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KRONECKER FUNCTION RINGS OF SEMISTAR-OPERATIONS

By

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1. Introduction

Let D be a commutative integral domain with quotient field K. Let $F(D)$ denote the set of non-zero fractional ideals of D in the sense of [K], i.e., non-zero R-submodules of K and let $F'(D)$ denote the subset of $F(D)$ consisting of all members A of $F(D)$ such that there exists some $0 \neq d \in D$ with $dA \subset D$. Let $f(D)$ be the set of finitely generated members of $F(D)$. Then $f(D) \subset F'(D) \subset F(D)$.

A mapping $A \to A^*$ of $F'(D)$ into $F'(D)$ is called a *star-operation* on D if the following conditions hold for all $a \in K - \{0\}$ and $A, B \in F'(D)$:

(1) $(a)^* = (a), (aA)^* = aA^*$;
(2) $A \subset A^*$; if $A \subset B$, then $A^* \subset B^*$; and
(3) $(A^*)^* = A^*$.

A fractional ideal $A \in F'(D)$ is called a $*$-*ideal* if $A = A^*$. We denote the set of all $*$-ideals of D by $F_*(D)$. A star-operation $*$ on D is said to be *of finite character* if $A^* = \bigcup\{J^* | J \in f(D) \text{ with } J \subset A\}$ for all $A \in F'(D)$. It is well known that if $*$ is a star-operation on D, then the mapping $A \to A^{*_f}$ of $F'(D)$ into $F'(D)$ given by $A^{*_f} = \bigcup\{J^* | J \in f(D) \text{ with } J \subset A\}$ is a finite character star-operation on D. Clearly we have $A^* = A^{*_f}$ for all $A \in f(D)$ and all star-operations $*$ on D.

The mapping on $F'(D)$ defined by $A \to A_v = (A^{-1})^{-1}$ is a star-operation on D and is called the *v-operation* on D, where $A^{-1} = \{x \in K | xA \subset D\}$. The *t-operation* on D is given by $A \to A_t = \bigcup\{J_v | J \in f(D) \text{ with } J \subset A\}$, that is, $t = v_f$. The reader can refer to [G, Sections 32 and 34] for the basic properties of star-operations and the v-operation.

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Let $A \to A^*$ be a star-operation on D. A $*$-ideal I is said to be *$*$-finite* if $I = A^*$ for some element A of $f(D)$. In $F_*(D)$, we define $A^* \times B^* = (AB)^* = (A^*B^*)^*$ for all $A^*, B^* \in F_*(D)$. A star-operation $*$ on D is said to be *arithmetisch brauchbar* (abbreviated a.b.) if for all $A^*, B^*, C^* \in F_*(D)$ such that A^* is $*$-finite, $A^* \times B^* \subset A^* \times C^*$ implies that $B^* \subset C^*$, and is said to be *endlich arithmetisch brauchbar* (e.a.b.), if for all $*$-finite A^*, B^*, $C^* \in F_*(D)$, $A^* \times B^* \subset A^* \times C^*$ implies $B^* \subset C^*$.

Let X be an indeterminate over D. For each polynomial $f \in D[X]$, we denote the fractional ideal of D generated by the coefficients of f by $c(f)$. It is well known that if $A \to A^*$ is an e.a.b. star-operation on D, then $D_* = \{0\} \cup \{f/g | f, g \in D[X] - \{0\}$ and $c(f)^* \subset c(g)^*\}$ is an integral domain with quotient field $K(X)$ such that $D_* \cap K = D$. Furthermore it is also known that D_* is a Bezout domain and for any finitely generated ideal A of D, we have $AD_* \cap K = A^*$ (cf. [G, Theorem (32.7)]). The integral domain D_* is called the *Kronecker function ring* of D with respect to the star-operation $*$.

In [OM] we introduced the notion of a *semistar-operation* on D. A mapping $A \to A^*$ on $F(D)$ is called a semistar-operation on D if the following conditions hold for all $a \in K - \{0\}$ and $A, B \in F(D)$:

(1) $(aA)^* = aA^*$;
(2) $A \subset A^*$; if $A \subset B$, then $A^* \subset B^*$; and
(3) $(A^*)^* = A^*$.

