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EINSTEIN-WEYL STRUCTURES ON ALMOST CONTACT METRIC MANIFOLDS

Dedicated to Professor Shûkichi Tanno on his sixtieth birthday

By

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1. Introduction

Pedersen and Swann [7] and Higa [2] studied the existence of an Einstein-Weyl structure on principal circle bundles over Einstein Kaehler manifolds of positive scalar curvature and obtained many examples of Einstein-Weyl spaces. In a Riemannian submersion $\pi : M \to N$ with totally geodesic fibers of dimension one over an Einstein manifold N, we studied the relation between an Einstein-Weyl structure and a Sasakian structure of M (cf. [5]).

On the other hand, in [4], we investigated some geometric structures of a Riemannian submersion $\pi : M \to N$, where M is a CR-submanifold of a locally conformal Kaehler manifold L. Let M be a leaf of the canonical foliation $\mathcal{M}$ given by the Lee form $\omega = 0$ of a locally conformal Kaehler manifold L. Then M admits an almost contact metric structure. We obtained a necessary and sufficient condition for the manifold M to admits a Sasakian structure.

In this paper, we shall study the existence of an Einstein-Weyl structure on an almost contact metric manifold. Let M be a complete and simply connected Sasakian manifold with constant ϕ-sectional curvature k. We show that if $k \geq 1$, then M admits an Einstein-Weyl structure.

Next, we assume that all local Kaehler metrics $g' = e^{-r}g$ of a locally conformal Kaehler manifold L have the same constant nonnegative holomorphic sectional curvature ρ. We show that if the induced almost contact metric structure of a leaf M of the canonical foliation $\mathcal{M}$ of L is Sasakian, then M admits an Einstein-Weyl structure. Finally, we discuss the existence of an Einstein-Weyl structure on a leaf of the canonical foliation of a complete and simply connected generalized Hopf manifold.

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2. Preliminaries

Firstly, we give the definition of an Einstein-Weyl space. Let (M,g) be a Riemannian manifold. Let D be a torsion-free affine connection on M. A manifold M is said to have an *Einstein-Weyl structure* if there exist a 1-form μ and a function Λ on M such that

$$Dg=\mu\otimes g \quad \text{and} \quad {}^{D}Ric(X,Y)+{}^{D}Ric(Y,X)=\Lambda g(X,Y), \tag{1}$$

where ${}^{D}Ric$ is the Ricci tensor of D. Since D is not a metric connection, the Ricci tensor is not necessarily symmetric. The Einstein-Weyl equation is conformally invariant. Let ∇ be the Levi-Civita connection of g. We define a vector field E by $g(X,E)=\mu(X)$. Then, since $Dg=\mu\otimes g$, we have

$$D_XY=\nabla_XY-\frac{1}{2}\mu(X)Y-\frac{1}{2}\mu(Y)X+\frac{1}{2}g(X,Y)E. \tag{2}$$

Let ${}^{D}R$ and R be the curvature tensor of D and ∇ respectively. Then, we have

$$\begin{aligned}{}^{D}R(X,Y)Z=R(X,Y)Z&-\frac{1}{2}\left\{\left[(\nabla_X\mu)Z+\frac{1}{2}\mu(X)\mu(Z)\right]Y\right.\\&-\left[(\nabla_Y\mu)Z+\frac{1}{2}\mu(Y)\mu(Z)\right]X+((\nabla_X\mu)Y)Z-((\nabla_Y\mu)X)Z\\&\left.-g(Y,Z)\left(\nabla_XE+\frac{1}{2}\mu(X)E\right)+g(X,Z)\left(\nabla_YE+\frac{1}{2}\mu(Y)E\right)\right\}\\&-\frac{1}{4}|\mu|^2(g(Y,Z)X-g(X,Z)Y),\end{aligned} \tag{3}$$

where X, Y and Z are any vector fields on M.

Next, we give the definition of an almost contact metric manifold. A Riemannian manifold (M,g) is said to be an *almost contact metric manifold* if there exist a tensor field ϕ of type $(1,1)$, a unit vector field V and a 1-form η such that

$$\begin{aligned}&\eta(V)=1,\quad \phi^2X=-X+\eta(X)V,\\&g(\phi X,\phi Y)=g(X,Y)-\eta(X)\eta(Y),\end{aligned} \tag{4}$$

for any vector fields X, Y on M (cf. [1]).

