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A CLASS OF REAL-ANALYTIC SURFACES IN THE 3-EUCLIDEAN SPACE

By

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Abstract. A smooth surface S in $\boldsymbol{R}^3$ is called *parallel curved* if there exists a plane in $\boldsymbol{R}^3$ such that at each point of S, there exists a principal direction parallel to the plane. For example, a plane, a cylinder and a round sphere are parallel curved. More generally, a surface of revolution is also parallel curved. The purposes of this paper are to study the behavior of the principal distributions on a real-analytic, parallel curved surface and to classify the connected, complete, real-analytic, embedded, parallel curved surfaces.

1. Introduction

Let S be a smooth surface in $\boldsymbol{R}^3$ and $\mathrm{Umb}(S)$ the set of the umbilical points of S. If $S\backslash\mathrm{Umb}(S) \neq \varnothing$, then there exists a one-dimensional continuous distribution on $S\backslash\mathrm{Umb}(S)$ which gives a principal direction at each points of $S\backslash\mathrm{Umb}(S)$. Such a distribution is called a *principal distribution* on S. Let p_0 be an isolated umbilical point of S. Then the indices of p_0 with respect to two principal distributions coincide with each other. The common number is called the *index* of p_0 on S and denoted by $\mathrm{ind}_{p_0}(S)$. Let (x, y) be local coordinates around p_0 such that p_0 corresponds to $(0,0)$ and r_0 a positive number such that p_0 is the only umbilical point on $\{x^2+y^2<r_0^2\}$, and let $\phi_{S;p_0}$ denote a continuous function on $(0,r_0)\times\boldsymbol{R}$ such that for any $(r,\theta)\in(0,r_0)\times\boldsymbol{R}$, a tangent vector $\cos\phi_{S;p_0}(r,\theta)\partial/\partial x+\sin\phi_{S;p_0}(r,\theta)\partial/\partial y$ is in a principal direction at $(r\cos\theta,\ r\sin\theta)$. Then the index $\mathrm{ind}_{p_0}(S)$ is represented as follows:

$$\mathrm{ind}_{p_0}(S)=\frac{\phi_{S;p_0}(r,\theta+2\pi)-\phi_{S;p_0}(r,\theta)}{2\pi}. \tag{1}$$

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Let $\mathscr{P}^k$ be the set of the homogeneous polynomials in two variables of degree $k \geqq 2$ and $\mathscr{P}_o^k$ the set of the elements of $\mathscr{P}^k$ such that on each of their graphs, the origin $o := (0,0,0)$ of $\boldsymbol{R}^3$ is an isolated umbilical point. For $g \in \mathscr{P}^k$ and for $\theta \in \boldsymbol{R}$, set $\tilde{g}(\theta) := g(\cos\theta, \sin\theta)$. In [1], we studied the behavior of the principal distributions around o on the graph G_g of $g \in \mathscr{P}_o^k$. Then we divided the study into two cases: $d\tilde{g}/d\theta \equiv 0$ and $d\tilde{g}/d\theta \not\equiv 0$. If $g \in \mathscr{P}_o^k$ satisfies $d\tilde{g}/d\theta \equiv 0$, then the "position vector field" $x\partial/\partial x + y\partial/\partial y$ is in a principal direction at each point of G_g, and from this together with formula (1), $\mathrm{ind}_o(\mathsf{G}_g) = 1$ follows. For $g \in \mathscr{P}_o^k$ satisfying $d\tilde{g}/d\theta \not\equiv 0$, we mainly paid attention to the relation between the behavior of the principal distributions and the behavior of the position vector field around a point at which the position vector field is in a principal direction, and we presented a way of computing $\mathrm{ind}_o(\mathsf{G}_g)$ and proved $\mathrm{ind}_o(\mathsf{G}_g) \in \{1 - k/2 + i\}_{i=0}^{[k/2]}$. In [2], we have further studied the behavior of the principal distributions in relation to the existence of other umbilical points than o, around a point at which the position vector field is in a principal direction. We may find such a point, because Euler's identity holds for any homogeneous polynomial. In order to study the behavior of the principal distributions around an isolated umbilical point on a general surface by a similar method, we need some other vector field than the position vector field.

For a smooth function f of two variables x, y, we set

$$p_f := \frac{\partial f}{\partial x}, \quad q_f := \frac{\partial f}{\partial y}, \quad r_f := \frac{\partial^2 f}{\partial x^2}, \quad s_f := \frac{\partial^2 f}{\partial x \partial y}, \quad t_f := \frac{\partial^2 f}{\partial y^2},$$

$$\mathrm{grad}_f := \begin{pmatrix} p_f \\ q_f \end{pmatrix}, \quad \mathrm{grad}_f^{\perp} := \begin{pmatrix} -q_f \\ p_f \end{pmatrix}, \quad \mathrm{Hess}_f := \begin{pmatrix} r_f & s_f \\ s_f & t_f \end{pmatrix}.$$

Let $\langle\, , \rangle$ be the scalar product in $\boldsymbol{R}^2$ and set

$$\varpi_f := \langle \mathrm{Hess}_f\, \mathrm{grad}_f, \mathrm{grad}_f^{\perp} \rangle.$$

In Section 2, we shall prove the following:

PROPOSITION 1.1. *Let f be a smooth function of two variables and G_f the graph of f. Then at a point of G_f, the gradient vector field of f is in a principal direction if and only if $\varpi_f = 0$ holds.*

For $g \in \mathscr{P}^k$, we see by Euler's identity $(k-1)\,\mathrm{grad}_g = \mathrm{Hess}_g{}^t(x, y)$ that

$$(k-1)\varpi_g = \det(\mathrm{Hess}_g)\frac{d\tilde{g}}{d\theta}(\theta) \tag{2}$$

holds at $(\cos\theta, \sin\theta)$ for any $\theta \in \boldsymbol{R}$. Therefore $\varpi_g \equiv 0$ holds if and only if $\det(\mathrm{Hess}_g) \equiv 0$ or $d\tilde{g}/d\theta \equiv 0$ holds. If $g \in \mathscr{P}^k$ satisfies $\det(\mathrm{Hess}_g) \equiv 0$, then there exists a vector ${}^t(\alpha, \beta) \in \boldsymbol{R}^2$ satisfying $g = (\alpha x + \beta y)^k$, which implies $g \notin \mathscr{P}_o^k$. Therefore we see that for $g \in \mathscr{P}_o^k$, $\varpi_g \equiv 0$ (resp. $\not\equiv 0$) is equivalent to $d\tilde{g}/d\theta \equiv 0$ (resp. $\not\equiv 0$) and this leads us to study the behavior of the principal distributions in relation to the behavior of the gradient vector field. In [2], we have carried out this on G_g for $g \in \mathscr{P}_o^k$.