It is apparent from the definition that semistar-operations may have many properties analogous to those of star-operations.

In section 2, we show that many of results in [G, Section 32] can be extended to the case of semistar-operation and that the condition "integrally closed" on D become unnecessary in our case.

In section 3, we treat semistar-operations in the case of commutative rings with zero-divisors.

2. The integral domain case

Let $A \to A^*$ be a semistar-operation on D. A fractional ideal $A \in F(D)$ is called a *$*$-ideal* if $A = A^*$, and the set of $*$-ideals of D is denoted by $F_*(D)$. In $F_*(D)$, we define the product of A^* and B^* by $A^* \times B^* = (AB)^* = (A^*B^*)^*$. A $*$-ideal I is called a *$*$-finite ideal* if $I = A^*$ for some element $A \in f(D)$. A semistar-operation $*$ on D is said to be *endlich arithmetisch brauchbar* (e.a.b.) if for all $*$-finite A^*, B^*, $C^* \in F_*(D)$, $A^* \times B^* \subset A^* \times C^*$ implies $B^* \subset C^*$ and is said to be *arithmetisch brauchbar* (a.b.) if for all $A^*, B^*, C^* \in F_*(D)$ such that A^* is $*$-finite, $A^* \times B^* \subset A^* \times C^*$ implies that $B^* \subset C^*$. For each polynomial $f \in D[X]$, we denote the fractional ideal of D generated by the coefficients of f by $c(f)$. The fractional ideal $c(f)$ is called the *content* of f. We assume that $A \to A^*$ is an e.a.b. semistar-operation on D. Then we have the following results.

LEMMA 1 (cf. [G, Lemma (32.6)]). *For all* $f, g \in D[X] - \{0\}$, $c(fg)^* = (c(f)c(g))^*$.

PROOF. This follows immediately from [G, Corollary (28.3)].

PROPOSITION 2 (cf. [G, Theorem (32.7)]). *Let* $D_* = \{0\} \cup \{f/g | f, g \in D[X] - \{0\}$ *and* $c(f)^* \subseteq c(g)^*\}$. *Then we have*

(a) D_* *is an integral domain with quotient field* $K(X)$ *such that* $D_* \cap K = D^*$.
(b) D_* *is a Bezout domain.*
(c) *if* A *is a finitely generated ideal of* D, *then* $AD_* \cap K = A^*$.

PROOF. (a) Clearly D_* is an integral domain with quotient field $K(X)$. Next, we shall show that $K \cap D_* = \bigcup\{a/b \in K | D \subset (b/a)^*\}$. If $a/b \in K \cap D_*$, then $(a)^* \subset (b)^*$, i.e., $(a) \subset (b)^*$, and so $D \subset 1/a \times (b)^* = (b/a)^*$. Conversely, if $D \subset (b/a)^*$, then $(a/b) \subset D^*$. Moreover, $D \subset (b/a)^*$ if and only if $a/b \in D^*$. Hence our assertion follows. The proofs of (b) and (c) are the same as in those of (b) and (c) of [G, Theorem (32.7)].

COROLLARY 3. *If* $*$ *is an e.a.b. semistar-operation on* D, *then* D^* *is integrally closed.*

PROOF. Since D_* is a Bezout domain, D_* is integrally closed and then our assertion follows from Proposition 2(a).

EXAMPLE 4. Let V be a valuation overring of D. Then $A \to A^* = AV$ is a semistar-operation on D and is denoted by $*_{(V)}$ in [OM]. In this case, $*_{(V)}$ is an e.a.b. semistar-operation on D and $D^{*(V)} = V$ is a valuation domain and is also integrally closed. Moreover, $D_{*_{(V)}} = \{0\} \cup \{f/g | f, g \in D[X] - \{0\}$ and $c(f)V \subset c(g)V\}$.

