

ON THE GAUSS MAP OF NULL SCROLLS

By

Soon Meen CHOI, U-Hang KI and Young Jin SUH

Abstract. The purpose of this paper is to characterize a class of non-degenerate ruled surfaces in \mathbf{R}_1^3 , which are said to be null scrolls, satisfying the condition $\Delta\xi = A\xi$, where ξ denote their Gauss maps and $A \in gl(3, \mathbf{R})$.

1. Introduction

Let $H^2(-1)$ (resp. $S_1^2(1)$) be the 2-dimensional hyperbolic space of constant curvature -1 (resp. the 2-dimensional de Sitter space of constant curvature 1) in the 3-dimensional Minkowski space \mathbf{R}_1^3 . Let M be a space-like surface (resp. time-like surface) in \mathbf{R}_1^3 and ξ a unit vector field normal to M . Then, for any point z in M , we regard $\xi(z)$ as a point in $H^2(-1)$ (resp. $S_1^2(1)$) by the parallel translation to the origin in the ambient space \mathbf{R}_1^3 . The map ξ of M into $H^2(-1)$ (resp. $S_1^2(1)$) is called the *Gauss map* of M . In this paper, we give a geometric characterization for a class of non-degenerate ruled surfaces in \mathbf{R}_1^3 satisfying $\Delta\xi = A\xi (A \in gl(3, \mathbf{R}))$.

Let \mathbf{R}^n denote the n -dimensional Euclidean space and $S_0^{n-1}(1/r^2)$ the hypersphere of \mathbf{R}^n centered at the origin with radius r . In the theory of minimal submanifolds in \mathbf{R}^n , Takahashi's theorem [11] is one of interesting results. The theorem gives an important relationship between the theory of minimal submanifolds in $S_0^{n-1}(1/r^2) (\subset \mathbf{R}^n)$ and that of eigenvalues of the Laplacian. From the viewpoint of this result, Chen [3], [4] generalized the notion of minimal submanifolds in $S_0^{n-1}(1/r^2)$ to that of submanifolds of finite type in \mathbf{R}^n , and developed the theory of them greatly. Let M be an m -dimensional Riemannian manifold, x an isometric immersion of M into \mathbf{R}^{m+1} and Δ the Laplacian of M . Generalizing the notion of minimal submanifolds in $S_0^{n-1}(1/r^2)$ another way,

1991 *Mathematics Subject Classification.* Primary 53C40; Secondary 53C15.

Key Words: framed null curve, null scroll, third curvature, Gauss map, Cartan framed curve.

This paper was supported by the grants from BSRI-97-1404 and TGRC-KOSEF.

Received November 22, 1996.

Revised February 10, 1997.

Garay [8] also studied hypersurfaces in \mathbf{R}^n satisfying the condition $\Delta x = Ax$, where A denotes a constant diagonal $(m+1) \times (m+1)$ matrix.

On the other hand, Chen and Piccinni [2] characterized n -dimensional submanifolds M in \mathbf{R}^m satisfying $\Delta G = \lambda G$ ($\lambda \in \mathbf{R}$), where $G : M \rightarrow G(n, m) \subset \mathbf{R}^N$ ($N = {}_m C_n$) denote the generalized Gauss maps of M . Baikoussis and Blair [1] also characterized surfaces in \mathbf{R}^3 satisfying $\Delta \xi = A\xi$ ($A \in gl(3, \mathbf{R})$), where ξ denote their Gauss maps.

As a Lorentzian version to [1], in [5] and [6], the first author has considered the Gauss maps ξ of space-like or time-like surfaces in \mathbf{R}_1^3 satisfying the following equation

$$\Delta \xi = A\xi, \quad A \in gl(3, \mathbf{R}),$$

where $gl(3, \mathbf{R})$ denotes the set of all real 3×3 -matrices. The first author has proved rigidity theorems only for surfaces of revolution and ruled surfaces along any non-null curve in \mathbf{R}_1^3 .

In this paper let us consider a null curve α with null frame $F = \{X, Y, Z\}$. Then (α, F) is called a *framed null curve with frame F* . A non-degenerate ruled surface M in \mathbf{R}_1^3 along α parametrized by

$$x(s, t) = \alpha(s) + tY(s)$$

is called a *null scroll*. It is a time-like surface. The purpose of this paper is to give a geometric characterization for null scrolls satisfying $\Delta \xi = A\xi$ in terms of the function k_0 and the third curvature k_3 (See §2).

