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A NOTE ON THE TYPE NUMBER OF REAL HYPERSURFACES IN $P_n(C)$

By

Hiroyuki KURIHARA and Ryoichi TAKAGI

1. Introduction

Let $P_n(C)$ denote an n-dimensional complex projective space with the Fubini-Study metric of constant holomorphic sectional curvature $4c$ and M a real hypersurface in $P_n(C)$ with the induced metric.

The problem with respect to the type number t, i.e., the rank of the second fundamental form of real hypersurfaces in $P_n(C)$ has been studied by many differential geometers ([1], [2] and [3] etc.).

The second named author [4] showed that there is a point p on M such that $t(p) \geq 2$ and M. Kimura and S. Maeda [1] gave an example of real hypersurface in $P_n(C)$ satisfying $t = 2$, which is non-complete. Y. J. Suh [3] proved that there is a point p on a complete real hypersurface M in $P_n(C)$ $(n \geq 3)$ such that $t(p) \geq 3$. According to [2], there is a point p on a complete real hypersurface M in $P_n(C)$ such that $t(p) \geq n$, but there is a mistake in deducation to lead a certain formula.

In this paper, we shall prove the following Main theorem

MAIN THEOREM. *Let M be a complete real hypersurface in $P_n(C)$ $(n \geq 4)$. Then there exists a point p on M such that $t(p) \geq 4$.*

2. Preliminaries

Let $P_n(C)$ $(n \geq 4)$ be a complex projective space with the metric of constant holomorphic sectional curvature $4c$ and M a real hypersurface in $P_n(C)$ with the induced metric. Choose a local field of orthonormal frames $e_1, \dots, e_{2n}$ in $P_n(C)$ such that $e_1, \dots, e_{2n-1}$, restricted to M, are tangent to M. We use the following convention on the range of indices unless otherwise stated: $A, B, \dots = 1, \dots, 2n$

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and $i,j,\ldots = 1,\ldots,2n-1$. We denote by ω^A and ω_B^A the canonical 1-forms and the connection forms, respectively. Then they satisfy

$$d\omega^A + \sum \omega_B^A \wedge \omega^B = 0, \quad \omega_B^A \wedge \omega_A^B = 0. \tag{2.1}$$

We restrict the forms under consideration to M. Then we have $\omega^{2n} = 0$ and by Cartan's lemma we may write as

$$\phi_i \equiv \omega_i^{2n} = \sum h_{ij}\omega^j, \quad h_{ij} = h_{ji}. \tag{2.2}$$

The quadratic form $\sum h_{ij}\omega^i \otimes \omega^j$ and the matrix $H = (h_{ij})$ is called second fundamental form and the shape operater of M for e_{2n}, respectively. Moveover, the curvature form Ω_j^i of M are defined by

$$\Omega_j^i = d\omega_j^i + \sum \omega_k^i \wedge \omega_j^k. \tag{2.3}$$

We denote by $\tilde{J}$ the complex structure of $P_n(\boldsymbol{C})$. Let (J_j^i, f_k) be the almost contact metric structure of M, i.e., $\tilde{J}(e_i) = \sum J_i^j e_j + f_i e_{2n}$. Then (J_j^i, f_k) satisfies

$$\begin{aligned} &\sum J_k^i J_j^k = f_i f_j - \delta_j^i, \quad \sum f_j J_i^j = 0, \\ &\sum f_i^2 = 1, \quad J_j^i + J_i^j = 0. \end{aligned} \tag{2.4}$$

The parallelism of $\tilde{J}$ implies

$$\begin{aligned} dJ_j^i &= \sum (J_k^i \omega_j^k - J_k^j \omega_i^k) - f_i \phi_j + f_j \phi_i, \\ df_i &= \sum (f_j \omega_i^j - J_i^j \phi_j). \end{aligned} \tag{2.5}$$

The equation of Gauss and Codazzi are given by

$$\Omega_j^i = \phi_i \wedge \phi_j + c\omega^i \wedge \omega^j + c\sum (J_k^i J_l^j + J_j^i J_l^k)\omega^k \wedge \omega^l, \tag{2.6}$$

$$d\phi_i = -\sum \phi_j \wedge \omega_i^j + c\sum (f_i J_k^j + f_j J_k^i)\omega^j \wedge \omega^k, \tag{2.7}$$

respectively.