REMARK 5. Let $S(D)$ be the set of all semistar-operations on D. For any two $*_1$, $*_2$ in $S(D)$, we define $*_1 \leq *_2$ if $A^{*_1} \subset A^{*_2}$ for all $A \in F(D)$. Let $*_1$ and $*_2$ be two e.a.b. semistar-operations on D. If $*_1 \leq *_2$, then $D_{*_1} \subset D_{*_2}$. In fact, if $f/g \in D_{*_1}$, then $c(f)^{*_1} \subset c(g)^{*_1}$, and then, by [OM, Lemma 16], we get $c(f)^{*_2} = (c(f)^{*_1})^{*_2} \subset (c(g)^{*_1})^{*_2} = c(g)^{*_2}$, and hence $f/g \in D_{*_2}$.

For any two $*_1, *_2 \in S(D), *_1$ and $*_2$ are said to be equivalent if $A^{*_1} = A^{*_2}$ for each $A \in f(D)$. If $*_1$ and $*_2$ are equivalent, then $*_1$ is e.a.b. iff $*_2$ is e.a.b.. Moreover, for any two e.a.b. semistar-operations $*_1, *_2$, it is easily seen that $*_1$ and $*_2$ are equivalent iff $D_{*_1} = D_{*_2}$.

DEFINITION 6. Let $\{D_{\lambda \in \Lambda}\}$ be a family of overrings of D. Then $A \to A^* = \bigcap_\lambda AD_\lambda$ is a semistar- operation on D (cf. [OM, Corollary 10]). This is called a semistar-operation of D *induced by overrings* $\{D_\lambda\}$ and is denoted by $*_{\{D_\lambda\}}$. If $\{V_\lambda\}$ is a family of valuation overrings of D, then a semistar-operation $A \to A^* = \bigcap_\lambda AV_\lambda$ is called a *w-operation* on D.

PROPOSITION 7 (cf. [G, Theorem (32.5)]). *Each w-operation of D is an a.b. semistar-operation on D.*

THEOREM 8 [cf. (G, Theorem (32.11)]). *Let* $\{D_{\lambda \in \Lambda}\}$ *be a family of overrings of D. Then* $D_{*\{D_\lambda\}} = \bigcap_{\lambda \in \Lambda} D_{*(D_\lambda)}$.

PROOF. Let $A^* = \bigcap AD_\lambda$ for all $A \in F(D)$. Let f and g be nonzero elements of $D[X]$. If $f/g \in \bigcap_{\lambda \in \Lambda} D_{*(D_\lambda)}$, then $c(f)D_\lambda \subseteq c(g)D_\lambda$ for all $\lambda \in \Lambda$. Then $c(f)^* = \bigcap_\lambda c(f)D_\lambda \subseteq \bigcap_\lambda c(g)D_\lambda = c(g)^*$ and so $f/g \in D_*$ and therefore $\bigcap_{\lambda \in \Lambda} D_{*(D_\lambda)} \subset D_*$. Conversely, if $f/g \in D_*$ then $c(f)^* \subseteq c(g)^*$ and so $c(f)D_\lambda = c(f)^*D_\lambda \subseteq c(g)^*D_\lambda = c(g)D_\lambda$ for each $\lambda \in \Lambda$. Hence $f/g \in \bigcap_{\lambda \in \Lambda} D_{*(D_\lambda)}$ and so $D_* \subseteq \bigcap_{\lambda \in \Lambda} D_{*(D_\lambda)}$. Thus $D_* = \bigcap_{\lambda \in \Lambda} D_{*(D_\lambda)}$.

Let v be a valuation on K and let V be the valuation overring of D associated with v. For each $a_0 + a_1X + \cdots + a_nX^n \in K(X)$, we define $\bar{v}(a_0 + a_1X + \cdots + a_nX^n) = \inf\{v(a_i) | a_i \neq 0\}$, then $\bar{v}$ is a valuation on $K(X)$. The valuation $\bar{v}$ is called the trivial extension of v to $K(X)$. Let W be the valuation ring associated with $\bar{v}$. Then, for any two elements f and g in $K[X] - \{0\}$, $f/g \in W$ if and only if $c(f)V \subset c(g)V$.

