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ON THE EXISTENCE OF POSTPROJECTIVE COMPONENTS IN THE AUSLANDER-REITEN QUIVER OF AN ALGEBRA

By

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Let k be an algebraically closed field and A be a basic finite-dimensional k-algebra of the form $A = kQ/I$, where Q is a quiver (= finite oriented graph) and I is an admissible ideal of the path algebra kQ, see [3]. In this work we assume that Q has no oriented cycles.

Let mod_A denote the category of finite dimensional left A-modules. For each indecomposable non-projective A-module X, the Auslander-Reiten translate $\tau_A X$ is an indecomposable non-injective module. The Auslander-Reiten quiver Γ_A has as vertices representatives of the isoclasses of the finite dimensional indecomposable A-modules, there are as many arrows from X to Y as $\dim_k \mathrm{rad}_A(X,Y)/\mathrm{rad}_A^2(X,Y)$. In this paper we do not distinguish between a module and its corresponding isoclass. A connected component $\mathcal{P}$ of Γ_A is *postprojective* if $\mathcal{P}$ has no oriented cycles and each module X in $\mathcal{P}$ has only finitely many predecessors in the path order of $\mathcal{P}$. Several important classes of algebras have postprojective components: hereditary algebras [3, 6], algebras satisfying the separation condition [1, 2], tilted algebras [8].

The aim of this work is to find necessary and sufficient conditions for the existence of postprojective components in Γ_A. In section 1 we give an algorithmic procedure to decide the existence of postprojective components. In section 2 we consider a one-point extension algebra $A = B[M]$ such that all indecomposable direct summands of M belong to postprojective components of Γ_B, then we give conditions that assure that the projective A-module P with rad $P = M$ lies in a postprojective component of Γ_A. In section 3 we consider some special cases. We recall that once identified a postprojective component $\mathcal{P}$ of Γ_A, the modules on $\mathcal{P}$ may be constructed using the *knitting procedure* [3]. In [5], an algorithmic procedure which makes essential use of the knitting procedure is given to construct all the postprojective components of Γ_A.

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1. Existence of postprojective components.

1.1. Let $A = kQ/I$ be a finite dimensional k-algebra such that the quiver Q has no oriented cycles. We may consider A as a k-category with objects the set of vertices Q_0 of Q and morphisms from $x, y \in Q_0$ the space $A(x,y) = e_y A e_x$, where e_x denotes the trivial path at the vertex x. For two vertices $x, y \in Q_0$ we write $y \leq x$ if there is a path from y to x *in* Q.

Let $x \in Q_0$, we denote by A^x the full subcategory of A whose vertices are those $y \in Q_0$ with $y \not\leq x$. Observe that the quiver Q^x of A^x is a convex (= path closed) subquiver of Q. The indecomposable projective A-module $P_x = Ae_x$ has radical rad P_x which is an A^x-module. We denote by rad $P_x = \oplus_{i=1}^{n_x} R_i^x$ the indecomposable decomposition of rad P_x.

1.2. A *path* in mod_A is a sequence $(X_0, \cdots, X_s)$ of (isomorphisms classes of) indecomposable A-modules $X_i, 0 \leq i \leq s$, such that there is a map $0 \neq f_i \in \text{Hom}_A(X_i, X_{i+1})$ which is not an isomorphism, $0 \leq i \leq s-1$. In this case we write $X_0 \leq X_s$ and we say that X_0 is a *predecessor* of X_s. If $s = 1$ and $X_0 = X_s$ we say that the path $(X_0, \cdots, X_s)$ is a *cycle*.

Following [4] we say that a module M is *directing* in mod_A provided there do not exist indecomposable direct summands M_1 and M_2 of M and an indecomposable non-projective module X such that $M_1 \leq \tau X$ and $X \leq M_2$. It is shown in [4] that an indecomposable module X is directing if and only if there are no cycles $(X_0, \cdots, X_s)$ with $X_0 = X = X_s$. The following result will be important in our work.