Let $\mathscr{A}_o^{(2)}$ be the set of the real-analytic functions defined on a connected neighborhood of $(0,0)$ in $\boldsymbol{R}^2$ such that for each $F \in \mathscr{A}_o^{(2)}$, $F(0,0) = p_F(0,0) = q_F(0,0) = 0$ hold, and $\mathscr{A}_o^2$ the set of the elements of $\mathscr{A}_o^{(2)}$ such that on each of their graphs, o is an isolated umbilical point. One of the purposes of this paper is to study the behavior of the principal distributions around o on the graph G_F of $F \in \mathscr{A}_o^{(2)}$ satisfying $\varpi_F \equiv 0$ and the index $\mathrm{ind}_o(\mathsf{G}_F)$ of o for $F \in \mathscr{A}_o^2$ satisfying $\varpi_F \equiv 0$. In Section 5, we shall prove the following:

Theorem 1.2. *Let F be an element of $\mathscr{A}_o^2$ satisfying $\varpi_F \equiv 0$. Then G_F is part of a surface of revolution such that o lies on the axis of rotation; at any point of G_F, the position vector field is in a principal direction and* $\mathrm{ind}_o(\mathsf{G}_F) = 1$ *holds.*

Theorem 1.3. *Let F be an element of $\mathscr{A}_o^{(2)} \backslash \mathscr{A}_o^2$ satisfying $\varpi_F \equiv 0$. Then one of the following holds:*

(1) *G_F is part of a plane or a round sphere;*

(2) *There exist a neighborhood U_o of $(0,0)$ in $\boldsymbol{R}^2$ and a real-analytic curve C_0 in U_o satisfying the following:*

 (a) $C_0 = \{(x,y) \in U_o; F(x,y) = 0\}$,

 (b) $C_0 = \mathrm{Umb}(\mathsf{G}_{F|_{U_o}})$ *or* $\mathrm{Umb}(\mathsf{G}_{F|_{U_o}}) = \varnothing$ *holds,*

 (c) *For any point $q \in C_0$ and for the plane $P_q^\perp$ in $\boldsymbol{R}^3$ normal to C_0 at q, the set $C_q^\perp := P_q^\perp \cap \mathsf{G}_{F|_{U_o}}$ is a real-analytic curve such that at each point of $C_q^\perp$, a tangent vector to $C_q^\perp$ is in a principal direction of G_F.*

Remark. For an integer $l \geqq 3$, let $\mathscr{A}_o^{(l)}$ be the subset of $\mathscr{A}_o^{(2)}$ such that for any $F \in \mathscr{A}_o^{(l)}$ and for non-negative integers $m, n \geqq 0$ satisfying $0 \leqq m+n < l$, $(\partial^{m+n}F/\partial x^m \partial y^n)(0,0) = 0$ holds. For each $F \in \mathscr{A}_o^2$,. there exists an element $f_F \in \mathscr{A}_o^{(3)}$ satisfying $\mathrm{Umb}(\mathsf{G}_{F-f_F}) = \mathsf{G}_{F-f_F}$, and there exists a homogeneous polynomial g_F of degree k_F satisfying $f_F - g_F \in \mathscr{A}_o^{(k_F+1)}$. Let $\mathscr{A}_{oo}^2$ be the subset of $\mathscr{A}_o^2$ such that each $F \in \mathscr{A}_{oo}^2$ satisfies $g_F \in \mathscr{P}_o^{k_F}$. In [3], we have mainly studied the behavior of the principal distributions around o on G_F for $F \in \mathscr{A}_{oo}^2$ satisfying $\varpi_F \not\equiv 0$ and proved $\mathrm{ind}_o(\mathsf{G}_{g_F}) \leqq \mathrm{ind}_o(\mathsf{G}_F) \leqq 1$ for $F \in \mathscr{A}_{oo}^2$.

The gradient vector field of a smooth function f is in a principal direction at a point of G_f if and only if there exists a principal direction at the same point parallel to the xy-plane. A smooth surface S in $\boldsymbol{R}^3$ is called *parallel curved* if there exists a plane P in $\boldsymbol{R}^3$ such that at each point of S, there exists a principal direction parallel to P; if S is parallel curved, then such a plane as P is called a *base plane* of S and the set of the base planes of S is denoted by $\mathscr{B}_S$. A plane, a cylinder and a round sphere are examples of parallel curved surfaces. More generally, a surface of revolution is also parallel curved. We see by Proposition 1.1 that a smooth function f satisfies $\varpi_f \equiv 0$ if and only if G_f is a parallel curved surface such that the xy-plane is an element of $\mathscr{B}_{\mathsf{G}_f}$. A surface does not have to be entirely represented as the graph of a function so that the surface is parallel curved. The other of the purposes of this paper is to classify the connected, complete, real-analytic, embedded, parallel curved surfaces.

Let C_b, C_g be real-analytic, simple curves in $\boldsymbol{R}^3$ with the unique intersection $p_{(C_b, C_g)}$ and contained in planes P_b, P_g, respectively. Then a pair (C_b, C_g) is called *generating* if we may choose as P_g the plane normal to C_b at $p_{(C_b, C_g)}$; if (C_b, C_g) is generating, then C_b and C_g are called the *base curve* and the *generating curve* of (C_b, C_g), respectively. In Section 4, we shall prove the following:

PROPOSITION 1.4. *Let (C_b, C_g) be a generating pair of which C_b (resp. C_g) is the base (resp. generating) curve. Then there exists a connected, real-analytic, parallel curved surface S_0 which contains a neighborhood of $p_{(C_b, C_g)}$ in $C_b \cup C_g$ and satisfies $P_b \in \mathscr{B}_{S_0}$. In addition, if $S_0^{(1)}$ and $S_0^{(2)}$ are such surfaces as S_0, then $S_0^{(1)} \cap S_0^{(2)}$ is also such a surface.*

For a generating pair (C_b, C_g), the maximum of such surfaces as S_0 in Proposition 1.4 is denoted by $S_{(C_b, C_g)}$. In Section 6, we shall prove the following:

THEOREM 1.5. *Let S be a connected, complete, real-analytic, embedded, parallel curved surface. Then S is homeomorphic to a sphere, a plane, a cylinder, or to a torus. In addition,*

(1) *if S is homeomorphic to a sphere, then S is a surface of revolution which crosses its axis of rotation at just two points;*
(2) *if S is homeomorphic to a plane, then one of the following holds:*
    (a) *S is a surface of revolution which crosses its axis of rotation at just one point,*
    (b) *$S = S_{(C_b, C_g)}$ holds, where (C_b, C_g) is a generating pair each element of which is isometric to $\boldsymbol{R}$;*

(3) *if S is homeomorphic to a cylinder, then $S = S_{(C_b, C_g)}$ holds, where (C_b, C_g) is a generating pair such that one of C_b and C_g is isometric to* $\boldsymbol{R}$ *and the other a simple closed curve*;

