_b, C_g) and that P_{xy} is a base plane of S_{C_b, C_g}. Let $\overline{C_0}$ be the intersection of the plane P_{xy} with the closure $\overline{S_{C_b, C_g}}$ of S_{C_b, C_g} in $\boldsymbol{R}^3$ and C_0 the interior of $\overline{C_0}$. Then we see that each connected component of $C_0 \backslash \{o\}$ is an embedded curve but that C_0 is not immersed at o. Let e_1, e_2 be the two ends of $\overline{C_0}$ and D_0 the domain bounded by $\overline{C_0}$ and the line segment determined by e_1, e_2. Then we see that the set

$$S_0 := S_{C_b, C_g} \cup C_0 \cup D_0$$

is a parallel curved surface such that P_{xy} is a base plane of S_0. In addition, we see that S_0 is not part of any surface of revolution and that S_0 is generated at any point of $S_0\backslash\{o\}$ but not generated at o.

Example 4.6 implies Theorem 1.3.

In order to prove Theorem 1.4, we shall use the following:

PROPOSITION 4.7. *Let S be a connected, complete, embedded, parallel curved surface which is uniquely generated at any parallel point of S with respect to a base plane. Then S is a canonical parallel curved surface generated by a generating pair (C_b, C_g) such that each of C_b and C_g is isometric to $\boldsymbol{R}$ or a simple closed curve. In particular, S is homeomorphic to a plane, a cylinder or a torus.*

5. Parallel Points of a Parallel Curved Surface

A parallel point p of S with respect to a base plane P_0 of S is called *isolated* if there exists a neighborhood of p in S in which p is the only one parallel point of S with respect to P_0; p is called *isolated in the weak sense* if the following hold:

(a) S is not generated at p by any generating pair such that its base curve is contained in the tangent plane at p;

(b) there exists a neighborhood U of p in S such that at each parallel point q of $U\backslash\{p\}$ with respect to P_0, S is uniquely generated by a generating pair such that its base curve is contained in the tangent plane at q.

By Proposition 3.4, we see that if p is isolated, then p is isolated in the weak sense.

EXAMPLE 5.1. Let S be part of an embedded surface of revolution such that a point p of S lies on its axis of rotation. Then p is a parallel point with respect to a plane normal to the axis. We see that if S is real-analytic and if S is not part of any plane, then p is isolated and that if S is not real-analytic, then p is not always isolated. We also see that even if p is not isolated, it is possible that p is isolated in the weak sense.

EXAMPLE 5.2. Let S_0 be as in Example 4.6. Then o is a parallel point with respect to a base plane P_{xy}. In addition, S_0 is not generated at o. However, since in any neighborhood of o, there exists another parallel point p with respect to P_0 than o such that S_0 is not uniquely generated at p, we see that o is not isolated in the weak sense.

We shall prove

PROPOSITION 5.3. *Let S be a connected, embedded, parallel curved surface and P_0 a base plane of S.*

(a) *If there exists a parallel point p of S with respect to P_0 which is isolated in the weak sense, then S is part of a surface of revolution such that p lies on its axis of rotation.*
(b) *In addition, if S is complete, then S is a surface of revolution which crosses its axis of rotation at just one point or just two points; correspondingly, S is homeomorphic to a plane or a sphere.*

We shall also prove

PROPOSITION 5.4. *Let S be an embedded, parallel curved surface and P_0 a base plane of S. Then for a parallel point p of S with respect to P_0,*

(a) *if p is a non-umbilical point, then S is uniquely generated at p by a generating pair such that its base curve is contained in the tangent plane at p;*
(b) *if p is an isolated umbilical point, then p is isolated in the weak sense.*

By (a) of Proposition 5.3 together with (b) of Proposition 5.4, we obtain Theorem 1.1. In addition, by Proposition 4.7, Proposition 5.3 and Proposition 5.4, we obtain Theorem 1.4.

PROOF OF PROPOSITION 5.3. Suppose that there exists a parallel point p with respect to P_0 which is isolated in the weak sense. Then let (x, y, z) be orthogonal coordinates on $\boldsymbol{R}^3$ satisfying the following:

(a) p corresponds to $(0, 0, 0)$;
(b) P_{xy} is tangent to S at p.