THEOREM. *Let M be a null scroll along the framed null curve with proper frame field. Then the Gauss map ξ of M satisfies*

$$\Delta \xi = A\xi, \quad A \in gl(3, \mathbf{R})$$

if and only if the mean curvature $H = (k_3/k_0)$ is constant. In this case, A is always equal to a scalar matrix.

A framed null curve (α, F) with the function $k_0 = 1$ and the first curvature $k_1 = 0$ is said to be a *Cartan framed null curve*. Moreover, for a Cartan framed null curve α with Cartan frame $F = \{X, Y, Z\}$ this kind of ruled surface is said to be a *B-scroll* (See Graves [9]).

COROLLARY. *Let M be a B-scroll along the framed null curve (α, F) . Then the Gauss map ξ of M satisfies the condition*

$$\Delta\xi = A\xi, \quad A \in gl(3, \mathbf{R})$$

if and only if the third curvature k_3 is constant.

2. Null scrolls in the Minkowski 3-space

Let us review the terminology and fundamental properties for a null scroll M in \mathbf{R}_1^3 . Here we refer to [7] and [9]. The purpose of this section is to represent the Laplacian Δ on M explicitly in terms of curvatures of the framed null curve, and to calculate the Gaussian curvature K and the mean curvature H of this null scroll.

\mathbf{R}_1^3 is by definition the 3-dimensional vector space \mathbf{R}^3 with the inner product of signature (1,2) given by

$$\langle x, y \rangle = -x_1y_1 + x_2y_2 + x_3y_3$$

for any column vectors $x = {}^t(x_1, x_2, x_3)$, $y = {}^t(y_1, y_2, y_3) \in \mathbf{R}^3$. Let $\{e_1, e_2, e_3\}$ be the standard orthonormal basis of \mathbf{R}_1^3 given by

$$e_1 = {}^t(1, 0, 0), \quad e_2 = {}^t(0, 1, 0), \quad e_3 = {}^t(0, 0, 1).$$

A basis $F = \{X, Y, Z\}$ of \mathbf{R}_1^3 is called a (*proper*) *null frame* if it satisfies the following conditions:

$$\langle X, X \rangle = \langle Y, Y \rangle = 0, \quad \langle X, Y \rangle = -1,$$

$$Z = X \times Y = \sum_{i=1}^3 \varepsilon_i \det[X, Y, e_i] e_i,$$

where $\varepsilon_1 = -1$, $\varepsilon_2 = \varepsilon_3 = 1$. Hence we obtain that

$$\langle X, Z \rangle = \langle Y, Z \rangle = 0, \quad \langle Z, Z \rangle = 1.$$

A vector V in \mathbf{R}_1^3 is said to be *null* if $\langle V, V \rangle = 0$.

Let $\alpha = \alpha(s)$ be a null curve in \mathbf{R}_1^3 , namely, a smooth curve whose tangent vectors $\alpha'(s)$ are null. For a given smooth positive function $k_0 = k_0(s)$ let us put $X = X(s) = k_0^{-1}\alpha'$. Then X is a null vector field along α . Moreover, there exists a null vector field $Y = Y(s)$ along α satisfying $\langle X, Y \rangle = -1$. Here if we put $Z = X \times Y$, then we can obtain a (*proper*) null frame field $F = \{X, Y, Z\}$ along α . In this case the pair (α, F) is said to be a (*proper*) *framed null curve*. A framed null curve (α, F) satisfies the following, so called the *Frenet equation*:

$$(2.1) \quad \begin{cases} X'(s) = k_1(s)X(s) + k_2(s)Z(s), \\ Y'(s) = -k_1(s)Y(s) + k_3(s)Z(s), \\ Z'(s) = k_3(s)X(s) + k_2(s)Y(s), \end{cases}$$

where $k_i = k_i(s)$, $i = 1, 2, 3$ are smooth functions defined by

$$k_1 = -\langle X', Y \rangle, \quad k_2 = \langle X', Z \rangle, \quad k_3 = \langle Y', Z \rangle.$$

The function k_i is called an i -th curvature of the framed curve. It follows from the fundamental theorem of ordinary differential equations that a framed null curve $(\alpha, F) = (\alpha(s), F(s))$ is uniquely determined by the functions $k_0 (> 0)$, k_1 , k_2 , k_3 and the initial condition.

A framed null curve (α, F) with $k_0 = 1$ and $k_1 = 0$ is called a *Cartan framed null curve* and the frame field F is called a *Cartan frame*.