(4) *if S is homeomorphic to a torus, then $S = S_{(C_b, C_g)}$ holds, where (C_b, C_g) is a generating pair each element of which is isometric to a simple closed curve.*

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2. Preliminaries

Let f be a smooth function of two variables x, y, and G_f the graph of f. We set

$$E_f := 1 + p_f^2, \qquad F_f := p_f q_f, \qquad G_f := 1 + q_f^2,$$

$$L_f := \frac{r_f}{\sqrt{\det(\mathrm{I}_f)}}, \quad M_f := \frac{s_f}{\sqrt{\det(\mathrm{I}_f)}}, \quad N_f := \frac{t_f}{\sqrt{\det(\mathrm{I}_f)}},$$

where $\det(\mathrm{I}_f) := E_f G_f - F_f^2$. The *Weingarten map* of G_f is a tensor field W_f on G_f of type $(1,1)$ satisfying

$$\left[\mathrm{W}_f\left(\frac{\partial}{\partial x}\right), \mathrm{W}_f\left(\frac{\partial}{\partial y}\right)\right] = \left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right]\mathrm{W}_f,$$

where

$$\mathrm{W}_f := \begin{pmatrix} E_f & F_f \\ F_f & G_f \end{pmatrix}^{-1} \begin{pmatrix} L_f & M_f \\ M_f & N_f \end{pmatrix}.$$

A *principal direction* of G_f at (x_0, y_0) is a one-dimensional eigenspace of $W_{f,(x_0,y_0)}$. Let PD_f be a symmetric tensor field on G_f of type $(0,2)$ represented in terms of the coordinates (x, y) as

$$\mathrm{PD}_f := \frac{1}{\sqrt{\det(\mathrm{I}_f)}}\{A_f\, dx^2 + 2B_f\, dxdy + C_f\, dy^2\},$$

where

$$A_f := E_f M_f - F_f L_f, \quad 2B_f := E_f N_f - G_f L_f, \quad C_f := F_f N_f - G_f M_f,$$

$$dx^2 := dx \otimes dx, \quad dxdy := \frac{1}{2}(dx \otimes dy + dy \otimes dx), \quad dy^2 := dy \otimes dy.$$

For vector fields V_1, V_2 on G_f, the following holds:

$$\frac{1}{2} \sum_{\{i,j\}=\{1,2\}} V_i \wedge \mathrm{W}_f(V_j) = \frac{\mathrm{PD}_f(V_1, V_2)}{\sqrt{\det(\mathrm{I}_f)}} \left(\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} \right).$$

Therefore we obtain

PROPOSITION 2.1. *A tangent vector v_0 to G_f at (x_0, y_0) is in a principal direction if and only if $\mathrm{PD}_{f,(x_0,y_0)}(v_0, v_0) = 0$ holds.*

Let $\mathrm{D}_f, \mathrm{N}_f$ be symmetric tensor fields on G_f of type $(0,2)$ represented in terms of the coordinates (x, y) as

$$\mathrm{D}_f := s_f\, dx^2 + (t_f - r_f)\, dxdy - s_f\, dy^2,$$

$$\mathrm{N}_f := (s_f p_f^2 - p_f q_f r_f)\, dx^2 + (t_f p_f^2 - r_f q_f^2)\, dxdy + (p_f q_f t_f - s_f q_f^2)\, dy^2.$$

Then we obtain $\det(\mathrm{I}_f)\mathrm{PD}_f = \mathrm{D}_f + \mathrm{N}_f$. For a vector field V on G_f, we set

$$\tilde{\mathrm{D}}_f(V) := \mathrm{D}_f(V, V), \quad \tilde{\mathrm{N}}_f(V) := \mathrm{N}_f(V, V),$$

$$\widetilde{\mathrm{PD}}_f(V) := \mathrm{PD}_f(V, V).$$

For $\phi \in \boldsymbol{R}$, we set

$$u_\phi := \begin{pmatrix} \cos\phi \\ \sin\phi \end{pmatrix}, \quad U_\phi := \cos\phi \frac{\partial}{\partial x} + \sin\phi \frac{\partial}{\partial y}.$$

Then we obtain

LEMMA 2.2. *For any $\phi \in \boldsymbol{R}$, the following hold:*

$$\tilde{\mathrm{D}}_f(U_\phi) = \langle \mathrm{Hess}_f\, u_\phi, u_{\phi+\pi/2} \rangle,$$

$$\tilde{\mathrm{N}}_f(U_\phi) = \langle \mathrm{grad}_f, u_\phi \rangle \langle \mathrm{grad}_f^\perp, \mathrm{Hess}_f\, u_\phi \rangle.$$

We set

$$\mathbf{Grad}_f := p_f \frac{\partial}{\partial x} + q_f \frac{\partial}{\partial y}, \quad \mathbf{Grad}_f^\perp := -q_f \frac{\partial}{\partial x} + p_f \frac{\partial}{\partial y}.$$

We shall prove

PROPOSITION 2.3. *At each point of* G_f, *the following conditions are mutually equivalent*:

(1) $\varpi_f = 0$;

(2) $A_f + C_f = 0$;

(3) $\mathbf{Grad}_f$ *is in a principal direction of* G_f;

(4) $\mathbf{Grad}_f^{\perp}$ *is in a principal direction of* G_f;

(5) *there exists a principal direction parallel to the xy-plane.*

PROOF. The following holds:

$$\varpi_f = (A_f + C_f)\sqrt{\det(\mathrm{I}_f)}.$$

Therefore we see that (1) is equivalent to (2). By Lemma 2.2, we obtain

$$\varpi_f = \widetilde{\mathrm{PD}}_f(\mathbf{Grad}_f) = -\det(\mathrm{I}_f)\widetilde{\mathrm{PD}}_f(\mathbf{Grad}_f^{\perp}).$$

Therefore we see by Proposition 2.1 that (1), (3) and (4) are mutually equivalent. It is easily seen that (4) is equivalent to (5). □

From Proposition 2.3, we obtain Proposition 1.1.

3. Parallel Curved Surfaces

Let S be a connected, real-analytic, embedded, parallel curved surface and for $P \in \mathscr{B}_S$, let $\Xi_{S,P}$ be the subset of S such that for any $q \in \Xi_{S,P}$, the tangent plane $T_q(S)$ to S at q is not parallel to P. We see that $\Xi_{S,P}$ is an open set of S. If there exists an element P_0 of $\mathscr{B}_S$ satisfying $\Xi_{S,P_0} = \varnothing$, then we see that S is part of a plane in $\boldsymbol{R}^3$. In the following, suppose $\Xi_{S,P} \neq \varnothing$ for any $P \in \mathscr{B}_S$.