Let f be a smooth function defined on $\{x^2 + y^2 < r_0^2\}$ for some $r_0 > 0$ satisfying the following:

(a) G_f is a neighborhood of p in S;
(b) at each parallel point q of $\mathsf{G}_f \backslash \{p\}$ with respect to P_0, S is uniquely generated by a generating pair such that its base curve is contained in the tangent plane at q.

Suppose that there exists a point q_0 of Ξ_{G_f, P_0} such that $P^{\perp}_{P_0, q_0}$ does not contain p. Then we see that there exists a point q_1 of $C_{P_0, q_0} \cap \mathsf{G}_f$ such that $P^{\perp}_{P_0, q_1}$ contains p. Since at any parallel point of $\mathsf{G}_f \backslash \{p\}$, S is uniquely generated, we see that there exists a simple curve C_b through p contained in $P_{xy} \cap \mathsf{G}_f$. The curve C_b is normal to $P_{P_0,q_1}^{\perp}$ at p. We set $C_g := P_{P_0,q_1}^{\perp} \cap \mathsf{G}_f$. Then we see that S is generated at p by a generating pair (C_b, C_g), which causes a contradiction. Therefore we see that for any $q_0 \in \Xi_{\mathsf{G}_f,P_0}$, $P_{P_0,q_0}^{\perp}$ contains p. Then we see that S is part of a surface of revolution such that p lies on its axis of rotation. Hence we obtain (a) of Proposition 5.3. In addition, by (a) of Proposition 5.3, we obtain (b) of Proposition 5.3. □

PROOF OF (a) OF PROPOSITION 5.4. Let (x, y, z) be as in the proof of Proposition 5.3 and f a smooth function defined on a neighborhood of p in P_{xy} such that any point of G_f is a non-umbilical point of S. Then not all the partial derivatives of f of order two at $(0,0)$ are equal to zero. In addition, since f satisfies $\varpi_f \equiv 0$, we may suppose that all the partial derivatives of $f - x^2$ of order two at $(0,0)$ are equal to zero. Then there exists a positive number $x_0 > 0$ satisfying $X_f(x) := (x, 0, f(x,0)) \in \Xi_{\mathsf{G}_f,P_0}$ for any $x \in (-x_0, x_0)\backslash\{0\}$. Let C_b be an integral curve of a principal distribution on G_f tangent to the y-axis at $(0,0,0)$. Then noticing Corollary 3.3 and Corollary 3.5, we obtain $C_b \cap C_{P_0,X_f(x)} = \emptyset$ for any $x \in (-x_0, x_0)\backslash\{0\}$ and we may suppose

$$\mathsf{G}_f = C_b \cup \bigcup_{x \in (-x_0,x_0)\backslash\{0\}} C_{P_0,X_f(x)}.$$

Therefore we obtain $C_b \subset P_{xy}$. We set $C_g := P_{xz} \cap \mathsf{G}_f$. Then by Corollary 3.3, we see that (C_b, C_g) is a generating pair such that G_f is generated at p by (C_b, C_g). We easily see that G_f is uniquely generated at p. Hence we obtain (a) of Proposition 5.4. □

PROOF OF (b) OF PROPOSITION 5.4. By Proposition 4.1 together with (a) of Proposition 5.4, we see that there exists a neighborhood U of p in S such that at each point of $U\backslash\{p\}$, S is uniquely generated by a generating pair the base curve of which is contained in a plane parallel to P_0. Suppose that S is generated at p by a generating pair (C_b, C_g) such that P_b is the tangent plane at p. We may suppose that any point of $C_b\backslash\{p\}$ is a non-umbilical point of S. Then by Proposition 3.4, we see that p is a non-umbilical point or a non-parallel point with respect to P_0, which causes a contradiction. Hence we obtain (b) of Proposition 5.4. □

6. Partial Differential Equations $\varpi = 0$ and $\varpi^{\perp} = 0$

Let f be a smooth function of two variables. From Proposition 2.2, we see that G_f is a parallel curved surface such that the xz-plane is its base plane if and only if f satisfies $\varpi_f^{\perp} \equiv 0$. From Proposition 4.2, we obtain the following proposition in relation to the existence of a solution for the partial differential equation $\varpi^{\perp} = 0$:

PROPOSITION 6.1. *Let I_1, I_2 be open intervals which contain 0 and a_1, a_2 smooth functions on I_1, I_2, respectively. Suppose $a_1(0) = a_2(0)$ and $a_1'(0) = 0$. Then there exist a neighborhood V of $(0,0)$ in $\boldsymbol{R}^2$ and a smooth function f defined on V satisfying the following*:

(a) $\varpi_f^{\perp} \equiv 0$ *on* V;
(b) $f(x,0) = a_1(x)$ *for any* $x \in I_1$ *satisfying* $(x,0) \in V$;
(c) $f(0,y) = a_2(y)$ *for any* $y \in I_2$ *satisfying* $(0,y) \in V$.