PROPOSITION 9 (cf. [G, Theorem (32.10)]). *Let* $A \to A^*$ *be an e.a.b. semistar-operation on D and let W be a valuation overring of* D_*, *then W is the trivial extension of* $V = W \cap K$ *to* $K(X)$.

Lemma 10. *If V is a valuation overring of D, then $D_{*_{(V)}}$ is the trivial extension of V to $K(X)$.*

Proof. Let f and g be non-zero elements of $D[X] - \{0\}$. Then $f/g \in D_{*_{(V)}}$ if and only if $c(f)^{*(V)} \subset c(g)^{*(V)}$, i.e., $c(f)V \subseteq c(g)V$. Hence $f/g \in D_{*_{(V)}}$ if and only if $f/g \in W$, the trivial extension of V to $K(X)$.

Corollary 11 (cf. [G, Theorem (32.11)]). *Let $\{V_\lambda\}$ be a family of valuation overrings of D and let $A \to A^* = \bigcap_\lambda AV_\lambda$ be a semistar-operation on D induced by $\{V_\lambda\}$. Then $D_* = \bigcap_\lambda W_\lambda$, where W_λ is the trivial extension of V_λ to $K(X)$.*

Proof. This follows from Theorem 8 and Lemma 10.

Proposition 12 (cf. [G, Theorem (32.12)]). *Each e.a.b. semistar-operation $*$ on D is equivalent to a w-operation on D.*

Proof. Since D_* is integrally closed, we have $D_* = \bigcap_\lambda W_\lambda$, where $\{W_\lambda\}$ is the family of valuation overrings of D_*. For each λ, we set $V_\lambda = W_\lambda \cap K$. Then V_λ is a valuation overring of D and by Proposition 9, W_λ is the trivial extension of V_λ to $K(X)$. Hence, if we set $A \to A^w = \bigcap_\lambda AV_\lambda$, then, by Corollary 11, $D_w = \bigcap W_\lambda = D_*$, and hence by Remark 5 w and $*$ are equivalent.

Corollary 13 (cf. [G, Corollary (32.13)]). *Each e.a.b. semistar-operation on D is equivalent to an a.b. semistar-operation on D.*

If $\{V_\lambda\}$ is the family of all valuation overrings of D, then $A \to A_b = \bigcap_\lambda AV_\lambda$ is an a.b. semistar-operation on D and is called the *b-operation* on D.

Corollary 14 (cf. [G, Corollary (32.14)]). *Each Kronecker function ring D_* of D contains D_b, the Kronecker function ring of D with respect to the b-operation.*

Proof. If $\{V_\lambda\}$ is the family of all valuation overrings of D and W_λ is the trivial extension of V_λ to $K(X)$, then $D_b = \bigcap_{\lambda\in\Lambda} W_\lambda$ by Corollary 11. Next, for each e.a.b. semistar-operation $*$ on D, Proposition 12 shows that $D_* = \bigcap_\lambda W_\lambda$, where W_λ is the trivial extension of a valuation overring V_λ of D, and so $D_* \supset D_b$ as desired.

Proposition 15 (cf. [G, Theorem (32.15)]). *Let D_b be the Kronecker function right of D with respect to the b-operation on D. Then*

(1) *If R is an overring of D and $*$ is a semistar-operation on R, then R_* contains D_b.*

(2) *If R is an overring of D_b, then R is a Kronecker function ring of $R \cap K$.*

PROOF. (1) It is evident that $D_b \subseteq R_b$. Then we have $D_b \subset R_*$, since $R_b \subset R_*$ by Corollary 14.

(2) Since D_b is a Bezout domain, R is also a Bezout domain by [C, Theorem 1.3], and so R is integrally closed. Then $R = \bigcap_\lambda W_\lambda$, where $\{W_\lambda\}$ is the family of valuation overrings of R. By Proposition 9, each W_λ is the trivial extension of $V_\lambda = W_\lambda \cap K$ to $K(X)$. Moreover, $R \cap K = \bigcap_\lambda V_\lambda$. Hence if we set $A^* = \bigcap_\lambda A V_\lambda$, then by Proposition 9, $(R \cap K)_* = \bigcap_\lambda W_\lambda = R$.