For an almost contact metric structure (ϕ,V,η,g) on M we put $\Phi(X,Y)=g(X,\phi Y)$. An almost contact metric structure is said to be:
Contact metric if $d\eta=\Phi$.
K-contact if $d\eta=\Phi$ and V is a Killing vector field with respect to g.

Sasakian if $d\eta = \Phi$ and $N_\phi + 2\,d\eta \otimes V = 0$, where $N_\phi(X, Y) = [\phi X, \phi Y] - \phi[\phi X, Y] - \phi[X, \phi Y] + \phi^2[X, Y]$.

If the Ricci tensor *Ric* of a contact metric manifold M is of the form

$$Ric(X, Y) = \beta g(X, Y) + \gamma\eta(X)\eta(Y), \tag{5}$$

β and γ being constant, then M is called *η-Einstein manifold.*

Next, let L be an almost Hermitian manifold with metric g, complex structure J and the fundamental 2-form Ω. The manifold L is said to be a *locally conformal Kaehler manifold* if every $x \in L$ has an open neighborhood U with a differentiable function $r : U \to \boldsymbol{R}$ such that $g'_U = e^{-r}g|_U$ is a Kaehler metric on U. If we can take $U = L$, the manifold is *globally conformal Kaehler*. The locally conformal Kaehler manifold L is characterized by

$$N_J = 0, \quad d\Omega = \omega \wedge \Omega, \quad d\omega = 0, \tag{6}$$

where N_J is the Nijenhuis tensor of J and ω is a globally defined 1-form on L. We call ω the *Lee form*. Since for $\dim L = 2$ we have $d\Omega = 0$, we may suppose $\dim L \geq 4$. Next we define a *Lee vector field B* by

$$g(X, B) = \omega(X). \tag{7}$$

The *Weyl connection* ${}^W\nabla$ is the linear connection defined by

$${}^W\nabla_X Y := \nabla_X Y - \frac{1}{2}\omega(X)Y - \frac{1}{2}\omega(Y)X + \frac{1}{2}g(X, Y)B, \tag{8}$$

where ∇ is the Levi-Civita connection of g. ${}^W\nabla$ is the Levi-Civita connection of a local Kaehler metric g'. It is shown in [9] that an almost Hermitian manifold L is a locally conformal Kaehler if and only if there is a closed 1-form ω on L such that

$${}^W\nabla_X J = 0. \tag{9}$$

Let L be a locally conformal Kaehler manifold. Let R^L be the curvature tensor field of the connection ∇ and ${}^W R$ the curvature tensor field of the Weyl connection ${}^W\nabla$. Since $d\omega = 0$, we obtain $(\nabla_X\omega)Y - (\nabla_Y\omega)X = 0$. Thus, we have

$$\begin{aligned} {}^W R(X, Y)Z = R^L(X, Y)Z - \frac{1}{2}\Big\{ &\Big[(\nabla_X\omega)Z + \frac{1}{2}\omega(X)\omega(Z)\Big]Y \\ &- \Big[(\nabla_Y\omega)Z + \frac{1}{2}\omega(Y)\omega(Z)\Big]X - g(Y, Z)\Big(\nabla_X B + \frac{1}{2}\omega(X)B\Big) \\ &+ g(X, Z)\Big(\nabla_Y B + \frac{1}{2}\omega(Y)B\Big)\Big\} - \frac{1}{4}|\omega|^2(g(Y, Z)X - g(X, Z)Y), \end{aligned} \tag{10}$$

where X, Y and Z are any vector fields on L (cf. [11]).

Let $\mathscr{M}$ be the foliation given by $\omega = 0$ of a locally conformal Kaehler manifold (L, J, g). $\mathscr{M}$ is called the *canonical foliation*.

A locally conformal Kaehler manifold (L, J, g) is said to be a *generalized Hopf manifold* if the Lee form is parallel, that is $\nabla\omega = 0$ $(\omega \neq 0)$.

Let M be a submanifold of a Riemannian manifold L. We denote by the same g the Riemannian metric tensor field induced on M from that of L. Let ∇^M denote covariant differentiation of M. Then the Gauss formula for M is written as

$$\nabla_X Y = \nabla_X^M Y + \sigma(X, Y) \tag{11}$$

for any tangent vector fields X, Y on M where σ denotes the second fundamental form of M in L.