In addition, by Proposition 3.1 together with Proposition 3.4, we obtain the following proposition in relation to the uniqueness of a solution for $\varpi^{\perp} = 0$:

PROPOSITION 6.2. *Let f_1, f_2 be smooth functions defined on a neighborhood V of $(0,0)$ in $\boldsymbol{R}^2$ satisfying the following*:

(a) $p_{f_i}(0,0) = 0$ *for* $i = 1,2$;
(b) $\varpi_{f_i}^{\perp} \equiv 0$ *on* V *for* $i = 1,2$;
(c) $f_1(x,0) = f_2(x,0)$ *for any* $x \in \boldsymbol{R}$ *satisfying* $(x,0) \in V$;
(d) $f_1(0,y) = f_2(0,y)$ *for any* $y \in \boldsymbol{R}$ *satisfying* $(0,y) \in V$.

Then there exists a neighborhood V' of $(0,0)$ in V satisfying $f_1 \equiv f_2$ on V'.

From Proposition 2.1, we see that G_f is a parallel curved surface such that the xy-plane is its base plane if and only if f satisfies $\varpi_f \equiv 0$. From Proposition 4.2, we obtain the following proposition in relation to the existence of a solution for the partial differential equation $\varpi = 0$:

PROPOSITION 6.3. *Let C_0 be a simple curve in P_{xy} through $(0,0)$ tangent to the y-axis at $(0,0)$. Let I_0 be an open interval which contains 0 and a_0 a smooth function on I_0 satisfying $a_0(0) = 0$. Then there exist a neighborhood U of $(0,0)$ in $\boldsymbol{R}^2$ and a smooth function f defined on U satisfying the following*:

(a) $\varpi_f \equiv 0$ *on* U;
(b) $f|_{C_0 \cap U} \equiv 0$;
(c) $f(x,0) = a_0(x)$ *for any* $x \in I_0$ *satisfying* $(x,0) \in U$.

We shall prove Theorem 1.2 in the present section. By Theorem 1.2, Corollary 3.5 and Proposition 6.2, we obtain the following proposition in relation to the uniqueness of a solution for $\varpi = 0$:

PROPOSITION 6.4. *Let f_1, f_2 be smooth functions defined on a neighborhood U of $(0,0)$ in $\boldsymbol{R}^2$ satisfying the following*:

(a) $\varpi_{f_i} \equiv 0$ *on* U *for* $i = 1, 2$;
(b) *there exists a simple curve in P_{xy} through $(0,0)$ tangent to the y-axis at $(0,0)$ on which $f_i \equiv 0$ for $i = 1, 2$*;
(c) $f_1(x, 0) = f_2(x, 0)$ *for any* $x \in \boldsymbol{R}$ *satisfying* $(x, 0) \in U$;
(d) *not all the partial derivatives of f_i at $(0,0)$ are equal to zero for $i = 1, 2$.*

Then there exists a neighborhood U' of $(0,0)$ in U satisfying $f_1 \equiv f_2$ on U'.

REMARK 6.5. Let C_0 and a_0 be as in Proposition 6.3 and set

$$C_b := C_0, \quad C_g := \{(x, 0, a_0(x)); x \in I_0\}.$$

Suppose that not all the derivatives of a_0 at 0 are equal to zero. Then from Proposition 6.4, we see that (C_b, C_g) is a generating pair such that the set $\Sigma_{(C_b, C_g)}$ determines the only one germ.

REMARK 6.6. Let C_0 and a_0 be as in Proposition 6.3. As we have seen in Example 4.5, if all the derivatives of a_0 at 0 are equal to zero, then $\Sigma_{(C_b, C_g)}$ always gives plural germs. This means that we may not remove condition (d) in Proposition 6.4.