For $P_0 \in \mathscr{B}_S$ and for $q \in \Xi_{S,P_0}$, let $P_{P_0,q}^{\perp}$ be the plane in $\boldsymbol{R}^3$ through q perpendicular to P_0 and to $T_q(S)$, and $C_{P_0,q}^{\perp}$ the connected component of $P_{P_0,q}^{\perp} \cap \Xi_{S,P_0}$ containing q. We shall prove

PROPOSITION 3.1. *The plane* $P_{P_0,q}^{\perp}$ *is perpendicular to* $T_p(S)$ *for each* $p \in C_{P_0,q}^{\perp}$.

PROOF. For each $q \in \Xi_{S,P_0}$, there exist orthogonal coordinates (ξ, v, ζ) on $\boldsymbol{R}^3$ satisfying the following:

(1) the point q corresponds to $(0,0,0)$;

(2) the $\xi\zeta$-plane $P_{\xi\zeta}$ is parallel to P_0;

(3) the $v\zeta$-plane $P_{v\zeta}$ is equal to $P_{P_0,q}^{\perp}$.

Then we see that the ξv-plane $P_{\xi v}$ is not perpendicular to $T_q(\Xi_{S,P_0})$. Therefore there exist two positive numbers $\xi_0, v_0 > 0$ and a real-analytic function $F^\perp$ defined on a neighborhood $U_{\xi_0, v_0} := (-\xi_0, \xi_0) \times (-v_0, v_0)$ of q in $P_{\xi v}$ such that the graph $\mathsf{G}_{F^\perp}$ of $F^\perp$ is a neighborhood of q in Ξ_{S, P_0}. The function $F^\perp$ satisfies $F^\perp(0,0) = (\partial F^\perp/\partial\xi)(0,0) = 0$. We see that at each point of $\mathsf{G}_{F^\perp}$, the tangent vector $\partial/\partial\xi$ is in a principal direction. Therefore by Proposition 2.1, we obtain

$$\frac{\partial^2 F^\perp}{\partial\xi\partial v}\left\{1 + \left(\frac{\partial F^\perp}{\partial\xi}\right)^2\right\} = \frac{\partial F^\perp}{\partial\xi}\frac{\partial F^\perp}{\partial v}\frac{\partial^2 F^\perp}{\partial\xi^2} \tag{3}$$

on U_{ξ_0, v_0}. We may represent $F^\perp$ as

$$F^\perp(\xi, v) := \sum_{i,j=0}^{\infty} \alpha_{ij}\xi^i v^j,$$

where $\alpha_{ij} \in \boldsymbol{R}$ and where $\alpha_{00} = \alpha_{10} = 0$. Then at $(0, v) \in U_{\xi_0, v_0}$, we may rewrite (3) into

$$\begin{aligned}&\left(\sum_{j=0}^{\infty}(j+1)\alpha_{1j+1}v^j\right) \times \left(1 + \left(\sum_{j=0}^{\infty}\alpha_{1j}v^j\right)^2\right)\\ &\quad = 2\left(\sum_{j=0}^{\infty}\alpha_{1j}v^j\right) \times \left(\sum_{j=0}^{\infty}(j+1)\alpha_{0j+1}v^j\right) \times \left(\sum_{j=0}^{\infty}\alpha_{2j}v^j\right).\end{aligned} \tag{4}$$

Since $\alpha_{10} = 0$, we obtain $\alpha_{11} = 0$. Generally, we see by (4) that if each element of $\{\alpha_{1k}\}_{k=0}^{j-1}$ for $j \in \boldsymbol{N}$ is equal to zero, then α_{1j} is also equal to zero. Therefore we obtain $\alpha_{1j} = 0$ for any $j \in \boldsymbol{N} \cup \{0\}$. Then for any $v \in (-v_0, v_0)$, $(\partial F^\perp/\partial\xi)(0, v) = 0$ holds. This implies that $T_{(0,v)}(\mathsf{G}_{F^\perp})$ is perpendicular to $P_{v\zeta}$. Noticing $P_{v\zeta} = P^\perp_{P_0,q}$, we obtain Proposition 3.1. $\square$

Corollary 3.2. *The following hold*:

(1) $C^\perp_{P_0,q}$ *is a real-analytic curve*;

(2) *A principal direction of S at each point of $C^\perp_{P_0,q}$ parallel to P_0 is perpendicular to $P^\perp_{P_0,q}$*;

(3) *A nonzero tangent vector to $C^\perp_{P_0,q}$ at each point of $C^\perp_{P_0,q}$ is in a principal direction of S and not parallel to P_0.*

We shall prove

Proposition 3.3. *Let F be an element of $\mathscr{A}_o^{(2)}$ satisfying $\varpi_F \equiv 0$. Then one of the following holds*:

(1) G_F *is part of a surface of revolution such that* o *lies on an axis of rotation*;
(2) *There exist a neighborhood* V_o *of* o *in the xy-plane* P_{xy} *and a positive number* $\varepsilon_0 > 0$ *and a real-analytic curve* γ_ε *in* V_o *for each* $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$ *satisfying the following*:
   (a) $V_o = \bigcup_{\varepsilon \in (-\varepsilon_0, \varepsilon_0)} \gamma_\varepsilon$,
   (b) *for any* $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$ *and for any* $(x, y) \in \gamma_\varepsilon$, $|F(x, y)| = |\varepsilon|$ *holds*,
   (c) *if a line* $l^\perp$ *in* P_{xy} *is normal to* γ_ε *at a point of* $l^\perp \cap \gamma_\varepsilon$ *for some* $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$, *then for any* $\varepsilon' \in (-\varepsilon_0, \varepsilon_0)$, $l^\perp$ *is normal to* $\gamma_{\varepsilon'}$ *at any point of* $l^\perp \cap \gamma_{\varepsilon'}$.

To prove Proposition 3.3, we need lemmas.

For any $\phi \in \boldsymbol{R}$, we set $u_\phi(x, y) := (\cos\phi)x + (\sin\phi)y$. For an element $F \in \mathscr{A}_o^{(2)}$, it is said that F is *of one-variable* if there exist a number $\phi_0 \in \boldsymbol{R}$ and a real-analytic function $f_{F,1}$ defined on a neighborhood of 0 in $\boldsymbol{R}$ satisfying $F = f_{F,1} \circ u_{\phi_0}$ around $(0,0)$, and it is said that F is *radial* if there exists a real-analytic function $f_{F,2}$ defined on a neighborhood of 0 in $\boldsymbol{R}$ satisfying $F = f_{F,2} \circ r^2$ around $(0,0)$, where $r(x, y) := \sqrt{x^2 + y^2}$. We shall prove

Lemma 3.4. *Let* g *be an element of* $\mathscr{P}^k$. *Then* $\varpi_g \equiv 0$ *holds if and only if* g *is of one-variable or radial.*

Proof. We see from equation (2) that $\varpi_g \equiv 0$ holds if and only if $\det(\mathrm{Hess}_g) \equiv 0$ or $d\tilde{g}/d\theta \equiv 0$ holds.