Let R^M be the Riemannian curvature tensor field of M. Then we have the equation of Gauss

(12)

$$R^L(W, Z, X, Y) = R^M(W, Z, X, Y) + g(\sigma(X, Z), \sigma(Y, W)) - g(\sigma(Y, Z), \sigma(X, W)).$$

3. Einstein-Weyl structures

Let (M, ϕ, V, η, g) be an almost contact metric manifold of dimension $2n+1$. Let ∇^M be the Levi-Civita connection of g and Ric^M be the Ricci tensor of ∇^M. Then we obtain the following result.

THEOREM 1. *Let (M, ϕ, V, η, g) be an almost contact metric manifold satisfying $\nabla_X^M V = -\phi(X)$. If $Ric^M(X, Y) = \beta g(X, Y) + \gamma\eta(X)\eta(Y)$, where β and γ are constant, and $\gamma \leq 0$, then M admits an Einstein-Weyl structure.*

PROOF. We define 1-form μ by $\mu = f\eta$, where f is a function on M. Let E be the dual vector field of μ. We define the connection D by

$$D_X Y = \nabla_X^M Y - \frac{1}{2}\mu(X)Y - \frac{1}{2}\mu(Y)X + \frac{1}{2}g(X, Y)E. \tag{13}$$

Then D is a torsion-free connection and $Dg = \mu \otimes g$. Let ${}^D Ric$ be the Ricci tensor of D. From (3), we obtain

(14)

$$\begin{aligned}{}^D Ric(X, Y) = {} & Ric^M(X, Y) + n(\nabla_X^M \mu)Y - \frac{1}{2}(\nabla_Y^M \mu)X \\ & + \frac{1}{4}(2n-1)\mu(X)\mu(Y) + g(X, Y)\left(\frac{1}{2}\operatorname{div} E - \frac{1}{4}(2n-1)|\mu|^2\right) \text{(cf. [7]).}\end{aligned}$$

Let $V, X_1, \phi X_1, \ldots, X_n, \phi X_n$ be an orthonormal basis of T_pM. Since $\nabla^M_X V = -\phi(X)$, for $j = 1, \ldots, n$, we get $\nabla^M_V V = 0$ and $g(\nabla^M_{X_j} X_j, V) = -g(\nabla^M_{X_j} V, X_j) = 0$. Thus we have $(\nabla^M_{X_j}\mu)V = X_j(f)$, $(\nabla^M_V \mu)(X_j) = 0$, $(\nabla^M_V \mu)V = V(f)$, $(\nabla^M_{X_i}\mu)X_j = 0$ for all i, j, $(\nabla^M_{X_i}\mu)(\phi X_j) = 0$ for $i \neq j$ and $(\nabla^M_{X_j}\mu)(\phi X_j) + (\nabla^M_{\phi X_j}\mu)X_j = 0$. From these, we have

$$(15)\qquad {}^DRic(X_j, V) + {}^DRic(V, X_j) = \frac{1}{2}(2n-1)X_j(f),$$

$$(16)\qquad {}^DRic(\phi X_j, V) + {}^DRic(V, \phi X_j) = \frac{1}{2}(2n-1)\phi X_j(f),$$

$$(17)\quad 2 \cdot {}^DRic(V, V) = 2(\beta+\gamma) + \frac{1}{2}(2n-1)(2V(f) + f^2) + \operatorname{div} E - \frac{1}{2}(2n-1)f^2,$$

$$(18)\qquad 2 \cdot {}^DRic(X_j, X_j) = 2\beta + \operatorname{div} E - \frac{1}{2}(2n-1)f^2,$$

and

$$(19)\qquad 2 \cdot {}^DRic(\phi X_j, \phi X_j) = 2\beta + \operatorname{div} E - \frac{1}{2}(2n-1)f^2.$$

We set $f^2 = -(4/(2n-1))\gamma$. Then $V(f) = 0$, $X_j(f) = \phi X_j(f) = 0$ for $j = 1, \ldots, n$ and $\operatorname{div} E = V \cdot \mu(V) = V(f) = 0$. Thus, by using equations (15)–(19), we obtain

$$(20)\qquad {}^DRic(X, Y) + {}^DRic(Y, X) = \Lambda g(X, Y),$$

where X, Y are tangent vectors of M and $\Lambda = 2(\beta + \gamma)$. Therefore M admits an Einstein-Weyl structure. ■