REMARK 6.7. By Theorem 1.2, Proposition 4.7, Proposition 5.3 and Proposition 5.4, we see that a connected, complete, real-analytic, embedded and parallel curved surface is homeomorphic to a sphere, a plane, a cylinder or a torus, which was already obtained in [2].

PROOF OF THEOREM 1.2. Let F be a smooth function as in the beginning of the second paragraph in Section 1 such that not all the partial derivatives of F at $(0,0)$ are equal to zero. Then there exists a homogeneous polynomial g of degree $k \geqq 2$ such that all the partial derivatives of $F - g$ of order less than $k + 1$ are equal to zero. By $\varpi_F \equiv 0$, we obtain $\varpi_g \equiv 0$. Then noticing Remark 1.6, we may suppose one of the following:

(a) k is even and g is equal to $(x^2 + y^2)^{k/2}$;
(b) g is equal to x^k.

If $g = (x^2 + y^2)^{k/2}$, then o is an isolated parallel point of G_F with respect to a base plane P_{xy}. Therefore from (a) of Proposition 5.3, we see that G_F is part of a surface of revolution such that o lies on its axis of rotation. Suppose $g = x^k$. Then there exists a positive number $x_0 > 0$ satisfying $X_F(x) := (x, 0, F(x, 0)) \in \Xi_{\mathsf{G}_F, P_{xy}}$ for any $x \in (-x_0, x_0)\backslash\{0\}$. Suppose that for each positive integer $n \in \boldsymbol{N}$, there exists a number $x_n \in (0, \min\{x_0, 1/n\})$ satisfying $q_F(x_n, 0) \neq 0$. Then noticing $\lim_{x \to 0} q_F(x, 0)/x^{k-1} = 0$, we see that there exists a positive integer $n_0 \in \boldsymbol{N}$ such that for each integer $n \geqq n_0$, $P^{\perp}_{P_{xy}, X_F(x_n)}$ is not normal to $C_{P_{xy}, X_F(x_1)}$ at an intersection, which causes a contradiction. Therefore we obtain $q_F(x, 0) = 0$ for any $x \in (-x_0, x_0)$. In particular, we see that P_{xz} is normal to G_F at any point of a neighborhood of o in $P_{xz} \cap \mathsf{G}_F$. Let $C_{b,+}$ (respectively, $C_{b,-}$) be the interior of the intersection of P_{xy} with the closure of

$$\bigcup_{x \in (0, x_0)} C_{P_{xy}, X_F(x)} \left(\text{respectively, } \bigcup_{x \in (-x_0, 0)} C_{P_{xy}, X_F(x)} \right)$$

in $\boldsymbol{R}^3$. Then $C_{b,+}$ and $C_{b,-}$ are smooth curves in P_{xy} tangent to the y-axis at o. We see by Proposition 3.4 that at any point of $C_{b,+}$ and $C_{b,-}$, not all the partial derivatives of F of order k are equal to zero. Suppose that for each neighborhood U_+ of o in $C_{b,+}$, there exists a point of $U_+\backslash\{o\}$ which is not contained in $C_{b,-}$. Then we may find a point p_+ of $U_+\backslash\{o\}$ such that the plane normal to U_+ at p_+ is not normal to $C_{b,-}$ at an intersection, which causes a contradiction. Therefore we see that there exists a neighborhood U of o in $C_{b,+}$ contained in $C_{b,-}$. We set $C_b := U$ and let C_g be a connected neighborhood of o in $P_{xz} \cap \mathsf{G}_F$ satisfying $C_g\backslash\{o\} \subset \Xi_{\mathsf{G}_F, P_{xy}}$. We see that (C_b, C_g) is a generating pair such that there exists a neighborhood of o in G_F which is a canonical parallel curved surface generated by (C_b, C_g). Noticing $C_g\backslash\{o\} \subset \Xi_{\mathsf{G}_F, P_{xy}}$, we obtain Theorem 1.2. □