REMARK 16. If W is a valuation overring of D_b, then W is the trivial extension of a valuation overring $V = W \cap K$ of D by Proposition 9. Conversely, if V is a valuation overring of D, then $D_{*_{(V)}}$ is the trivial extension of V to $K(X)$ and $D_{*_{(V)}} \cap K = V$ by Lemma 10. Hence there is a one-to-one correspondence between valuation overrings of D and valuation overrings of D_b. If R is a Bezout domain, then the set of valuation overrings of R is in one-to-one correspondence with the set of proper prime ideals of R (cf. [C, Theorem 1.3 and Proposition 1.5]).

PROPOSITION 17 (cf. [G, Proposition (32.16)]). *Let D_b be the Kronecker function ring of D with respect to the b-operation. Then $\dim D_b = \dim_v D$, where $\dim_v D$ is the valuative dimension of D.*

LEMMA 18 (cf. [G, Lemma (32.17)]). *Let $A \to A^*$ be a semistar-operation on D. If A is an invertible fractional ideal of D, then, for each $B \in F(D)$, $(AB)^* = AB^*$.*

PROPOSITION 19 (cf. [G, Proposition (32.18)]). *D be a Prüfer domain. Then each semistar-operation on D is arithmetisch brauchbar. If $*_1$ and $*_2$ are semistar-operations on D such that $D^{*_1} = D^{*_2}$, then $*_1$ and $*_2$ are equivalent.*

PROOF. Let $A, B, C \in F(D)$ with $A \in f(D)$. Suppose $(AB)^* \subseteq (AC)^*$. It follows from Lemma 18 that $AB^* = (AB)^* \subset (AC)^* = AC^*$, since D is Prüfer and A is invertible. Then, $B^* = A^{-1}AB^* \subseteq A^{-1}AC^* = C^*$, which implies that $*$ is arithmetisch brauchbar. Let $*_1$ and $*_2$ be two semistar-operations on D such that $D^{*_1} = D^{*_2}$. Then, by Lemma 18, we have $A^{*_1} = (AD)^{*_1} = AD^{*_1} = AD^{*_2} = (AD)^{*_2} = A^{*_2}$ for all $A \in f(D)$. Hence $*_1$ and $*_2$ are equivalent.

Proposition 20. *Let* $\{D_\alpha | \alpha \in A\}$ *and* $\{D_\beta | \beta \in B\}$ *be two families of overrings of a Prüfer domain D such that* $\bigcap\{D_\alpha | \alpha \in A\} = \bigcap\{D_\beta | \beta \in B\}$. *Then* $\{D_\alpha\}$ *and* $\{D_\beta\}$ *induce equivalent semistar-operations on D.*

Proof. Set $A^{*_1} = \bigcap AD_\alpha$ and $A^{*_2} = \bigcap AD_\beta$ for all $A \in F(D)$. Then clearly $D^{*_1} = \bigcap D_\alpha = \bigcap D_\beta = D^{*_2}$. Next, if $A \in f(D)$, then A is invertible and so, by Lemma 18, we have $A^{*_1} = (AD)^{*_1} = AD^{*_1} = AD^{*_2} = A^{*_2}$. Thus $*_1$ and $*_2$ are equivalent as wanted.

We shall now state our main results of this section.

Lemma 21. *Let T be a Bezout overring of D. Then the semistar-operation* $*_{(T)}$ *on D is arithmetisch brauchbar.*

Proof. Let A, B and C be in $F(D)$, with A finitely generated. Suppose $(AB)_{*_{(T)}} \subseteq (AC)_{*_{(T)}}$. Then $ATBT = (AB)_{*_{(T)}} \subseteq (AC)_{*_{(T)}} = ATCT$. Since AT is principal, $(AT)(BT) \subseteq (AT)(CT)$ implies $BT \subseteq CT$. Hence $*_{(T)}$ is an a.b. semistar-operation on D.