If $d\tilde{g}/d\theta \equiv 0$, then g is radial (see [1]). Suppose $\det(\mathrm{Hess}_g) \equiv 0$ and $d\tilde{g}/d\theta \not\equiv 0$, and let $\tilde{g}$ attain a nonzero extremum at $\theta_0 \in \boldsymbol{R}$. If we represent g as

$$g := \sum_{i=0}^{k} a_i u_{\theta_0}(x, y)^{k-i} u_{\theta_0+\pi/2}(x, y)^i,$$

then by $(d\tilde{g}/d\theta)(\theta_0) = 0$, we obtain $a_1 = 0$. In addition, by $\det(\mathrm{Hess}_g) \equiv 0$, we obtain $a_i = 0$ for each $i \in \{2, \dots, k\}$. Therefore we see that g is of one-variable.

If g is of one-variable (resp. radial), then by direct computation, we obtain $\det(\mathrm{Hess}_g) \equiv 0$ (resp. $d\tilde{g}/d\theta \equiv 0$).

Hence we have proved Lemma 3.4. □

For integers $k_1, k_2, k_3 \geqq 2$, let g_1, g_2, g_3 be elements of $\mathscr{P}^{k_1}, \mathscr{P}^{k_2}, \mathscr{P}^{k_3}$, respectively. We set

$$t_{g_1,g_2,g_3} := \langle \mathrm{Hess}_{g_1} \operatorname{grad}_{g_2}, \operatorname{grad}^{\perp}_{g_3} \rangle,$$

$$T_{g_1,g_2,g_3} := \sum_{\{j_1,j_2,j_3\}=\{1,2,3\}} t_{g_{j_1},g_{j_2},g_{j_3}}.$$

We shall prove

LEMMA 3.5. *Suppose $k_3 \geqq k_2 \geqq k_1$ and that g_1 and g_2 are radial. Then g_3 is also radial if and only if $T_{g_1,g_2,g_3} \equiv 0$ holds.*

PROOF. If g_1 and g_2 are radial, then k_1 and k_2 are even. If we set $g_j = r^{k_j}$ for $j = 1, 2$, then we obtain

$$t_{g_{j_1},g_{j_2},g_3} = -k_1k_2(k_{j_1} - 1)r^{k_1+k_2-4}(xq_{g_3} - yp_{g_3}),$$

$$t_{g_{j_1},g_3,g_{j_2}} = k_1k_2r^{k_1+k_2-4}(xq_{g_3} - yp_{g_3}),$$

$$t_{g_3,g_{j_1},g_{j_2}} = k_1k_2(k_3 - 1)r^{k_1+k_2-4}(xq_{g_3} - yp_{g_3}),$$

where $\{j_1, j_2\} = \{1, 2\}$. Therefore we obtain

$$T_{g_1,g_2,g_3} = k_1k_2(2k_3 - k_1 - k_2 + 2)r^{k_1+k_2-4}(xq_{g_3} - yp_{g_3}).$$

Since $k_3 \geqq k_2 \geqq k_1$, we see that $T_{g_1,g_2,g_3} \equiv 0$ is equivalent to $xq_{g_3} \equiv yp_{g_3}$. In addition, noticing that $xq_{g_3} \equiv yp_{g_3}$ is equivalent to $d\tilde{g}_3/d\theta \equiv 0$, we see that $T_{g_1,g_2,g_3} \equiv 0$ holds if and only if g_3 is radial. Hence we have proved Lemma 3.5. □

PROOF OF PROPOSITION 3.3. We may represent $F \in \mathscr{A}_o^{(2)}$ as $F := \sum_{i \geqq 2} F^{(i)}$, where $F^{(i)} \in \mathscr{P}^i$. We suppose $F \not\equiv 0$ and set

$$I_F := \{i_0 \in N; F^{(i_0)} \neq 0\}, \quad m_F := \min I_F.$$

Then we may represent ϖ_F as

$$\varpi_F = \sum_{j_1,j_2,j_3 \in I_F} t_{F^{(j_1)},F^{(j_2)},F^{(j_3)}},$$

and we obtain $\varpi_F^{(3m_F-4)} = \varpi_{F^{(m_F)}}$. Therefore by Lemma 3.4, we see that if $F \in \mathscr{A}_o^{(2)}$ satisfies $\varpi_F \equiv 0$, then $F^{(m_F)}$ is of one-variable or radial. If $I_F = \{m_F\}$, then we obtain Proposition 3.3.

Suppose $I_F \neq \{m_F\}$ and that I_F is a finite set. Then set $n := \sharp I_F$ and let $i_1, \dots, i_n$ be the integers satisfying $i_1 < \cdots < i_n$ and $I_F = \{i_j\}_{j=1}^n$. If $F^{(i_1)}, \dots, F^{(i_j)}$

are radial for $j \in \{1, \dots, n-1\}$, then we see by Lemma 3.5 that $F^{(i_{j+1})}$ is also radial. Therefore we see that if $F^{(m_F)}$ is radial, then F is also radial. If I_F is an infinite set, then we obtain the same result. Hence we see that if $F^{(m_F)}$ is radial, then G_F is part of a surfece of revolution such that o lies on an axis of rotation.

Suppose $I_F \neq \{m_F\}$ and that $F^{(m_F)}$ is of one-variable. Then we may suppose $F^{(m_F)} = x^{m_F}$. For each $q \in \mathsf{G}_F$, let $\Pi_q^{\perp}$ be the set of the planes in $\boldsymbol{R}^3$ through q such that each $P^{\perp} \in \Pi_q^{\perp}$ is perpendicular to P_{xy} and to $T_p(\mathsf{G}_F)$ for any point p of the connected component of $P^{\perp} \cap \mathsf{G}_F$ containing q. By Proposition 3.1, we obtain $\sharp \Pi_q^{\perp} = 1$ for any $q \in \Xi_{\mathsf{G}_F, P_{xy}}$. In addition, we shall prove

LEMMA 3.6. *If F is not of one-variable, then the following hold*:
(1) *For each $q \in \mathsf{G}_F$, $\sharp \Pi_q^{\perp} = 1$ holds*;
(2) *the xz-plane P_{xz} is the only one element of $\Pi_o^{\perp}$*.