THEOREM [4, 7]. *Let* $x \in Q_0$. *Then* P_x *is directing in* mod_A *if and only if* rad P_x *is directing in* mod_A.

Moreover, if x is a source, then P_x *is directing in* mod_A *if and only if* rad P_x *is directing in* mod_{A^x}.

1.3. We state our main result which provides an algorithmic criterion for the existence of postprojective components.

THEOREM. *Let* $A = kQ/I$ *be a finite dimensional k-algebra such that* Q *has no* *oriented cycles. Then* Γ_A *has a postprojective component if and only if for each vertex* $x \in Q_0$ *one of the following conditions is satisfied:*

(1x) *there is a postprojective component* $\mathcal{P}$ *of* Γ_{A^x} *such that* $R_i^x \notin \mathcal{P}$ *for every* $1 \leq i \leq n_x$;

(2x) *for each* $1 \leq i \leq n_x$ *the set of predecessors* $\{Y \in \Gamma_{A^x} : Y \leq R_i^x\}$ *of* R_i^x *in* mod_{A^x} *is finite and formed by directing modules. Moreover, if x is a source, then* rad P_x *is directing in* mod_{A^x}.

We prove the theorem in (1.5) after some preparation. In (1.8) we give some examples.

1.4. LEMMA. *Assume that all* $x \in Q_0$ *the condition* (2x) *is satisfied, then* Γ_A *has a postprojective component.*

PROOF: We claim that for every $x \in Q_0$ the following condition is satisfied:

(3x): for each $1 \leq i \leq n_x$, the set of predecessors $\{X \in \Gamma_A : X \leq R_i^x\}$ of R_i^x in mod_A is finite and formed by directing modules.

Indeed, let X be a predecessor of R_i^x in Γ_A and assume that X is not an A^x-module. We may assume that x is minimal with this property in the path order of Q. Then there exists a vertex $y \leq x$ in Q such that $X(y) \neq 0$. Therefore in mod_A we get

$$P_y \leq X \leq R_i^x \leq P_x \leq P_y.$$

Since (2y) is satisfied, then by (1.2) y is not a source in Q. Let z be a proper predecessor of y in Q. Therefore, P_y is a non-directing predecessor of some R_j^z. By (2z), P_y is not an A^z-module, contradicting the minimality of x.

Then we are in position to repeat the argument given in [2, theorem (2.5)] to prove the existence of a postprojective component. For the sake of completeness we sketch the argument. We construct inductively full subquivers C_n of Γ_A satisfying:

i) C_n is finite, connected, contains no oriented cycle and is closed under predecessors.

ii) $\tau_A^{-1} C_n \cup C_n \subset C_{n+1}$.

Then $\bigcup_n C_n$ forms the wanted postprojective component.

Set $C_0 = \{S\}$ where S is a simple projective A-module. Assume C_n to be defined and let $M_1, \cdots, M_t$ be the modules in C_n with $\tau_A^{-1} M_i \notin C_n$. We may assume that $M_i \leq M_j$ implies $i \leq j$. If $t = 0$, set $C_{n+1} = C_n$. Otherwise we define full subquivers $D_i (0 \leq i \leq t)$ of Γ_A satisfying $D_0 = C_n, D_i \cup \{\tau_A^{-1} M_{i+1}\} \subset D_{i+1}$ and condition (i) imposed on D_i. Then $D_{n+1} = C_t$ will satisfy conditions (i) and (ii).

Indeed, assume D_i is well defined. Take the almost split sequence $0 \to M_{i+1} \to X \to \tau_A^{-1} M_{i+1} \to 0$ and define D_{i+1} as the full subquiver of Γ_A with vertices D_i and all predecessors of $\tau_A^{-1} M_{i+1}$. It is enough to show that for each indecomposable direct summand Y of X, the set of predecessors $\{Z \in \Gamma_A : Z \leq Y\}$ is finite and formed by directing modules. If Y is not projective, then $\tau_A Y \in C_n$ whence Y belongs to D_i and we are done. If $Y = P_y$ is projective, then (3y) is satisfied. By (1.2), we get the result. □