Theorem 22. *Let T be a Bezout overring of D. Then* $D_{*_{(T)}}$ *is a Bezout overring of* $D[X]$ *and* $D_{*_{(T)}} \cap K = T$.

Proof. Since $*_{(T)}$ is e.a.b. by Lemma 21, $D_{*_{(T)}}$ is a Bezout domain and $D_{*_{(T)}} \cap K = D^{*_{(T)}} = T$ by Proposition 2.

Proposition 23 (cf. [G, Proposition (32.19)]). *Let D be a Prüfer domain, and let* $\{D_\alpha\}$ *be the set of overrings of D. The mapping* $D_\alpha \longrightarrow (D_\alpha)_b$ *is a one-to-one mapping from the set* $\{D_\alpha\}$ *onto the set of overrings of* D_b

Proof. Let R be an overring of D_b. Then $R = \bigcap W_\lambda$, where $\{W_\lambda\}$ is a family of valuation overrings of R. If we set $V_\lambda = W_\lambda \cap K$, then V_λ is a valuation overring of $R \cap K$ and $R \cap K = \bigcap V_\lambda$. Set $A^* = \bigcap AV_\lambda$ for all $A \in F(D)$. Then, by Proposition 20, $*$ is equivalent to the b-operation on $R \cap K$. By Proposition 15(2), we have $R = (R \cap K)_* = (R \cap K)_b$. Thus the mapping $\pi : D_\alpha \longrightarrow (D_\alpha)_b$ is surjective.

Next, let D_α and D_β be two overrings of D and assume that $(D_\alpha)_b = (D_\beta)_b$. Then, by Proposition 2(a), $D_\alpha = (D_\alpha)_b = (D_\alpha)_b \cap K = (D_\beta)_b \cap K = (D_\beta)_b = D_\beta$, because, by [G, Theorem (23.4)], D_α and D_β are both integrally closed. Thus π is also injective and our proof is complete.

PROPOSITION 24 (cf. [G, Exercise 12, p. 409]). *Let V be a rank one valuation ring of the form $K(X)+M$, where M is the maximal ideal of V. If $J=K+M$, then J admits a unique S-representation and J has a unique Kronecker function ring, but J is not a Prüfer domain.*

PROOF. First, by [G2, Theorem A i), p. 561], J is not a Prüfer domain. Next, by [BG, Theorem 3.1], each overring of J is of the form either $D_\lambda + M$ or V, where $\{D_\lambda\}$ is the family of subrings of $K(X)$ containing K. Moreover, by [G2, Theorem A h)], $D_\lambda + M$ is a valuation ring of V if and only if D_λ is a valuation ring on $K(X)$. Now, by [G, Exercise 4, p. 249], the family of nontrivial valuation rings on $K(X)$ containing K is $\{K[X^{-1}]_{(X^{-1})}\} \cup \{K[X]_{(P(X))} | P(x)$ is prime in $K[X]\}$. In above, $K[X^{-1}]_{(X^{-1})}$ is the valuation ring of the valuation v_∞, where $v_\infty(0)=\infty$ and $v_\infty(f(X)) = -deg\, f(X)$ for each $f(X) \neq 0$ in $K[X]$, and $K[X]_{(P(X))}$ is the valuation ring of the $P(X)$-adic valuation on $K(X)$. Then $\{K[X^{-1}]_{(X^{-1})}+M\} \cup \{K[X]_{(P(X))}+M | P(X)$ is prime in $K[X]\}$ gives a unique S-representation of $J=K+M$, and our assertion follows.

PROPOSITION 25. *Let $\{V_\alpha | \alpha \in A\}$ be a family of valuation overrings of D and let $\{W_\beta | \beta \in B\}$ be the family of all valuation rings W on L such that $W \cap K$ is in $\{V_\alpha\}$. Assume that L is an algebraic extension field of K and denote by J the integral closure of D_* in L, where $D_* = \cap \{V_\alpha | \alpha \in A\}$. Then*

(1) $J = \bigcap \{W_\beta | \beta \in B\}$.