3. Formulas

Let M be a real hypersurface in $P_n(\boldsymbol{C})$. In this section, we assume that the rank of second fundamental form is not larger than m on an open set U. In the sequel, we use the following convention on the range of indices: $a,b,\ldots = 1,\ldots,m$ and $r,s,\ldots = m+1,\ldots,2n-1$. Then for an arbitary point p in U we can take a local field of orthonormal frames $\{e_1, \dots, e_{2n-1}\}$ on a neighborhood of p such that the 1-forms ϕ_i can be written as

$$\phi_a = \sum h_{ab}\omega^b, \quad \phi_r = 0. \tag{3.1}$$

Here, we put

$$\omega_r^a = \sum A_{rb}^a \omega^b + \sum B_{rs}^a \omega^s. \tag{3.2}$$

Taking the exterior derivative of $\phi_r = 0$ and using (2.7) and (3.1), we have

$$\sum h_{ab}\omega^b \wedge \omega_r^a - c\sum (f_r J_j^i + f_i J_j^r)\omega^i \wedge \omega^j = 0,$$

which, together with (3.2), implies

$$\sum (h_{ac}A_{rb}^c - h_{bc}A_{ra}^c) - cf_a J_b^r + cf_b J_a^r - 2cf_r J_b^a = 0, \tag{3.3}$$

$$\sum h_{ab}B_{rs}^b - cf_a J_s^r + cf_s J_a^r - 2cf_r J_s^a = 0, \tag{3.4}$$

$$f_s J_t^r - f_t J_s^r + 2f_r J_t^s = 0. \tag{3.5}$$

The above equation (3.5) is equivalent to

$$f_r J_t^s = 0. \tag{3.6}$$

Similarly, taking the exterior derivative of $\phi_a = \sum h_{ab}\omega^b$ and making use of (2.1), (2.7), (3.1), (3.2) and (3.4), we get

$$\begin{aligned} dh_{ab} &- \sum (h_{ac}\omega_b^c + h_{bc}\omega_a^c - \sum h_{ac}A_{rb}^c \omega^r) \\ &+ c\sum (f_b J_r^a \omega^r - f_r J_b^a \omega^r + 2f_a J_r^b \omega^r) \equiv 0 \quad (mod\ \omega^a). \end{aligned} \tag{3.7}$$

Here, we denote by T the maximal value of the type number t.

The following two Lemmas are proved in [2] and [3].

LEMMA 3.1 ([3]). *Assume that there exists a point $p \in M$ such that $\tilde{J}(\ker H_p) \perp \ker H_p$. Then $t(p) \geq n - 1$. Furthermore, the equality holds if and only if $\tilde{J}((\ker H_p)^{\perp}) \subset \ker H_p$, where $(\ker H_p)^{\perp}$ denotes the set of all vectors normal to $\ker H_p$.*

LEMMA 3.2 ([2]). *If $\tilde{J}(\ker H_{|U}) \perp \ker H_{|U}$, then $T \geq n$ on U.*

We shall take T as m in above. In the remainder of this section we restrict the forms under consideration to the following open set V_T defined by

$$V_T = \{p \in M \mid J^r_s(p) \neq 0, t(p) = T\}.$$

From (3.6) we have $f_r = 0$. Thus we may set $f_1 = 1$ and $f_2 = \cdots = f_T = 0$. This and (2.4) show