PROOF. By $\varpi_F \equiv 0$, we obtain $q_{F^{(i)}}(x, 0) = 0$ for any $x \in \boldsymbol{R}$ and for any $i \in I_F$. Therefore we obtain $P_{xz} \in \Pi_o^{\perp}$ and $P_{xz} = P_{P_{xy}, q}^{\perp}$ for any $q \in P_{xz} \cap \Xi_{\mathsf{G}_F, P_{xy}}$. We easily see that for any $\phi \in (-\pi/2, \pi/2) \backslash \{0\}$, the plane perpendicular to P_{xy} and determined by u_{ϕ} is not an element of $\Pi_o^{\perp}$. Therefore we see that for each $q \in \mathsf{G}_F$, $\sharp \Pi_q^{\perp} = 1$ or $= 2$ holds and that if $\sharp \Pi_q^{\perp} = 2$, then the two elements of $\Pi_q^{\perp}$ are perpendicular to each other. Suppose that there exists a point $q_0 \in \mathsf{G}_F$ satisfying $\sharp \Pi_{q_0}^{\perp} = 2$. Then we see that for any $q \in \mathsf{G}_F$, an element of $\Pi_q^{\perp}$ is parallel or perpendicular to P_{xz}. Therefore by Proposition 2.3 and by Corollary 3.2, we see that each of $\partial/\partial x$ and $\partial/\partial y$ is in a principal direction at each point of G_F and that F is of one-variable. Therefore we obtain $\sharp \Pi_q^{\perp} = 1$ for any $q \in \mathsf{G}_F$. Particularly, $\Pi_o^{\perp} = \{P_{xz}\}$ holds and we have proved Lemma 3.6. □

Suppose that $F^{(m_F)}$ is of one-variable and that F is not of one-variable. Then for each $q \in \mathsf{G}_F$, we denote by $P_q^{\perp}$ the only one element of $\Pi_q^{\perp}$. Then we may find a positive number $y_0 > 0$ and an open line segment l_y in P_{xy} through $(0, y)$ for each $y \in (-y_0, y_0)$ satisfying the following:
(1) $l_y \subset P_{(0,y,F(0,y))}^{\perp}$ holds for any $y \in (-y_0, y_0)$;
(2) $\tilde{V}_o := \bigcup_{y \in (-y_0, y_0)} l_y$ is a neighborhood of o in P_{xy}.
In addition, we may find a real-analytic vector field on $\tilde{V}_o$ nonzero and tangent to l_y for some $y \in (-y_0, y_0)$ at each point of $\tilde{V}_o$. Therefore we may find a neighborhood V_o of o in P_{xy} and a positive number $\varepsilon_0 > 0$ and a real-analytic curve γ_{ε} in V_o for each $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$ satisfying (a)~(c) of (2) of Proposition 3.3. If F is of one-variable, then we may easily obtain the same result. Hence we have proved Proposition 3.3. □

4. Generating Pairs

Let P be a plane in $\boldsymbol{R}^3$ and Γ_P the set of the real-analytic, simple curves in P, and for $C \in \Gamma_P$ and for a point $p \in C$, let $L_p^\perp$ be the line in P normal to C at p. Then for $C, \tilde{C} \in \Gamma_P$, we write $C \sim \tilde{C}$ if and only if there exists a continuous bijective map $\alpha_{C,\tilde{C}}$ from C onto $\tilde{C}$ satisfying $L^\perp_{\alpha_{C,\tilde{C}}(p)} = L_p^\perp$ for any $p \in C$. It is seen that $\sim$ is an equivalence relation in Γ_P. We denote by Γ_C the equivalence class of $C \in \Gamma_P$, and by $\Sigma(C)$ the connected component of the set $\bigcup_{C' \in \Gamma_C} C'$ containing C. We immediately obtain

LEMMA 4.1. *For $C \in \Gamma_P$ and for each $p \in C$, there exists a neighborhood O_p of p in C such that $\Sigma(O_p)$ is a domain of P.*

For each plane P in $\boldsymbol{R}^3$, we denote by Proj_P the map from $\boldsymbol{R}^3$ onto P such that if a line L is perpendicular to P, then $\mathrm{Proj}_P(L)$ consists of the only one point of $P \cap L$. Then by Corollary 3.2 and by Proposition 3.3, we obtain

PROPOSITION 4.2. *Let S be a connected, real-analytic, embedded, parallel curved surface and P_0 an element of $\mathscr{B}_S$. Then for any $q \in S$, just one of the following holds:*

(1) *S is part of a surface of revolution such that the line through q perpendicular to P_0 is an axis of rotation of S;*

(2) *There exists a neighborhood U_q of q in S such that if P_1 and P_2 are base planes of S parallel to P_0 and satisfying $P_i \cap U_q \neq \emptyset$ for $i = 1, 2$, then each connected component C_i of $\mathrm{Proj}_{P_0}(P_i \cap U_q)$ is an element of Γ_{P_0} satisfying $C_1 \sim C_2$.*

COROLLARY 4.3. *Let S be a real-analytic, embedded, parallel curved surface and P_0 an element of $\mathscr{B}_S$ and q a point of S for which (2) of Proposition 4.2 holds. Then there exists a generating pair (C_b, C_g) of which C_b (resp. C_g) is the base (resp. generating) curve and which satisfies $q = p_{(C_b, C_g)}$, $C_b, C_g \subset S$ and that P_b is parallel to P_0.*

PROOF OF PROPOSITION 1.4. Let (C_b, C_g) be a generating pair and P_b, P_g planes satisfying $C_b \subset P_b$, $C_g \subset P_g$ and that P_g is normal to C_b at $p_{(C_b, C_g)}$, and $P^\perp$ the plane through $p_{(C_b, C_g)}$ perpendicular to P_b and to P_g. If $C_g \subset P_b$, then we see that a connected, real-analytic, parallel curved surface S_0 which contains a neighborhood of $p_{(C_b, C_g)}$ in $C_b \cup C_g$ and satisfies $P_b \in \mathscr{B}_{S_0}$ is part of P_b. In the following, suppose $C_g \not\subseteq P_b$. Then by Lemma 4.1, we see that there exist neighborhoods O_b, O_g of $p_{(C_b,C_g)}$ in C_b, C_g, respectively satisfying $\mathrm{Proj}_{P_b}(O_g) \subset \Sigma(O_b)$ and the condition that $\mathrm{Proj}_{P^\perp}$ embeds each connected component of $O_g \backslash \{p_{(C_b,C_g)}\}$ into $P^\perp$. For O_b, O_g, there exists a real-analytic surface S satisfying $O_b, O_g \subset S$ and the condition that if P is a plane parallel to P_b and satisfying $P \cap O_g \neq \emptyset$, then each connected component of $\mathrm{Proj}_{P_b}(P \cap S)$ is an element of Γ_{O_b}. The minimum of such surfaces as S is denoted by S_{O_b,O_g}. Then we see that P_b is not parallel to $T_q(S_{O_b,O_g})$ for any $q \in S_{O_b,O_g} \backslash O_b$. For each $q \in S_{O_b,O_g} \backslash O_b$, let (ξ, v, ζ) be orthogonal coordinates on $\boldsymbol{R}^3$ satisfying the following:

(1) the point q corresponds to $(0,0,0)$;

(2) $P_{\xi\zeta}$ is parallel to P_b;

(3) $P_{v\zeta}$ is perpendicular to P_b and to $T_q(S_{O_b,O_g})$.