Let $(\alpha, F) = (\alpha(s), F(s))$ be a null curve with frame $F = \{X, Y, Z\}$. A ruled surface M along α parametrized by

$$x(s, t) = \alpha(s) + tY(s), \quad s \in I, t \in J$$

is called a *null scroll*. It is a time-like surface. Furthermore, for a Cartan framed null curve α with Cartan frame $F = \{X, Y, Z\}$ the ruled surfaces is called a *B-scroll*.

From the Frenet equation (2.1), the natural frame $\{x_s, x_t\}$ on the null scroll M is obtained by

$$x_s = k_0X - k_1tY + k_3tZ, \quad x_t = Y,$$

and the first fundamental form g on M is given by

$$g = g_{11}(ds)^2 + 2g_{12} ds \cdot dt + g_{22}(dt)^2, \\ g_{11} = 2k_0k_1t + k_3^2t^2, \quad g_{12} = -k_0, \quad g_{22} = 0.$$

Hence the null scroll M is a time like surface, namely, $\det g < 0$ everywhere on M . Let g^{ij} ($i, j = 1, 2$) denote the components of the inverse matrix g^{-1} :

$$(2.2) \quad g^{-1} = -\frac{1}{k_0^2} \begin{pmatrix} 0 & k_0 \\ k_0 & (k_3t)^2 + 2k_0k_1t \end{pmatrix}.$$

One can show that the Laplacian Δ of M is expressed as

$$(2.3) \quad \Delta = -\frac{1}{\sqrt{|\mathfrak{G}|}} \sum_{i,j} \frac{\partial}{\partial x_i} \left(\sqrt{|\mathfrak{G}|} g^{ij} \frac{\partial}{\partial x_j} \right) \\ = -\frac{1}{k_0} \left[\frac{\partial}{\partial s} \left(-\frac{\partial}{\partial t} \right) + \frac{\partial}{\partial t} \left\{ \left(-\frac{\partial}{\partial s} \right) - \left(\frac{k_3^2t^2 + 2k_0k_1t}{k_0} \right) \frac{\partial}{\partial t} \right\} \right] \\ = \frac{2}{k_0} \frac{\partial^2}{\partial s \partial t} + \frac{2}{k_0^2} (k_3^2t + k_0k_1) \frac{\partial}{\partial t} + \frac{1}{k_0^2} (k_3^2t^2 + 2k_0k_1t) \frac{\partial^2}{\partial t^2},$$

where \mathfrak{G} denotes the determinant of (g_{ij}) .

Let ξ be the unit normal vector field on the null scroll M in \mathbf{R}_1^3 defined by

$$(2.4) \quad \xi = -\frac{k_3}{k_0} tY - Z.$$

Then, it is a space-like normal vector field to M . Thus, for any point x in M , we can regard $\xi(s)$ as a point in $\mathbf{S}_1^2(1)$ by the parallel translation to the origin in the ambient space \mathbf{R}_1^3 . The map ξ of M into $\mathbf{S}_1^2(1)$ is called the *Gauss map* of M in \mathbf{R}_1^3 . So, the components h_{ij} , $i, j = 1, 2$, of the second fundamental form of M in \mathbf{R}_1^3 are given by

$$h_{12} = g(x_{st}, \xi) = -k_3, \quad h_{22} = g(x_{tt}, \xi) = 0$$

since $x_{st} = x_{ts} = Y' = -k_1 Y + k_3 Z$, $x_{tt} = 0$. Accordingly, the Gaussian curvature K and the mean curvature H of the null scroll M is given by respectively

$$H = \frac{1}{2} \sum_{ij} g^{ij} h_{ij} = g^{12} h_{12} = \frac{k_3}{k_0},$$

and

$$K = \frac{-h_{11}h_{22} + h_{12}^2}{g_{11}g_{22} - g_{12}^2} = -\left(\frac{k_3}{k_0}\right)^2.$$

From the last formula we can assert

PROPOSITION. *A null scroll M along the framed null curve α in \mathbf{R}_1^3 is flat if and only if the third curvature k_3 of α vanishes identically.*

3. Proof of Theorem

In this section, let us prove the Theorem in the introduction.