1.5. PROOF OF THE THEOREM. Let $\mathcal{P}$ be a postprojective component of Γ_A. Let $x \in Q_0$. If the projective module P_x belongs to $\mathcal{P}$, then (2x) is satisfied. Assume that $P_x \notin \mathcal{P}$. We show that $\mathcal{P}$ is formed by A^x-modules. Let $X \in \mathcal{P}$ and assume $X(y) \neq 0$ for some $y \leq x$ in Q. Then $P_x \leq P_y \leq X$ in mod_A, which implies $P_x \in \mathcal{P}$, a contradiction. Hence $\mathcal{P}$ is a postprojective component in Γ_{A^x} and $R_i^x \notin \mathcal{P}$ for $1 \leq i \leq n_x$, that is (1x) is satisfied.

Conversely, assume that for each $x \in Q_0$, one of the conditions (1x) or (2x) is satisfied. If for every $x \in Q_0$, (2x) is satisfied then (1.4) implies the result.

Assume that for $x \in Q_0$, (2x) is not satisfied. Choose a minimal such x in the path order in Q. By hypothesis (1x) is satisfied, that is, there is a postprojective component $\mathcal{P}$ of Γ_{A^x} such that $R_i^x \notin \mathcal{P}$ for every $1 \leq i \leq n_x$. We shall prove that $\mathcal{P}$ is a component of Γ_A. For this purpose it is enough to show that x is a source in Q.

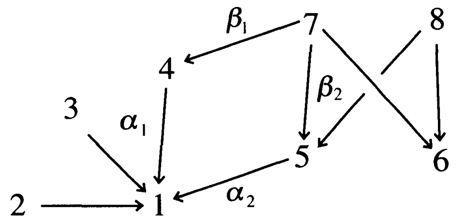
Assume $y \leq x$ is a source in Q and $y \neq x$. The minimality of x implies that (2y) is satisfied. We will show that (2x) is also satisfied which yields the wanted contradiction. Indeed, let X be a predecessor of R_i^x in mod_A. Then $X \leq R_i^x \leq P_x \leq P_y$, implies that X is a predecessor of R_j^y for some $1 \leq j \leq n_y$. Moreover, since P_y is directing in mod_A, then X is an A^y-module. Thus $\{X \in \Gamma_A : X \leq R_i^x\}$ is finite and formed by directing modules. Our theorem is proved. □

1.6. COROLLARY. *Let $A = kQ/I$ be as above and assume Q is connected. Then all indecomposable projective modules belong to a postprojective component if and only if for every $x \in Q_0$ the condition (2x) is satisfied.*

PROOF. The "only if" direction is clear. For the converse, assume that for every $x \in Q_0$, the condition (2x) is satisfied. By the theorem there is a postprojective component $\mathcal{P}$ of Γ_A. Clearly we may assume that Q is connected (otherwise we take a postprojective component for each maximal connected full subcategory of A). Let x_0 be a sink in Q such that the projective $P_{x_0} \in \mathcal{P}$. Let

$x \in Q_0$ and fix a walk $x_0 \overset{\alpha_1}{—} x_1 —— \cdots \overset{\alpha_s}{—} x_s = x$ in Q (that is, each α_i is an arrow in Q with some orientation). By induction, we may assume that $P_{x_{s-1}} \in \mathcal{P}$. If $x_{s-1} \xrightarrow{\alpha_s} x_s$, then P_x is a predecessor of $P_{x_{s-1}}$ and $P_x \in \mathcal{P}$. Thus, assume that $x_s \xrightarrow{\alpha_s} x_{s-1}$. Then there is a morphism $f : P_{x_{s-1}} \to \operatorname{rad} P_x$. Since (2x) is satisfied, then f is a linear combination of compositions of finitely many irreductible maps. Hence $R_i^x \in \mathcal{P}$ for some $1 \leq i \leq n_x$. Thus $P_x \in \mathcal{P}$ and we are done. □

1.7. EXAMPLES. Consider the algebra $A = kQ/I$ given by the quiver



and the ideal $I = \langle \alpha_1\beta_1 - \alpha_2\beta_2 \rangle$. The quiver Γ_A has no postprojective component but for every proper full convex subcategory B of A, the quiver Γ_B has a postprojective component. Consider for example A as the one-point extension $A = B[M]$ where $B = A/Ae_7$ and $M = \operatorname{rad} P_7$. Then B is an hereditary algebra and $M = M_1 \oplus P_6$ where P_6 is a postprojective B-module and M_1 is a regular B-module. Therefore M is not directing and both conditions (1x) and (2x) are not satisfied for $x = 7$.