Then there exist two positive numbers $\xi_0, v_0 > 0$ and a real-analytic function $F^\perp$ defined on a neighborhood $U_{\xi_0,v_0} := (-\xi_0, \xi_0) \times (-v_0, v_0)$ of q in $P_{\xi v}$ such that the graph $\mathsf{G}_{F^\perp}$ of $F^\perp$ is a neighborhood of q in $S_{O_b,O_g} \backslash O_b$. Then we obtain

$$\frac{\partial F^\perp}{\partial \xi}(0, v) = \frac{\partial^2 F^\perp}{\partial \xi \partial v}(0, v) = 0$$

for any $v \in (-v_0, v_0)$. Therefore by Proposition 2.1, we see that each of $\partial/\partial\xi$ and $\partial/\partial v$ is in a principal direction at $(0, v, F^\perp(0, v))$ for any $v \in (-v_0, v_0)$. Since $\partial/\partial\xi$ is parallel to P_b, we see that $S_{O_b,O_g} \backslash O_b$ is a parallel curved surface satisfying $P_b \in \mathscr{B}_{S_{O_b,O_g} \backslash O_b}$. Then we see that a tangent vector to O_b at each point of O_b is in a principal direction of S_{O_b,O_g}. Therefore $S_0 := S_{O_b,O_g}$ is a parallel curved surface which contains a neighborhood $O_b \cup O_g$ of $p_{(C_b,C_g)}$ in $C_b \cup C_g$ and satisfies $P_b \in \mathscr{B}_{S_0}$. It is clear that if $S_0^{(1)}$ and $S_0^{(2)}$ are parallel curved surfaces which contain a neighborhood of $p_{(C_b,C_g)}$ in $C_b \cup C_g$ and satisfy $P_b \in \mathscr{B}_{S_0^{(i)}}$ for $i = 1, 2$, then $S_0^{(1)} \cap S_0^{(2)}$ is also such a surface as $S_0^{(i)}$. Hence we have proved Proposition 1.4. □

5. Proof of Theorem 1.2 and Theorem 1.3

Suppose that $F \in \mathscr{A}_o^{(2)}$ satisfies $\varpi_F \equiv 0$ and (1) of Proposition 3.3. Then F is radial. Then the following hold:

$$\mathrm{grad}_F = 2\frac{df_{F,2}}{d\rho} \circ r^2 \begin{pmatrix} x \\ y \end{pmatrix},$$

$$\mathrm{Hess}_F = 2\frac{df_{F,2}}{d\rho} \circ r^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + 4\frac{d^2 f_{F,2}}{d\rho^2} \circ r^2 \begin{pmatrix} x^2 & xy \\ xy & y^2 \end{pmatrix}.$$

Therefore by Lemma 2.2, we obtain

$$\begin{aligned}&\det(\mathrm{I}_F)\widetilde{\mathrm{PD}}_F(\mathrm{U}_\phi)\\&=4\left\{-\frac{d^2f_{F,2}}{d\rho^2}\circ r^2+2\left[\frac{df_{F,2}}{d\rho}\circ r^2\right]^3\right\}u_\phi(x,y)u_\phi(-y,x).\end{aligned}$$

This implies that the position vector field $x\partial/\partial x+y\partial/\partial y$ is in a principal direction at any $(x,y)\in\mathsf{G}_F$. If $d^2f_{F,2}/d\rho^2\not\equiv 2(df_{F,2}/d\rho)^3$, then we obtain $F\in\mathscr{A}_o^2$ and by formula (1), we obtain $\mathrm{ind}_o(\mathsf{G}_F)=1$. If $d^2f_{F,2}/d\rho^2\equiv 2(df_{F,2}/d\rho)^3$, then $f_{F,2}\equiv 0$ holds or there exists a positive number $a_F>0$ satisfying $f_{F,2}=\sqrt{a_F}-\sqrt{a_F-\rho}$ or $=-\sqrt{a_F}+\sqrt{a_F-\rho}$. Therefore we see that G_F is part of a plane or a round sphere.

Suppose that $F\in\mathscr{A}_o^{(2)}$ satisfies $\varpi_F\equiv 0$ and (2) of Proposition 3.3. Then we see that there exist a neighborhood V_o of o in P_{xy} and a real-analytic curve γ_0 in V_o satisfying $\gamma_0=\{(x,y)\in V_o; F(x,y)=0\}$. For each $F_0\in\mathscr{A}_o^{(2)}$ and for each $q_0:=(x_0,y_0)\in\gamma_0$, we set $f_{F_0,q_0}(x,y):=F_0(x-x_0,y-y_0)$. The function f_{F_0,q_0} is defined on a neighborhood of q_0 in P_{xy}. We shall prove

LEMMA 5.1. *For each $q_0\in\gamma_0$, there exists an element F_{q_0} of $\mathscr{A}_o^{(2)}$ satisfying $\mathsf{G}_{f_{F_{q_0},q_0}}\subset\mathsf{G}_F$ and $m_{F_{q_0}}=m_F$.*

PROOF. There exist positive numbers $u_0,v_0>0$ and a real-analytic map Φ from $U_{u_0,v_0}:=(-u_0,u_0)\times(-v_0,v_0)$ into V_o satisfying the following:

(1) The Jacobian of Φ is nonsingular at each point of U_{u_0,v_0};
(2) for any $u'\in(-u_0,u_0)$, Φ maps the open line segment $\{u=u'\}$ in U_{u_0,v_0} into γ_ε for some $\varepsilon\in(-\varepsilon_0,\varepsilon_0)$;
(3) for any $v'\in(-v_0,v_0)$, Φ maps the open line segment $\{v=v'\}$ in U_{u_0,v_0} into l_y for some $y\in(-y_0,y_0)$.

Then the function $F\circ\Phi$ on U_{u_0,v_0} is of one-variable. This implies Lemma 5.1. □

Suppose $m_F=2$. Then $\mathrm{Umb}(\mathsf{G}_F)\cap\gamma_0=\varnothing$ holds. Therefore by Corollary 3.2, Proposition 3.3 and by Lemma 3.6, we may find a neighborhood U_o of $(0,0)$ in $\boldsymbol{R}^2$ and a real-analytic curve C_0 in U_o satisfying (a), (c) of (2) of Theorem 1.3 and $\mathrm{Umb}(\mathsf{G}_{F|_{U_o}})=\varnothing$.