REMARK 1. K-contact manifold and Sasakian manifold satisfy the condition $\nabla^M_X V = -\phi(X)$ (cf. [1]). Let $\pi : M^{2n+1} \to N^{2n}$ be a Riemannian submersion with totally geodesic fibers of dimension one over an Einstein manifold N^{2n} such that $Ric^N(\tilde{X}, \tilde{Y}) = c\tilde{g}(\tilde{X}, \tilde{Y})$. Moreover, we assume that $(M^{2n+1}, \phi, V, \eta, g)$ is a standard Sasakian manifold. Let $V, X_1, \phi X_1, \ldots, X_n, \phi X_n$ be an orthonormal basis of T_pM. Then we get $Ric^M(X_j, X_j) = Ric^M(\phi X_j, \phi X_j) = c - 2$, $Ric^M(V, V) = 2n$, $Ric^M(X_j, V) = Ric^M(\phi X_j, V) = 0$, $Ric^M(X_i, \phi X_j) = 0$ for all i, j and $Ric^M(X_i, X_j) = Ric^M(\phi X_i, \phi X_j) = 0$ for $i \neq j$. (cf. [5]). Therefore M is an η-Einstein manifold such that $Ric^M(X, Y) = (c-2)g(X, Y) + (2n - c + 2)\eta(X)\eta(Y)$. If the scalar curvature $\tilde{s}$ of N is $\tilde{s} \geq 4n(n+1)$, then $2n - c + 2 = (1/(2n))(4n^2 + 4n - \tilde{s}) \leq 0$. Thus, as a corollary of Theorem 1, we obtain Theorem 2 in [5].

Next, let (M, ϕ, V, η, g) be a Sasakian manifold of constant ϕ-sectional curvature k and $\dim M = 2n + 1$. The curvature of M is

$$(21)\quad R^M(X,Y,Z,W) = \frac{k+3}{4}\{g(X,Z)g(Y,W) - g(X,W)g(Y,Z)\}$$
$$-\frac{k-1}{4}\{\eta(X)\eta(Z)g(Y,W) - \eta(Y)\eta(Z)g(X,W)$$
$$+ g(X,Z)\eta(Y)g(V,W) - g(Y,Z)\eta(X)g(V,W)$$
$$+ g(\phi Y,Z)g(\phi X,W) - g(\phi X,Z)g(\phi Y,W)$$
$$+ 2g(X,\phi Y)g(\phi Z,W)\} \quad \text{(cf. [12])}.$$

From this equation, we get

$$(22)\qquad Ric^M(X,Y) = \beta g(X,Y) + \gamma\eta(X)\eta(Y),$$

where $\beta = ((n+1)/2)k + ((3n-1)/2)$ and $\gamma = -((n+1)/2)(k-1)$.

Let S^{2n+1} be the unit sphere in C^{n+1} and τ be real number such that $\tau > 0$. Let (J,h) be the flat Kaehler structure on C^{n+1} and U be the unit normal vector field of S^{2n+1} in C^{n+1}. We define tensor field ϕ and 1-form $\bar{\eta}$ on S^{2n+1} by $J(X) = \phi(X) - \bar{\eta}(X)U$. And we put $V = (1/\tau)JU$, $\eta = \tau\bar{\eta}$ and $g = \tau h' + \tau(\tau-1)\bar{\eta}\otimes\bar{\eta}$, where h' is the induced metric on S^{2n+1} by h. Then (ϕ,V,η,g) is a Sasakian structure with constant ϕ-sectional curvature $k = (4/\tau) - 3$ and we denote S^{2n+1} with this structure by $S^{2n+1}(k)$ (cf. [8]). Thus we have (22) and $\nabla^S_X V = -\phi(X)$, where ∇^S is the Levi-Civita connection of g. From a Theorem of Tanno [8] and Theorem 1, we obtain the following result.

THEOREM 2. *Let M be a complete and simply connected Sasakian manifold with constant ϕ-sectional curvature k. If $k \geq 1$, then M admits an Einstein-Weyl structure.*

REMARK 2. Let (ϕ,V,η,g) be a Sasakian structure of the sphere $(S^{2n+1}(k),g)$. In the Hopf fibration $\pi : (S^{2n+1}(k),g) \to (P_n(C)(k+3),\tilde{g})$, for constant $a \neq 0$, we define a Riemannian metric g_a by $g_a = \pi^*\tilde{g} + a^2\eta\otimes\eta$ (cf. [2]). We set $\tilde{V} = (1/a)V$ and $\tilde{\eta} = a\eta$. Then $d\tilde{\eta} = a\tilde{\Phi}$, where $\tilde{\Phi}(X,Y) = g_a(X,\phi Y)$. Let A be the integrability tensor of the Riemannian submersion $\pi_a : (S^{2n+1}(k),g_a) \to (P_n(C)(k+3),\tilde{g})$ with totally geodesic fibers. Let R^S and R^P denote the curvature tensors of $(S^{2n+1}(k),g_a)$ and $(P_n(C)(k+3),\tilde{g})$ respectively. We recall the following curvature identity.