$$J^1_a = 0, \quad J^1_r = 0. \tag{3.8}$$

Furthermore, the fact that $df_a = 0$ and $df_r = 0$ tells us

$$\omega^1_a = -\sum J^a_b \phi_b, \tag{3.9}$$

$$A^1_{ra} = \sum h_{ab} J^b_r, \tag{3.10}$$

$$B^1_{rs} = 0, \tag{3.11}$$

where we have used (2.5), (3.1), and (3.2). The above equation (3.9) yields

$$\omega^1_a \equiv 0 \quad (mod\ \omega^a). \tag{3.12}$$

From (3.4), we have

$$\sum h_{ab} B^b_{rs} = c f_a J^r_s. \tag{3.13}$$

Moreover, from (3.11) and (3.13), it follows that (cf. [3])

$$det(h_{ab}) = 0 \quad (a, b = 2, \ldots, T). \tag{3.14}$$

Thus, for a suitable choice of a field $\{e_a\}$ of orthonormal frames, we may set

$$h_{ab} = \lambda_a \delta_{ab} \quad (a, b = 2, \ldots, T). \tag{3.15}$$

Combining (3.15) with (3.14), we can set $\lambda_2 = 0$. Since $det(h_{ab}) = -(h_{12})^2 \lambda_3 \cdots \lambda_T$, it follows that

$$h_{12} \neq 0 \quad \text{and} \quad h_{aa} = \lambda_a \neq 0 \quad (a = 3, \ldots, T), \tag{3.16}$$

because $det(h_{ab})$ does not vanish on V_T.

On the other hand, the equation (3.10), together with (3.8) and (3.15), yields

$$A^1_{r2} = 0. \tag{3.17}$$

Now put $a = 2$ and $b \geq 3$ in (3.3). Then using (3.10), (3.15) and (3.16), we find

$$A^b_{r2} = h_{12} J^b_r \quad (b \geq 3). \tag{3.18}$$

Similarly, put $a = 1$ and $b = 2$ in (2.4). Then we obtain

$$\sum (h_{1a} A^a_{r2} - h_{2a} A^a_{r1}) + cJ^2_r = 0.$$

It follows from (3.10), (3.15), (3.17) and (3.18) that the above equation can be reformed as

$$h_{12} A^2_{r2} = h_{12} \sum h_{1a} J^a_r - h_{12} \sum_{a \geq 3} h_{1a} J^a_r - cJ^2_r. \tag{3.19}$$

We put $a = 2$ and $b \geq 3$ in (3.7) and take account of (3.12) and (3.15). Then we have

$$h_{bb} \omega^b_2 - h_{12} \sum A^1_{rb} \omega^r \equiv 0 \quad (mod\ \omega^a).$$

which, together with (3.8), (3.10) and (3.16), leads to

$$\omega^b_2 \equiv h_{12} \sum J^b_r \omega^r \quad \text{for } b \geq 3 \quad (mod\ \omega^a). \tag{3.20}$$

Put $a = 1$ and $b = 2$ in (3.7). Then from (3.12) it follows that

$$dh_{12} - \sum (h_{1b} \omega^b_2 - \sum h_{1b} A^b_{r2} \omega^r) + 2c \sum J^2_r \omega^r \equiv 0 \quad (mod\ \omega^a).$$

Combining this equation with (3.8), (3.12) and (3.17)~(3.20), we get

$$dh_{12} + ((h_{12})^2 + c) \sum J^2_r \omega^r \equiv 0 \quad (mod\ \omega^a). \tag{3.21}$$

On the other hand, from (3.13) we have

$$h_{a1} B^1_{rs} + \sum_{b \geq 2} h_{ab} B^b_{rs} = 0 \quad \text{for } a \neq 1.$$

Using (3.11) and (3.15), we obtain

$$\lambda_a B^a_{rs} = 0.$$

This equation yields

$$B^a_{rs} = 0 \quad \text{for } a \neq 2. \tag{3.22}$$

Similarly, from (3.4), we find

$$h_{12} B^2_{rs} = cJ^r_s,$$

which, together with (3.17), lead to

$$B^2_{rs} = \frac{c}{h_{12}} J^r_s. \tag{3.23}$$

4. The proof of Main theorem

In this section, we keep the notation in section 3 unless otherwise stated. If $\tilde{J}(\ker H) \perp \ker H$ on a non-empty open set, then Lemma 3.2 proves Main theorem. Therefore, we have only consider the case where the open set V_T defined section 3 is not empty. It is known that $T \geq 3$ (cf. [3]). Assume M is complete and $T = 3$ and derive a contradiction.