(2) *Let $*'$ and $*$ be semistar-operations on J and D induced by $\{W_\beta\}$ and $\{V_\alpha\}$ respectively. Then $J_{*'}$ is the integral closure of D_* in $L(X)$.*

PROOF. (1) follows from [G, Exercise 14, p. 409].

(2) Let $\overline{V}_\alpha$ and $\overline{W}_\beta$ be the trivial extension of V_α and W_β to $K(X)$ and $L(X)$ respectively. Then, by Corollary 11, $D_* = \bigcap\{\overline{V}_\alpha | \alpha \in A\}$ and $J_{*'} = \bigcap\{\overline{W}_\beta | \beta \in B\}$. It is easily seen that if $\overline{W}_\beta \cap K = V_\alpha$, then $\overline{W}_\beta \cap K(X) = \overline{V}_\alpha$. Next, let W be a valuation ring on $L(X)$ such that $W \cap K(X) \in \{\overline{V}_\alpha | \alpha \in A\}$. Then by [G, Theorem (19.16)], W and $W \cap K(X)$ have the same rank, since $L(X)/K(X)$ is algebraic. Moreover, $W \cap L \in \{W_\beta | \beta \in B\}$, because $W \cap K \in \{V_\alpha | \alpha \in A\}$. Let $\overline{W}$ be the trivial extension of $W \cap L$ to $L(X)$ and let M be the maximal ideal of $W \cap L$. Then, by [BJ, Theorem 3.6.20], $\overline{W} = (W \cap L)[X]_{M[X]}$. By [K, Theorems

39 and 68], we have height $(M) = \text{height}(M[X])$, and so $\overline{W}$ and $W \cap L$ have the same rank. On the other hand, W and $W \cap L$ also have the same rank. Then, since $W \supset \overline{W}$, we have $W = \overline{W}$. Hence our assertion also follows from [G, Exercise 14, p. 409].

DEFINITION 26. Let $\{M_\beta | \beta \in B\}$ be the set of maximal ideals of D and set $S = D[X] - \cup\{M_\beta[X] | \beta \in B\}$, where X is an indeterminate over D. Then we denote by $D(X)$ the quotient ring $D[X]_S$. Then $\{M_\beta D(X) | \beta \in B\}$ is the set of maximal ideals of $D(X)$.

PROPOSITION 27 (cf. [G. Theorem (33.3)]). *If D' is the integral closure of D, then $D(X)$ is contained in J, the Kronecker function ring of D' with respect to the b-operation.*

PROOF. Let $\{V_\alpha | \alpha \in A\}$ be the set of valuation overrings of D. Then $D' = \bigcap\{V_\alpha | \alpha \in A\}$. Here, by [G, Corollary (19.7)(2)], we may assume that each V_α is centered on a maximal ideal of D. By Corollary 11, $J = (D')_b = \bigcap\{W_\alpha | \alpha \in A\}$, where W_α is the trivial extension of V_α to $K(X)$ and, by [BJ, Theorem 3.6.20], $W_\alpha = V_\alpha[X]_{P_\alpha[X]} = V_\alpha(X)$, where P_α is the maximal ideal of V_α. Now, let $\{M_\beta | \beta \in B\}$ be the set of maximal ideals of D. Then, by [G, Theorem (33.3)], $D(X) = \bigcap\{D[X]_{M_\beta[X]} | \beta \in B\} = \bigcap\{D_{M_\beta}(X) | \beta \in B\}$. If $P_\alpha \cap D = M_\beta$, then $D_{M_\beta}(X) \subset V_\alpha(X)$, and so each $W_\alpha = V_\alpha(X)$ contains some $D_{M_\beta}(X)$. Hence $D(X) \subset J = (D')_b$ as wanted.

Let $*$ be an e.a.b. semistar-operation on a domain D. We set $U^* = \{g \in D^*[X] | c(g)^* = D^*\}$. Then U^* is a multiplicative system of $D^*[X]$.