$$(23)\qquad R^S(W,Z,X,Y) = R^P(\tilde{W},\tilde{Z},\tilde{X},\tilde{Y}) - g_a(A_Y Z, A_X W)$$
$$+ g_a(A_X Z, A_Y W) + 2g_a(A_X Y, A_Z W),$$

where $\tilde{X}$, $\tilde{Y}$, $\tilde{Z}$ and $\tilde{W}$ are tangent vector fields on $P_n(C)(k+3)$ and X, Y, Z and W are the horizontal lifts of $\tilde{X}$, $\tilde{Y}$, $\tilde{Z}$ and $\tilde{W}$ respectively (cf. [6]). For a function f on $S^{2n+1}(k)$, we put $\mu = f\eta$. We define the connection $\bar{D}$, by

$$\bar{D}_X Y = \nabla^a_X Y - \frac{1}{2}\mu(X)Y - \frac{1}{2}\mu(Y)X + \frac{1}{2}g_a(X, Y)E,$$

where ∇^a is the Levi-Civita connection of g_a and E is the dual vector field of μ. Let ${}^{\bar{D}}Ric^a$ be the Ricci tensor of $\bar{D}$. Since $A_X Y = (1/2)\mathscr{V}[X, Y]$ (cf. [6]) and $\tilde{V}$ is a unit vertical vector field, we get $A_X Y = -d\tilde{\eta}(X, Y)\tilde{V} = -ag_a(X, \phi Y)\tilde{V}$, where X, Y are horizontal vector fields. Using $|\mu|^2 = (f^2/a^2)$, for an orthonormal basis $\tilde{V}, X_1, \phi X_1, \dots, X_n, \phi X_n$ of $T_pS^{2n+1}(k)$, we get following.

$$2 \cdot {}^{\bar{D}}Ric^a(\tilde{V}, \tilde{V}) = 4na^2 + \frac{1}{2}(2n-1)\left(\frac{2}{a}\tilde{V}(f) + \frac{f^2}{a^2}\right) + \operatorname{div} E - \frac{1}{2}(2n-1)\frac{f^2}{a^2},$$

$$2 \cdot {}^{\bar{D}}Ric^a(X_j, X_j) = (n+1)(k+3) - 4a^2 + \operatorname{div} E - \frac{1}{2}(2n-1)\frac{f^2}{a^2}$$

and

$${}^{\bar{D}}Ric^a(X_j, \tilde{V}) + {}^{\bar{D}}Ric^a(\tilde{V}, X_j) = \frac{1}{2}(2n-1)X_j\left(\frac{f}{a}\right).$$

For ϕX_j, we also have same equations.

Let constant a be $(k+3)/4 \geq a^2$ and set $f^2 = (2(n+1)a^2/(2n-1)) \times (k+3-4a^2)$. Then we have

$${}^{\bar{D}}Ric^a(X, Y) + {}^{\bar{D}}Ric^a(Y, X) = \Lambda g_a(X, Y),$$

where $\Lambda = 4na^2$. Therefore, for $k > -3$, $(S^{2n+1}(k), g_a, \mu)$ admits an Einstein-Weyl structure but not Sasakian for $a \neq 1$.

4. Foliations of locally conformal Kaehler manifolds

As an application of Theorem 1 and Theorem 2, we consider the canonical foliation of a locally conformal Kaehler manifold. Let (L, J, g) be a locally conformal Kaehler manifold of dimension $2n+2$ and ω the Lee form and $\mathscr{M}$ the canonical foliation given by $\omega = 0$. Let M be a leaf of the canonical foliation $\mathscr{M}$, that is M is an orientable real hypersurface of L. Let B be the Lee vector field. We set $C = (B/|\omega|)$, $V = JC$, $\eta(X) = g(X, V)$ and $JX = \phi X - \eta(X)C$. It is known that every orientable real hypersurface of an almost Hermitian manifold has an almost contact metric structure (ϕ, V, η, g) (cf. [1], [10]).

We show the following theorem.