Suppose $m_F\geqq 3$. Then $\gamma_0\subset\mathrm{Umb}(\mathsf{G}_F)$ holds. There exist real-analytic functions θ_F,c_F on γ_0 such that an element $\tilde{F}_{q_0}\in\mathscr{A}_o^{(2)}$ defined for each $q_0\in\gamma_0$ by

$$\tilde{F}_{q_0}(x,y) := F_{q_0}(x\cos\theta_F(q_0) - y\sin\theta_F(q_0), x\sin\theta_F(q_0) + y\cos\theta_F(q_0))$$

satisfies $m_{\tilde{F}_{q_0}} = m_F$ and $\tilde{F}_{q_0}^{(m_F)} = c_F(q_0)x^{m_F}$. We may suppose that there exist a neighborhood V_o' of o in V_o and a neighborhood γ_0' of o in γ_0 such that for any $(x,y) \in V_o'$ and for any $q_0 \in \gamma_0'$, $\Psi_F(x,y,q_0) := \tilde{F}_{q_0}(x,y)$ makes sense. Then we see that the function Ψ_F is real-analytic on $V_o' \times \gamma_0'$. Therefore we may find a continuous function $\tilde{x}$ on γ_0' satisfying $\tilde{x}(q_0) > 0$ and $(x, 0, \tilde{F}_{q_0}(x,0)) \notin \mathrm{Umb}(\mathbf{G}_{\tilde{F}_{q_0}})$ for any $x \in (-\tilde{x}(q_0), \tilde{x}(q_0))\backslash\{0\}$ and for any $q_0 \in \gamma_0'$. Then by Corollary 3.2, Proposition 3.3 and by Lemma 3.6, we may find a neighborhood U_o of $(0,0)$ in $\boldsymbol{R}^2$ and a real-analytic curve C_0 in U_o satisfying (a), (c) of (2) of Theorem 1.3 and $C_0 = \mathrm{Umb}(\mathbf{G}_{F|_{U_o}})$. Hence we have proved Theorem 1.2 and Theorem 1.3.

6. Classification

In this section, let S be a connected, complete, real-analytic, embedded, parallel curved surface.

Suppose that there exists an element P_0 of $\mathscr{B}_S$ satisfying $\Xi_{S,P_0} = S$. Then for each $q \in S$, we see by Corollary 3.2 that $C_{P_0,q}^{\perp}$ is isometric to $\boldsymbol{R}$. There exists the element $P_{P_0,q} \in \mathscr{B}_S$ satisfying $q \in P_{P_0,q}$ and the condition that $P_{P_0,q}$ is parallel to P_0. Then by Proposition 4.2, we see that $P_{P_0,q} \cap S$ is a real-analytic curve isometric to $\boldsymbol{R}$ or to a simple closed curve. Therefore we obtain

Proposition 6.1. *Let S be a connected, complete, real-analytic, embedded, parallel curved surface satisfying $\Xi_{S,P_0} = S$ for some $P_0 \in \mathscr{B}_S$. Then there exists a generating pair (C_b, C_g) of which C_b (resp. C_g) is the base (resp. generating) curve and which satisfies the following*:

(1) *P_b is parallel to P_0*;

(2) *C_b is isometric to $\boldsymbol{R}$ or to a simple closed curve*;

(3) *C_g is isometric to $\boldsymbol{R}$*;

(4) $S = S_{(C_b, C_g)}$.

Then S is homeomorphic to a plane or to a cylinder.

Suppose $\Xi_{S,P_0} \neq S$ and $\Xi_{S,P_0} \neq \emptyset$ for $P_0 \in \mathscr{B}_S$. Then for $P_0 \in \mathscr{B}_S$ and for $q \in \Xi_{S,P_0}$, we see by Corollary 3.2 and by Proposition 4.2 that the connected component of $P_{P_0,q}^{\perp} \cap S$ containing q is a real-analytic curve isometric to $\boldsymbol{R}$ or to a simple closed curve. There exists the element $P_{P_0,q} \in \mathscr{B}_S$ satisfying $q \in P_{P_0,q}$ and that $P_{P_0,q}$ is parallel to P_0. Then by Proposition 4.2, we see that the connected component of $P_{P_0,q} \cap S$ containing q is a real-analytic curve isometric to $\boldsymbol{R}$ or to a simple closed curve. We shall prove

LEMMA 6.2. *Let P_0 be an element of $\mathscr{B}_S$ and q_0 a point of Ξ_{S,P_0} such that some connected component of $P_{P_0,q_0} \cap S$ shares plural points with some connected component of $P^{\perp}_{P_0,q_0} \cap S$. Then S is a surface of revolution such that a line perpendicular to P_0 is an axis of rotation of S.*

PROOF. Let $O_{q_0}, O^{\perp}_{q_0}$ be domains in $P_{P_0,q_0} \cap S$, $P^{\perp}_{P_0,q_0} \cap S$, respectively satisfying $O_{q_0} \cap O^{\perp}_{q_0} = \emptyset$ and $\sharp(\bar{O}_{q_0} \cap \bar{O}^{\perp}_{q_0}) = 2$, and q_1, q_2 two points of S satisfying $\bar{O}_{q_0} \cap \bar{O}^{\perp}_{q_0} = \{q_1, q_2\}$. Then by Proposition 3.1, we see that there exists the only one point p_0 of $S \backslash \Xi_{S,P_0}$ satisfying $P^{\perp}_{P_0,q} \cap O^{\perp}_{q_0} = \{p_0\}$ for any $q \in O_{q_0}$. By Proposition 4.2, we see that S is a surface of revolution such that the line through p_0 perpendicular to P_0 is an axis of rotation of S. Hence we have proved Lemma 6.2. □

By Lemma 6.2, we obtain

PROPOSITION 6.3. *Let S be a connected, complete, real-analytic, embedded, parallel curved surface satisfying $\Xi_{S,P_0} \neq S$ and $\Xi_{S,P_0} \neq \emptyset$ for any $P_0 \in \mathscr{B}_S$. Then one of the following holds:*

(1) *S is a surface of revolution such that the number of the intersections of S with its axis of rotation is equal to one or two, and then S is homeomorphic to a plane or to a sphere,*

(2) *There exists a generating pair (C_b, C_g) of which C_b (resp. C_g) is the base (resp. generating) curve and which satisfies the following:*

(a) *each of C_b and C_g is isometric to $\boldsymbol{R}$ or to a simple closed curve,*

(b) *$S = S_{(C_b, C_g)}$,*

and then S is homeomorphic to a plane, a cylinder or to a torus.

Using Proposition 6.1 and Proposition 6.3, we obtain Theorem 1.5.

REMARK. If C_b is a circumference in each of Proposition 6.1 and Proposition 6.3, then S is a surface of revolution and its axis of rotation is perpendicular to P_b.

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