PROPOSITION 28 (cf. [G, Theorem (33.4)]). *Let D be a domain with quotient field L, let X be an indeterminate over D, and let D_b be the Kronecker function ring of D with respect to the b-operation. The following conditions are equivalent:*

(1) *D^b is a Prüfer domain.*
(2) *$D^b[X]_{U^b} = D_b$.*
(3) *$D^b[X]_{U^b}$ is a Prüfer domain.*
(4) *D_b is a quotient ring of $D^b[X]_{U^b}$.*

PROPOSITION 29 (cf. [G, Theorem (34.11)]. *Let D be a Prüfer v-multiplication ring with quotient field L. Then D is a v-domain, and if H is the group of divisor classes of D of finite type, the H is order isomorphic to the group of divisibility of D_v.*

PROPOSITION 30 (cf. [G, Exercise 6, p. 430]). *Assume that D is a v-domain, that X is a set of indeterminates over D. Then the following conditions are equivalent*:

(1) *D is a Prüfer v-multiplication ring.*

(2) *D_v is a quotient ring of $D[X]$.*

PROPOSITION 31 (cf. [G, Proposition (36.7)]). *Let D be a domain which is not a field. The following conditions are equivalent*:

(1) *D^b is almost Dedekind.*

(2) *$D^b[X]_{U^b}$ is almost Dedekind.*

(3) *D_b is almost Dedekind.*

PROPOSITION 32 (cf. [G, Proposition (38.7)]). *In an integral domain D the following conditions are equivalent*:

(a) *D^b is a Dedekind domain.*

(b) *$D^b[X]_{U^b}$ is Dedekind.*

(c) *D_b is Dedekind.*

(d) *D_b is Noetherian.*

(e) *D_b is a PID.*

PROPOSITION 33 (cf. [G, Corollary (44.12)]). *If D a Krull domain with quotient field K, then D_v is a PID.*

PROPOSITION 34 (cf. [G, Exercise 21, p. 558]). *Assume that D admits a Kronecker function ring D_* which is a PID. Then D^* is a Krull domain.*

3. The case of commutative rings with zero-divisors

Let R be a commutative ring with zero-divisors. A non-zero-divisor of R is called a *regular element* of R and an ideal I of R is said to be *regular* if it contains a regular element of R.

DEFINITION 35. A commutative ring R is called a *Marot ring* if each regular ideal of R is generated by regular elements. Let $f(X)$ be a regular element of a polynomial ring $R[X]$. The ideal of R generated by the coefficients of $f(X)$ is called the *content* of $f(X)$ and is denoted by $c(f)$.

DEFINITION 36. A commutative ring R is said to have the *Property A* if for any regular element $f(X)$ of $R[X]$, the content ideal $c(f)$ is a regular ideal of R.

Hereafter, a commutative ring R will denote a Marot ring with the property A and the total quotient ring of R will be denoted by K. Let $F(R)$ be the set of nonzero R-submodules of K and let $F'(R)$ be the subset of $F(R)$ consisting of all members I of $F(R)$ such that there exists a regular element d of R with $dI \subseteq R$. Let $f(R)$ be the subset of finitely generated members of $F(R)$.

A mapping $A \to A^*$ of $F(R)$ into $F(R)$ is called a semistar-operation on R if the following conditions hold for all regular elements $a \in K$ and $I, J \in F(R)$:

(1) $(aI)^* = aI^*$;

(2) $I \subset I^*$; if $I \subset J$, then $I^* \subset J^*$; and

(3) $(I^*)^* = I^*$.

R is called a *Bezout ring* if every finitely generated regular ideal of R is a principal ideal.

Lemma 37. *Let T be a Bezout overring of R, then $A \to A^{*(T)} = AT$ is an a.b. semistar-operation of R.*

Proof. The proof is the same of that in Lemma 20.

Let $R_* = \{0\} \cup \{f/g | f, g \in R[X], g$ is regular and $c(f)^* \subseteq c(g)^*\}$. Then we have the following.

Theorem 38. *Let T be a Bezout overring of R. Then $R_{*(T)}$ is a Bezout overring of $R[X]$ and $R_{*(T)} \cap K = T$.*

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