THEOREM 3. *Let (L, g) be a locally conformal Kaehler manifold and assume that all its local Kaehler metrics $g' = e^{-r}g$ have the same constant nonnegative holomorphic sectional curvature ρ. Let M be a leaf of the canonical foliation $\mathscr{M}$. If the induced almost contact metric structure of M is Sasakian, then M admits an Einstein-Weyl structure.*

PROOF. It is known that M admits a Sasakian structure if and only if

$$\sigma(X, Y) = -\left(\left(\frac{1}{2}|\omega| - 1\right)g(X, Y) + \alpha\eta(X)\eta(Y)\right)C, \tag{24}$$

where α is a function on M (cf. [4]). Let $C, V, X_1, JX_1, \ldots, X_n, JX_n$ be an orthonormal basis of T_pL such that $V, X_1, \phi X_1, \ldots, X_n, \phi X_n$ $(\phi X_i = JX_i)$ form an orthonormal basis of T_pM. We denote the Ricci tensor of ∇ of L by Ric^L. Using (12), for $X, Y \in T_pM$, the Ricci tensor Ric^M of ∇^M of M is

$$\begin{aligned} Ric^M(X, Y) = {} & Ric^L(X, Y) - R^L(C, Y, C, X) - \{g(\sigma(V, Y), \sigma(X, V)) \\ & - g(\sigma(X, Y), \sigma(V, V)) + \sum_{i=1}^{n}[g(\sigma(X_i, Y), \sigma(X, X_i)) \\ & - g(\sigma(X, Y), \sigma(X_i, X_i)) + g(\sigma(\phi X_i, Y), \sigma(X, \phi X_i)) \\ & - g(\sigma(X, Y), \sigma(\phi X_i, \phi X_i))]\}. \end{aligned} \tag{25}$$

Since the local Kaehler metrics g' have the same constant nonnegative holomorphic sectional curvature ρ, the curvature tensor of g' is given by

$$\begin{aligned} R'(X, Y, Z, W) = \frac{\rho}{4}\{ & g'(X, Z)g'(Y, W) - g'(X, W)g'(Y, Z) \\ & + g'(X, JZ)g'(Y, JW) - g'(X, JW)g'(Y, JZ) \\ & + 2g'(X, JY)g'(Z, JW)\} \quad \text{(cf. [12])}. \end{aligned} \tag{26}$$

We put ${}^{W}R(X, Y, Z, W) = g({}^{W}R(Z, W)Y, X)$. Since $R'(X, Y, Z, W) = e^{-r\,W}R(X, Y, Z, W)$, we get

$$\begin{aligned} {}^{W}R(X, Y, Z, W) = \frac{\rho}{4}e^{-r}\{ & g(X, Z)g(Y, W) - g(X, W)g(Y, Z) \\ & + g(X, JZ)g(Y, JW) - g(X, JW)g(Y, JZ) \\ & + 2g(X, JY)g(Z, JW)\}. \end{aligned} \tag{27}$$

From this equation, we get

$$ {}^{W}Ric(X, Y) = \frac{\rho}{4} e^{-r}(2n+4) g(X, Y). \tag{28} $$

From (10), for any vector fields X, Y tangent to M, we have

$$ \begin{aligned} R^{L}(C, Y, C, X) &= \frac{\rho}{4} e^{-r}(g(X, Y) + 3\eta(X)\eta(Y)) - \frac{1}{2} g(((\nabla_X \omega)Y)C, C) \\ &\quad - \frac{1}{2} g(X, Y) g(\nabla_C B, C) \end{aligned} \tag{29} $$

and

(30)

$$ Ric^{L}(X, Y) = \frac{\rho}{4} e^{-r}(2n+4) g(X, Y) - n(\nabla_X \omega)Y - \left(\frac{1}{2} \operatorname{div} B - \frac{1}{2} n|\omega|^2\right) g(X, Y). $$

From (11) and (24), we have

$$ (\nabla_X \omega)Y = -\omega(\nabla_X Y) = \left(\left(\frac{1}{2}|\omega| - 1\right) g(X, Y) + \alpha \eta(X)\eta(Y)\right)|\omega| \tag{31} $$

and

$$ \begin{aligned} \operatorname{div} B &= g(\nabla_C B, C) + g(\nabla_V B, V) + \sum_{i=1}^{n} g(\nabla_{X_i} B, X_i) + \sum_{i=1}^{n} g(\nabla_{JX_i} B, JX_i) \\ &= g(\nabla_C B, C) + (2n+1)|\omega|\left(\frac{1}{2}|\omega| - 1\right) + \alpha|\omega|. \end{aligned} \tag{32} $$