It is also interesting to consider $A = C[N]$ where $C = A/Ae_2$ and $N = \operatorname{rad} P_2$. Then Γ_C has a postprojective component $\mathcal{P}$ and $N = P_1$ is an indecomposable module in $\mathcal{P}$. In this case the projective C-module P_7 belongs to $\mathcal{P}$. In section 2 we will consider more carefully this kind of situation.

Finally, we observe that in our example for every convex subcategory B of A (including $B = A$), the Auslander-Reiten quiver Γ_B has a preinjective component.

1.8. Let $A = kQ/I$ be an algebra as above. Let $x \in Q_0$ and consider the connected components $Q_1^x, \cdots, Q_{s_x}^x$ of the quiver Q^x associated with the algebra A^x. Recall that the vertex x is said to be *separating* if for each $1 \leq j \leq s_x$ the quiver Q_j^x contains the support of at most one $R_i^x (1 \leq i \leq n_x)$; thus $s_x \geq n_x$. The algebra A satisfies the *separation condition* if all $x \in Q_0$ are separating, see [1, 2]. Observe that with A also A^x satisfies the separation condition.

COROLLARY [2]. *If A satisfies the separation condition, then Γ_A has a postprojective component.*

PROOF. Let $x \in Q_0$. Consider $A^x = A_1^x \amalg \cdots \amalg A_{n_x}^x$ where A_j^x is the full convex subalgebra of A with connected quiver Q_j^x. Since also A_j^x satisfies the separation condition, by induction hypothesis, the Auslander-Reiten quiver of A_j^x has a postprojective component $\mathcal{P}_j$. For each $1 \le i \le n_x$, we may assume that R_i^x is an A_i^x-module.

If $R_i^x \notin \mathcal{P}_i$ for some i, then $R_j^x \notin \mathcal{P}_i$ for every $1 \le j \le n_x$. In this case (1x) is satisfied. Otherwise, $R_i^x \in \mathcal{P}_i$ for all $1 \le i \le n_x$. Then clearly (2x) is satisfied. Hence (1.3) implies the result.

2. One-point extensions using postprojective modules.

2.1. Let $A = kQ/I$ be a finite dimensional k-algebra such that Q has no oriented cycles. Let a be a source in Q and consider the quotient $B = A/Ae_a$. For the B-module $M = \operatorname{rad} P_a$, we have $A = B[M]$. Let $\mathcal{P}$ be a postprojective component of Γ_B and assume that all indecomposable direct summands of M belong to $\mathcal{P}$. In this section we consider the problem of when P_a belongs to a postprojective component of Γ_A.

We recall that for two A-modules X, Y we have $\operatorname{rad}_A^\infty(X,Y) = \bigcap_{m \ge 0} \operatorname{rad}_A^m(X,Y)$. We say that an irreducible map $h: X \to Y$ in $\mathcal{P}$ is *M-finite* if $h \notin \operatorname{rad}_A^\infty(X,Y)$. An indecomposable B-module $X \in \mathcal{P}$ is *M-finite* if there is a walk $M_i = X_0 \overset{\alpha_1}{—} —\cdots \overset{\alpha_s}{—} X_s = X$ in $\mathcal{P}$ (where M_i is an indecomposable direct summand of M) such that each α_i is M-finite, $1 \le i \le s$. Of course, if a map or a module is not M-finite we say that it is *M-infinite*.

The following characterization is useful.