LEMMA 4.1. *$J_r^2 \neq 0$ on any non-empty open subset of V_3.*

PROOF. If there exist an open subset of V_3 such that $J_r^2 = 0$, then from (2.4) we get

$$J_3^2 = \pm 1, \quad J_i^3 = 0 \quad \text{for } i \neq 2.$$

Taking account of the coefficient of ω^s in $dJ_r^3 = 0$, and using (2.5), (3.2) and (3.22) we find

$$B_{rs}^2 = 0.$$

This implies $J_s^r = 0$, which contradicts the fact that rank $J = 2n - 2 \geq 4$. □

Thus, owing to Lemma 4.1, we have

$$\forall p \in V_3, \forall U(p), \exists q \in U(p) \quad \text{such that } J_r^2(q) \neq 0, \tag{4.1}$$

where $U(p)$ denotes a neighborhood of p.

Moveover, we consider the open set V_3' defined by

$$V_3' = \{p \in V_3 \mid J_r^2(p) \neq 0\}.$$

Since V_3' is dense subset of V_3 by (4.1), any equality obtained on V_3' holds also on V_3. Hence, we may assume $V_3 = V_3'$ whenever we treat equalities.

On the other hand, for a suitable choice of a field $\{e_r\}$ of orthonormal frames, we can set

$$J_5^2 = \cdots = J_{2n-1}^2 = J_6^3 = \cdots = J_{2n-1}^3 = 0. \tag{4.2}$$

For simplicity, we put $\alpha = J_3^2$ and $\beta = J_4^2$. Then from (2.4) and (4.2), we obtain

$$\begin{aligned} \alpha^2 + \beta^2 &= 1, \\ \beta J_3^4 &= 0. \end{aligned} \tag{4.3}$$

Since $\beta \neq 0$ on V_3', above equation implies

$$J_3^4 = 0 \quad \text{on } V_3. \tag{4.4}$$

From (2.4), (4.2) and (4.4), we get

$$\sum (J_i^3)^2 = \alpha^2 + (J_5^3)^2 = 1,$$

which yields

$$J_5^3 = \pm\beta.$$

We may assume

$$J_5^3 = \beta, \tag{4.5}$$

by taking $-e_5$ instead of e_5 if necessary. Similarly, from (2.4), (4.2), (4.4), (4.5) and the equation $\sum J_i^3 J_4^i = 0$, we have

$$J_4^5 = \alpha. \tag{4.6}$$

It follows from (2.4), (4.2), (4.4)~(4.6) and the equation $\sum (J_i^4)^2 = 1$, that

$$J_6^4 = \cdots = J_{2n-1}^4 = J_6^5 = \cdots = J_{2n-1}^5 = 0. \tag{4.7}$$

Hence, we obtain the following matrix

$$(J_j^i) = \left(\begin{array}{ccccc|c} 0 & 0 & 0 & 0 & 0 & \\ 0 & 0 & \alpha & \beta & 0 & \\ 0 & -\alpha & 0 & 0 & \beta & 0 \\ 0 & -\beta & 0 & 0 & -\alpha & \\ 0 & 0 & -\beta & \alpha & 0 & \\ \hline & & 0 & & & * \end{array}\right) \tag{4.8}$$

Lemma 4.2. *β has not zero points everywhere on V_3.*

Proof. Taking the exterior derivative of $J_5^2 = 0$ and making use of (3.20), (3.22) and (4.8), we have

$$\beta(\omega_5^4 + h_{12}\beta\omega^5) + \alpha^2 \frac{c}{h_{12}} \omega^5 \equiv 0 \quad (mod\ \omega^a).$$

Then if there exists a point p on V_3 such that $\beta(p) = 0$, we get $\alpha(p) = 0$. This contradicts (4.3). □

On the other hand, we put $F = h_{12}$, then the equation (3.21) is equivalent to

$$dF + (F^2 + c)\beta\omega^4 \equiv 0 \quad (mod\ \omega^a). \tag{4.9}$$

Let p be any point of V_3 and let $\gamma : I \to V_3$ be a maximal integral curve of the unit vector field e_4 on V_3 through p. Assume that I has an infimum or a supremum, say t_0.