By using (29), (30), (31) and (32), we have

$$ \begin{aligned} &Ric^{L}(X, Y) - R^{L}(C, Y, C, X) \\ &= \frac{\rho}{4} e^{-r}\{(2n+3) g(X, Y) - 3\eta(X)\eta(Y)\} \\ &\quad + \left(-\frac{n}{2}|\omega|^2 + 2n|\omega| - \frac{1}{2}\alpha|\omega|\right) g(X, Y) - \left(n|\omega| - \frac{1}{2}|\omega|\right)\alpha \eta(X)\eta(Y). \end{aligned} \tag{33} $$

From (24), we obtain

$$ \begin{aligned} &g(\sigma(V, Y), \sigma(X, V)) + \sum_{i=1}^{n} [g(\sigma(X_i, X), \sigma(X_i, Y)) + g(\sigma(\phi X_i, X), \sigma(\phi X_i, Y))] \\ &= \left(\frac{1}{2}|\omega| - 1\right)^2 g(X, Y) + (\alpha^2 + \alpha|\omega| - 2\alpha)\eta(X)\eta(Y). \end{aligned} \tag{34} $$

Thus, since $\sigma(V, V) = -((1/2)|\omega| - 1 + \alpha)C$ and $\sigma(X_i, X_i) = -((1/2)|\omega| - 1)C$, we have

$$(35)\quad -\{g(\sigma(V, Y), \sigma(X, V)) - g(\sigma(X, Y), \sigma(V, V)) + \sum_{i=1}^{n} [g(\sigma(X_i, Y), \sigma(X, X_i))$$
$$- g(\sigma(X, Y), \sigma(X_i, X_i)) + g(\sigma(\phi X_i, Y), \sigma(X, \phi X_i)) - g(\sigma(X, Y), \sigma(\phi X_i, \phi X_i))]\}$$
$$= -\left(\frac{1}{2}|\omega| - 1\right)^2 g(X, Y) - (\alpha^2 + \alpha|\omega| - 2\alpha)\eta(X)\eta(Y)$$
$$+ (2n+1)\left\{\left(\frac{1}{2}|\omega| - 1\right)^2 g(X, Y) + \left(\frac{1}{2}|\omega| - 1\right)\alpha\eta(X)\eta(Y)\right\}$$
$$+ \alpha^2\eta(X)\eta(Y) + \left(\frac{1}{2}|\omega| - 1\right)\alpha g(X, Y)$$
$$= \left(\frac{n}{2}|\omega|^2 - 2n|\omega| + \frac{1}{2}\alpha|\omega| + 2n - \alpha\right)g(X, Y)$$
$$+ \left(n|\omega| - \frac{1}{2}|\omega| - 2n + 1\right)\alpha\eta(X)\eta(Y).$$

Using (33) and (35), from (25), we have

$$(36)\qquad Ric^M(X, Y) = \beta g(X, Y) + \gamma\eta(X)\eta(Y),$$

where $\beta = 2n - \alpha + (\rho/4)e^{-r}(2n+3)$ and $\gamma = \alpha(1-2n) - (3/4)\rho e^{-r}$. It is known that if all local Kaehler metrics $g' = e^{-r}g$ of L have the same constant holomorphic sectional curvature ρ, then $\rho = 0$ or L is a globally conformal Kaehler manifold (cf. [11]). Since $B \in TM^{\perp}$, for tangent vector field X on M, $X(r) = dr(X) = \omega(X) = 0$. Hence r is constant on M. Since M is a Sasakian manifold, β and γ are constant on M and $\beta + \gamma = 2n$ (cf. [12]). Thus, $\alpha = (\rho/4)e^{-r} = \text{constant}$ on M and $\beta = 2n + (\rho/2)e^{-r}(n+1)$, $\gamma = -(\rho/2)e^{-r}(n+1) \leq 0$. Therefore, from Theorem 1, M admits an Einstein-Weyl structure. ■

EXAMPLE. Let H_λ^{n+1} be a Hopf manifold. H_λ^{n+1} is isometric to $S^1 \times S^{2n+1}$, where S^{2n+1} is the unit sphere in C^{n+1} with constant curvature 1. Let ω be the Lee form of H_λ^{n+1}. The sphere S^{2n+1} is a leaf of the canonical foliation given by the Lee form $\omega = 0$. Since the local Kaehler metric of H_λ^{n+1} is flat and S^{2n+1} admits a Sasakian structure, S^{2n+1} admits an Einstein-Weyl structure.