LEMMA. *Let $h: X \to Y$ be a map in* mod$A$ *with X and Y indecomposable modules. Then $h \in \operatorname{rad}_A^\infty(X,Y)$ if and only if there are infinitely many A-modules $L_n, n \in N$, without common direct summands and morphisms $f_n: X \to L_n, g_n: L_n \to Y$ with $g_n f_n = h$.*

PROOF. Assume that $h \in \operatorname{rad}^\infty(X,Y)$. We construct the modules L_n inductively. For $n = 1$, we set $L_1 = X$. Assume we have already constructed $L_1, \cdots, L_n$ as in the statement. Let m be the maximal of $\dim_k C$ for C an indecomposable direct summand of some $L_i, 1 \le i \le n$. By the Harada-Sai Lemma, there is a number $N(m)$ such that for every chain $C_1 \to C_2 \to \cdots \to C_s$ of non isomorphisms between indecomposable modules with $\dim_k C_i \le m+1$, if $s \ge N(m)$, then the composition of the chain is zero. Since $h \in \operatorname{rad}_A^{N(m)}(X,Y)$, then h may be written as a linear combination $h=\sum_{i=1}^{r} h_i$, where h_i is composition of $N(m)$ non-isomorphisms between indecomposable modules. Therefore each h_i factorizes through some indecomposable module Z_i with $\dim_k Z_i \geq m+1$, $1 \leq i \leq r$. We can define $L_{n+1}=\oplus_{i=1}^{r} Z_i$.

For the converse, define inductively the finite set of indecomposable modules $X^{(n)}$ in the following way. The set $X^{(1)}$ is formed by those indecomposable modules which are direct summands of the module Z, where $X \to Z$ is a source map in the category mod$A$. If $X^{(n)}$ is defined, then $X^{(n+1)}$ is formed by those modules in $Z^{(1)}$ for Z in $X^{(n)}$. For any n, choose an m such that the module L_m has no direct summands in $X^{(n)}$. Consider the factorization $h=g_m f_m$ with $f_m : X \to L_m, g_m : L_m \to Z$. Using the properties of source maps, we get that f_m lies in $\operatorname{rad}_A^n(X, L_m)$. Hence $h \in \operatorname{rad}_A^\infty(X,Z)$.

2.2. Consider the directed vector space category $\operatorname{Hom}_A(M,\mathcal{P})$, see [3, 6]. Denote by $|X|=\operatorname{Hom}_A(M,X), X\in\mathcal{P}$. Then the full subcategory of $\operatorname{Hom}_A(M,\mathcal{P})$ whose objects are those $|X|\neq 0$ with $X\in\mathcal{P}$, form a poset $\mathcal{P}_M$. Indeed, $|X|\leq|Y|$ in $\mathcal{P}_M$ implies that $X\leq Y$ in $\mathcal{P}$.

A subposet $\mathcal{V}$ of $\mathcal{P}_M$ is said to be of *finite type* if for each $|X|\in\mathcal{V}$, $\dim_k|X|\leq 1$ and $\mathcal{V}$ does not contain as a full subposet one of the posets (1,1,1,1), (2,2,2), (1,3,3), (1,2,5) or (N, 4) of Kleiner's list.

If $\mathcal{P}_M$ is representation-infinite there is a infinite family of triples $Y_\lambda=(V,Y,\gamma_\lambda : V\to \operatorname{Hom}_B(M,Y))$ where $V\in \operatorname{mod}_k, Y$ is a B-module whose indecomposable direct summands X have $|X|\in\mathcal{P}_M$ and γ_λ is linear, corresponding to indecomposable pairwise non-isomorphic A-modules. A module $X\in\mathcal{P}$ is said to be *M-representation-infinite* if there are infinitely many pairwise non-isomorphic indecomposable A-modules of the form $(V,Y,\gamma : V\to \operatorname{Hom}_B(M,Y))$ where $V\in\operatorname{mod}_k$, Y is a B-module with X as a direct summand and γ is linear.