7. Construction of a Compact, Orientable, Parallel Curved Surface of Genus $g \geqq 2$

Let g be an integer not less than two, and let $C_1, C_2, \dots, C_g$ be circles in P_{xy} with radius two and D_i the open disc bounded by C_i ($i \in \{1, 2, \dots, g\}$). We set $\overline{D_i} := D_i \cup C_i$ for $i \in \{1, 2, \dots, g\}$ and suppose $\overline{D_i} \cap \overline{D_j} = \varnothing$ for arbitrary distinct two $i, j \in \{1, 2, \dots, g\}$. Let C_0 be a circle such that the open disc D_0 bounded by C_0 contains $\overline{D_1}, \overline{D_2}, \dots, \overline{D_g}$. We set $S_g^{(0)} := C_0 \cup D_0 \backslash \bigcup_{i=1}^{g} D_i$. Let $\mathrm{Proj}^{(2)}$ be a map from $\boldsymbol{R}^3$ onto $\{z = 2\}$ defined by $\mathrm{Proj}^{(2)}(x, y, z) := (x, y, 2)$ for each $(x, y, z) \in \boldsymbol{R}^3$. We set $S_g^{(2)} := \mathrm{Proj}^{(2)}(S_g^{(0)})$.

Let k be a smooth function on $(-2, 2)$ defined by

$$k(t) := a_0 \exp\left(\frac{1}{t^2 - 4}\right),$$

where

$$a_0 := \frac{\pi}{\int_{-2}^{2} \exp\left(\frac{1}{t^2 - 4}\right) dt}.$$

Then k satisfies

$$\int_{-2}^{2} k(t)\, dt = \pi. \tag{2}$$

For a sufficiently small positive number $\varepsilon_0 > 0$, let γ be a map from $(-2 - \varepsilon_0, 2 + \varepsilon_0)$ into P_{xz} satisfying $|\gamma'| \equiv 1$ on $(-2 - \varepsilon_0, 2 + \varepsilon_0)$, $|\gamma''| = k$ on $(-2, 2)$ and

$$\gamma(t) = \begin{cases} (2 + t, 0, 0) & \text{for } t \in (-2 - \varepsilon_0, -2], \\ (2 - t, 0, c) & \text{for } t \in [2, 2 + \varepsilon_0), \end{cases} \tag{3}$$

where $c > 0$. Noticing (2), we see that such a map as γ exists. We represent γ as $\gamma = (\gamma_1, 0, \gamma_3)$ and we set

$$\Gamma := \left(\gamma_1, 0, \frac{2}{c}\gamma_3\right).$$

Then from (3), we see that for $t \in [2, 2 + \varepsilon_0)$, $\Gamma(t)$ is contained in the plane $\{z = 2\}$. We set $C^\perp := \Gamma((-2, 2))$.

For each $i \in \{0, 1, \ldots, g\}$ and each $p \in C_i$, let Ψ_p be an isometry of $\boldsymbol{R}^3$ satisfying the following:

(a) $\Psi_p(0, 0, 0) = p$,
(b) for a sufficiently small $\varepsilon_0 > 0$, $\Psi_p(t, 0, 0)$ is contained in $S_g^{(0)}$ for any $t \in (-\varepsilon_0, 0]$,
(c) $(d\Psi_p)_{(0,0,0)}((\partial/\partial x)_{(0,0,0)})$ is normal to C_i,
(d) $(d\Psi_p)_{(0,0,0)}((\partial/\partial z)_{(0,0,0)}) = (\partial/\partial z)_p$.

We set

$$S_i^\perp := \bigcup_{p \in C_i} \Psi_p(C^\perp), \quad S_g^{(\perp)} := \bigcup_{i=0}^{g} S_i^\perp, \quad S_g := S_g^{(0)} \cup S_g^{(\perp)} \cup S_g^{(2)}.$$

Then we see that S_g is a connected, compact, orientable, embedded, parallel curved surface of genus g. Since there exist embedded, parallel curved surfaces homeomorphic to a sphere and a torus, respectively, we obtain Theorem 1.5.

References

[1] Ando, N., The behavior of the principal distributions around an isolated umbilical point, J. Math. Soc. Japan **53** (2001), 237–260.
[2] Ando, N., A class of real-analytic surfaces in the 3-Euclidean space, Tsukuba J. Math. **26** (2002), 251–267.
[3] Ando, N., The behavior of the principal distributions on a real-analytic surface, J. Math. Soc. Japan **56** (2004), 201–214.
[4] Ando, N., The index of an isolated umbilical point on a surface, (a survey), Contemporary Mathematics **308** "Differential Geometry and Integrable Systems" by American Mathematical Society, 1–11.
[5] Ando, N., An umbilical point on a non-real-analytic surface, Hiroshima Math. J. **33** (2003), 1–14.

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