Next, we shall consider generalized Hopf manifolds.

Let (M, ϕ, V, η, g) be a Sasakian manifold and $L = M \times \boldsymbol{R}$. We set

$$(37) \qquad \begin{aligned} &J\left(X, x\frac{\partial}{\partial s}\right) = \left(\phi X + xV, -\eta(X)\frac{\partial}{\partial s}\right), \\ &h\left(\left(X, x\frac{\partial}{\partial s}\right), \left(Y, y\frac{\partial}{\partial s}\right)\right) = g(X, Y) + xy, \end{aligned}$$

where X, Y are vector fields on M and x, y functions on L. Then (L, J, h) is a generalized Hopf manifold with Lee form $\omega = 2\,ds$ and Lee vector field $B = 2(\partial/(\partial s))$ (cf. [10]). Therefore $S^{2n+1}(k) \times \boldsymbol{R}$ is a generalized Hopf manifold.

Let (L, J, h) be a generalized Hopf manifold. Then $|\omega|$ is constant. We set

$$(38) \qquad c = \frac{|\omega|}{2}, \qquad u = \frac{\omega}{|\omega|}, \qquad v = -u \circ J, \qquad C = \frac{B}{|\omega|}, \qquad \tilde{V} = JC.$$

Then we have $dv = c(\Omega + 2u \wedge v)$.

Let $\mathscr{M}$ be the canonical foliation of a generalized Hopf manifold (L, J, h) and M be a leaf of $\mathscr{M}$. Then M is a totally geodesic submanifold of L. We set

$$(39) \qquad V = \frac{1}{c}\tilde{V}_{|M}, \qquad \eta = cv, \qquad g = c^2 h_{|M}, \qquad \phi = J + \eta \otimes C_{|M}.$$

Then (M, ϕ, V, η, g) admits a Sasakian structure (cf. [10]).

From a Theorem of Vaisman [10] and Theorem 2, we have following.

THEOREM 4. *Let L be a complete and simply connected generalized Hopf manifold and every leaf M of the canonical foliation $\mathscr{M}$ be of constant ϕ-sectional curvature k. If $k \geq 1$, then M admits an Einstein-Weyl structure.*

REMARK 3. In a generalized Hopf manifold (L, h), if all the local Kaehler metrics $g' = e^{-r}h$ have the same constant holomorphic sectional curvature ρ, then every leaf M of the canonical foliation is of constant ϕ-sectional curvature $k = (1/c^2)\rho e^{-r} + 1$, where $c = (|\omega|/2)$. But converse is not true.

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References

[1] D. E. Blair, Contact manifolds in Riemannian geometry, Lecture Notes in Math. **509**, Springer, Berlin, (1976).

[2] T. Higa, Weyl manifolds and Einstein-Weyl manifolds, Comm. Math. Univ. Saint Pauli, **42** (1993) 143–160.

[3] J. C. Marrero and J. Rocha, Locally conformal Kaehler submersion, Geom. Dedicata, **52** (1994), 271–286.

[4] F. Narita, CR-submanifolds of locally conformal Kaehler manifolds and Riemannian submersions, Colloq. Math., **70** (1996), 165–179.

[5] F. Narita, Riemannian submersions and Riemannian manifolds with Einstein-Weyl structures, Geom. Dedicata, **65** (1997), 103–116.

[6] B. O'Neill, The fundamental equations of a submersion, Michigan Math. J., **13** (1966), 1–20.

[7] H. Pedersen and A. Swann, Riemannian submersions, four-manifolds and Einstein-Weyl geometry, Proc. London Math., **66** (1993), 381–399.

[8] S. Tanno, Sasakian manifolds with constant ϕ-holomorphic sectional curvature Tōhoku Math. J., **21** (1969), 501–507.

[9] I. Vaisman, On locally conformal almost Kähler manifolds, Israel J. Math., **24** (1976), 338–351.

[10] I. Vaisman, Locally conformal Kähler manifolds with parallel Lee form, Rend. Mat. Roma, **12** (1979), 263–284.

[11] I. Vaisman, Some curvature properties of locally conformal Kaehler manifolds, Trans. Amer. Math. Soc., (2) **259** (1980), 439–447.

[12] K. Yano and M. Kon, Structures on manifolds, World Sci. Singapore, (1984).

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