LEMMA. *Let $h: X\to Y$ be an irreducible map in $\mathcal{P}$. Then h is M-infinite if and only if the following two conditions hold*

i) *X is M-representation-infinite;*

ii) *there is a morphism $0\neq g\in\operatorname{Hom}_B(M,X)$ with $hg=0$.*

PROOF. First assume that $h\in\operatorname{rad}_A^\infty(X,Y)$. Then there are infinitely many A-modules $L_n=(V_n,Z_n,\gamma_n : V_n\to\operatorname{Hom}_B(M,Z_n))$, $n\in N$ without common direct summands and morphisms $f_n : X\to L_n, g_n : L_n\to Y$ with $g_n f_n = h$. Fix $n\in N$ and let $Z_n = X^a\oplus Y^b\oplus Z_n'$ be such that X and Y are not summands of Z_n'. The following diagrams commute:

$$\begin{pmatrix}\lambda_i\\ h'_j\\ *\end{pmatrix} = f_n \qquad \begin{array}{ccc} X & \xrightarrow{\ h\ } & Y \\ {\scriptstyle f_n}\downarrow & \nearrow & {\scriptstyle g_n = g(h''_i,\mu_j,*)} \\ Z_n = X^a \oplus Y^b \oplus Z'_n & & \end{array} \qquad \begin{array}{ccc} V_n & \xrightarrow{\ \gamma_n\ } & \operatorname{Hom}_B(M,Z_n) \\ \downarrow & & \downarrow {\scriptstyle \operatorname{Hom}(M,g_n)} \\ 0 & \longrightarrow & \operatorname{Hom}_B(M,Y) \end{array}$$

with $\lambda_i \in k, h''_i \in \operatorname{Hom}_B(X,Y)\ (1 \le i \le a), \mu_j \in k,\ h'_j \in \operatorname{Hom}_B(X,Y)\ (1 \le j \le b)$. Without loss of generality we may assume that $V_n \neq 0$ and $(0,X,0)$, $(0,Y,0)$ are not direct summands of L_n. First we show that $\mu_j = 0\,(1 \le j \le b)$. Otherwise there is some $0 \neq \upsilon \in V_n$ and $\gamma_n(\upsilon) = (\upsilon'_i, \upsilon''_j, *)$ with $\upsilon''_{j_0} \neq 0$ and $\operatorname{Hom}(M, \mu_{j_0})(\upsilon''_{j_0}) \neq 0$ for some j_0, a contradiction. Since h is irreducible as a B-morphism, then $a > 0, \lambda_{i_0} \neq 0$ and h''_{i_0} is a non-zero multiple of h for some $1 \le i_0 \le a$. This shows (i). Moreover, there is some $0 \neq \omega \in V_n$ with $\gamma_n(\omega) = (\omega'_i, \omega''_j, *)$ and $0 \neq \omega'_{i_0} \in \operatorname{Hom}_B(M,X)$. Therefore $\omega'_{i_0} h''_{i_0} = \operatorname{Hom}(M, h''_{i_0}(\omega'_{i_0})) = 0$ and condition (ii) holds.

For the converse, consider an infinite family $L_n = (V_n, Z_n, \gamma_n)$ of pairwise non-isomorphic indecomposable A-modules $(n \in N)$ such that X is a direct summand of Z_n. Let $Z_n = X \oplus Z'_n$ and $\sigma_n : X \to Z_n$ be the canonical inclusion. Assume first that $\dim_k |X| = 1$. Then for the A-morphism $g_n = (0, h\pi_n) : L_n \to Y$ where $\pi_n : Z_n \to X$ is the canonical projection, we get $g_n \sigma_n = h$. This may only happen if $h \in \operatorname{rad}_A^\infty(X,Y)$. Now, assume that $\dim_k |X| \ge 2$ and take $b \in \operatorname{Hom}_B(M,X)$ such that g, b are linearly independent. Then we may choose $Z_n = X \oplus X, V = k$ and $\gamma_n : k \to \operatorname{Hom}_B(M,X)^2$, $1 \mapsto (\lambda_n g, b)$ for some $\lambda_n \neq 0$. Again, if $g_n = (0, h\pi_n)$: $L_n \to Y$ where $\pi_n : X \oplus X \to X$ is the first canonical projection, we get $g_n \sigma_n = h$. We are done. □