LEMMA 4.3.

$$\lim_{t \to t_0} h_{33}(\gamma(t)) \neq 0.$$

PROOF. Put $a = b = 3$ in (3.7). Then we get

$$dh_{33} - 2\sum h_{3c}\omega_3^c + \sum h_{3c}A_{r3}^c\omega^r \equiv 0 \quad (mod\ \omega^a).$$

From (3.8), (3.10), (3.12) and (3.15), it follows that

$$dh_{33} + h_{33}\sum(h_{31}J_r^3 + A_{r3}^3)\omega^r \equiv 0 \quad (mod\ \omega^a). \tag{4.10}$$

We restrict the forms under consideration to γ. Then (4.10), together with (4.4), becomes

$$\frac{dh_{33}}{dt} + h_{33}A_{43}^3 = 0, \quad t \in I.$$

On the otherhand, since M is complete, there exists a limit point $\lim_{t \to t_0} \gamma(t)$ on M. Suppose that $\lim_{t \to t_0} h_{33}(\gamma(t)) = 0$. Then from the above differential equation, we have $h_{33} = 0$ on γ. This contradicts (3.16). □

LEMMA 4.4.

$$\lim_{t \to t_0} F(\gamma(t)) = 0.$$

PROOF. Assume that $\lim_{t \to t_0} F(\gamma(t)) \neq 0$. Owing to Lemma 4.3, we see $t(\gamma(t_0)) = 3$. Since γ is maximal, we have $J_s^r(\gamma(t_0)) = 0$. Then by Lemma 3.1, we obtain

$$t(\gamma(t_0)) \geq n - 1 \geq 4 \quad \text{for } n \geq 5,$$

which is a contradiction. For a case where $n = 4$, also by using Lemma 3.1 we get $f_a(\gamma(t_0)) = 0$. This also contradicts $f_1(\gamma(t_0)) = 1$. □

Put $t_1 = \inf I(\geq -\infty)$ and $t_0 = \sup I(\leq \infty)$. Then there are four possibilities of an open interval (t_1, t_0). Namely, the interval I is one of the following:

$$(1) -\infty < t_1, t_0 < \infty,$$

$$(2) -\infty = t_1, t_0 < \infty,$$

$$(3) -\infty < t_1, t_0 = \infty,$$

$$(4) -\infty = t_1, t_0 = \infty.$$

Case (1):

Owing to Lemma 4.4 it is seen that there exist a real number t' such that $t_1 < t' < t_0$, $dF = 0$ at $\gamma(t') \in V_3$. Then (4.9) gives $\beta(\gamma(t')) = 0$. This contradicts Lemma 4.2.

Case (2), (3), (4):

Taking the exterior derivative of $J_4^2 = \beta$ and using (2.5) and (4.8), we have

$$d\beta \equiv -\frac{c}{F}\alpha^2\omega^4 \quad (mod\ \omega^a).$$

We restrict the forms under consideration to γ. Then above equation becomes

$$\frac{d\beta}{dt} = -\frac{c}{F}\alpha^2, \quad t \in I. \tag{4.11}$$

Put $g = F\beta$ and from (4.9) and (4.11), we have

$$\frac{dg}{dt} = -g^2 - c, \quad t \in I. \tag{4.12}$$

Then solving (4.12), we get

$$g(\gamma(t)) = -\sqrt{c}\tan\sqrt{c}(t - t_2), \tag{4.13}$$

where t_2 is a constant. However, (4.13) is defined only for a finite interval, which is contradiction.

It completes the proof of Main theorem.

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Graduate School of Science and Technology
Chiba University
Chiba-Shi, 263-8522
Japan

Department of Mathematics and Informatics
Chiba University
Chiba-Shi, 263-8522
Japan