Since $\xi = -(k_3/k_0)tY - Z$, by applying the Frenet equation (2.1), the Laplacian of ξ is calculated as follows:

$$\begin{aligned} (3.1) \quad \Delta\xi &= \frac{2}{k_0} \frac{\partial^2 \xi}{\partial s \partial t} + \frac{2}{k_0^2} (k_3^2 t + k_0 k_1) \frac{\partial \xi}{\partial t} + \frac{1}{k_0^2} (k_3^2 t^2 + 2k_0 k_1 t) \frac{\partial^2 \xi}{\partial t^2} \\ &= \frac{2}{k_0} \left\{ \frac{1}{k_0^2} (k_0' k_3 - k_0 k_3' + k_0 k_1 k_3) Y - \frac{k_3^2}{k_0} Z \right\} + \frac{2}{k_0^2} (k_3^2 t + k_0 k_1) \left(-\frac{k_3}{k_0} Y \right) \\ &= -\frac{2}{k_0} \left(\frac{k_3}{k_0} \right)' Y + 2 \left(\frac{k_3}{k_0} \right)^2 \xi. \end{aligned}$$

This implies that if the mean curvature $H = (k_3/k_0)$ is constant, then the Gauss map ξ of M is of 1-type:

$$\Delta\xi = 2H^2\xi,$$

namely, the Gauss map $\xi : M \rightarrow \mathcal{S}_1^2(1)$ is harmonic (cf. [10]). Thus the Gauss map satisfied the following formula in Theorem

$$(3.2) \quad \Delta\xi = A\xi, \quad A \in gl(3, \mathbf{R}).$$

Now let us consider the converse. Assume that the Gauss map ξ of the null scroll M satisfies (3.2). Then, for the matrix A we have by (2.4), (3.1) and (3.2)

$$\frac{k_3}{k_0} tAY + AZ = 2 \left\{ \frac{1}{k_0} \left(\frac{k_3}{k_0} \right)' + \left(\frac{k_3}{k_0} \right)^3 t \right\} Y + 2 \left(\frac{k_3}{k_0} \right)^2 Z$$

for the parameter t . Then we have

$$(3.3) \quad \frac{k_3}{k_0} AY = 2 \left(\frac{k_3}{k_0} \right)^3 Y,$$

$$(3.4) \quad AZ = \frac{2}{k_0} \left(\frac{k_3}{k_0} \right)' Y + 2 \left(\frac{k_3}{k_0} \right)^2 Z.$$

We put $k = (k_3/k_0)$. Differentiating (3.3) with respect to the parameter s , we get

$$(3.5) \quad k'AY + k(AY)' = 2(3k^2k'Y + k^3Y').$$

On the other hand, the Frenet equation (2.1) gives

$$(AY)' = AY' = -k_1AY + k_3AZ.$$

From this together with (3.3), (3.4) and (3.5) we have $kk'Y = 0$, which implies that k^2 is constant. It completes the proof of Theorem.

References

- [1] C. Baikoussis and D. E. Blair, On the Gauss map of ruled surfaces, *Glasgow. Math. J.* **34** **16** (1992), 355–359.
- [2] B. Y. Chen and Piccinni, Submanifolds with finite type Gauss map, *Bull. Austral. Math. Soc.* **11** (1987), 161–186.
- [3] B. Y. Chen, Submanifolds of finite type and applications, *Proc. of TGRC.* **3** (1993), 1–48.
- [4] ———, Total curvature and submanifolds of finite type, *World Sciences*, Singapore, 1984.
- [5] S. M. Choi, On the Gauss map of ruled surfaces in a 3-dimensional Minkowski space, *Tsukuba J. Math.* **19** (1995), 285–304.

- [6] S. M. Choi, On the Gauss map of surfaces of revolution in a 3-dimensional Minkowski space, *Tsukuba J. Math.* **19** (1995), 351–367.
- [7] M. Dajczer and K. Nomizu, On flat surfaces in S_1^3 and H_1^3 , *Manifolds and Lie Groups*, Univ. Notre Dame, Birkhäuser, 1981, pp. 71–108.
- [8] O. J. Garay, An extension of Takahashi's Theorem, *Geometriae Dedicata* **34** (1990), 105–112.
- [9] L. K. Graves, Codimension one isometric immersions between Lorentz space, *Trans. Amer. Math. Soc.* **252** (1979), 367–392.
- [10] E. A. Ruh and J. Vilms, The tension field of the Gauss map, *Trans. Amer. Math. Soc.* **149** (1970), 569–573.
- [11] T. Takahashi, Minimal immersions of Riemannian manifolds, *J. Math. Soc. Japan* **18** (1966), 380–385.

Topology and Geometry Research Center,
Kyungpook National University, 702-701,
KOREA

Department of Mathematics,
Kyungpook National University, 702-701,
KOREA

Department of Mathematics,
Kyungpook National University, 702-701,
KOREA