2.3. The main result in this section is the following:

THEOREM. *Let $A = B[M]$ be a one-point extension algebra with $M = \operatorname{rad} P_a$ for a source a of Q. Assume that all indecomposable direct summands of M belong to a postprojective component $\mathcal{P}$ of Γ_B.*

If P_a belongs to a postprojective component of Γ_A then the following conditions hold:

a) *M is directing;*

b) *for every irreducible map $h : X \to Y$ in $\mathcal{P}$ such that Y is M-finite, then h is M-finite;*

c) for every indecomposable projective B-module $P_y \in \mathcal{P}$ which is M-finite, the set of predecessors of P_y in Γ_A is finite and formed by directing modules.

Conversely, if conditions (*a*) and (*c*) *hold, then P_a belongs to a postprojective component of Γ_A.*

PROOF. Assume first that $\mathcal{P}'$ is a postprojective component of Γ_A containing P_a. Therefore M is directing.

Let $Y \in \mathcal{P}$ be M-finite, we show that $Y \in \mathcal{P}'$. Indeed, consider a chain of irreducible maps $M \overset{\alpha_1}{\text{---}} X_1 \overset{\alpha_2}{\text{---}} X_2 \text{---} \cdots \overset{\alpha_s}{\text{---}} X_s = Y$ with α_i being M-finite. By induction we may assume that $X_{s-1} \in \mathcal{P}'$. If $X_s \xrightarrow{\alpha_s} X_{s-1}$, then clearly $X_s \in \mathcal{P}'$. If $X_{s-1} \xrightarrow{\alpha_s} X_s$ and $X_s \notin \mathcal{P}'$, then $\alpha_s \in \operatorname{rad}_A^\infty(X_{s-1}, X_s)$, which is a contradiction. Therefore $Y \in \mathcal{P}'$.

We show (b): let $h: X \to Y$ be an irreducible map in $\mathcal{P}$ and assume Y to be M-finite. Then $Y \in \mathcal{P}'$ and also $X \in \mathcal{P}'$. Since $\mathcal{P}'$ is postprojective, $h \notin \operatorname{rad}_A^\infty(X,Y)$. And (c): let $P_y \in \mathcal{P}$ be M-finite. Then $P_y \in \mathcal{P}'$ and therefore P_y has only finitely many predecessors in Γ_A, all of them directing.

For the converse we proceed as in (1.4) to construct a postprojective component $\mathcal{P}'$ of Γ_A. Indeed, we define inductively full subquivers C_n of Γ_A satisfying: (i) C_n is finite, connected, contains no oriented cycle and is closed under predecessors and (ii) $\tau_A^{-1} C_n \cup C_n \subset C_{n+1}$.

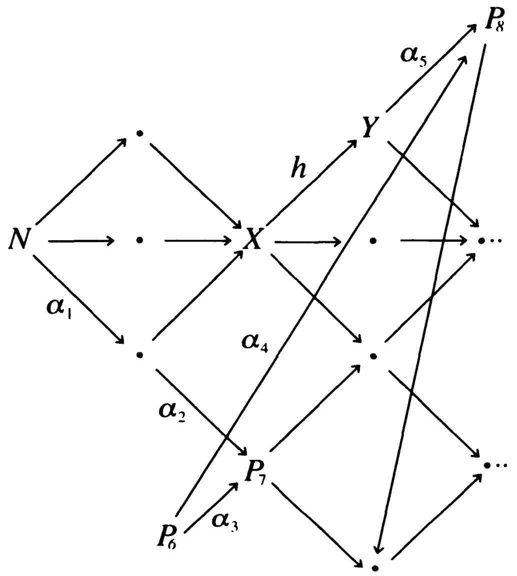
Let S be a simple projective in $\mathcal{P}$, then set $C_0 = \{S\}$. Assume C_n is well defined and let $X_1, \cdots, X_t$ be those modules in C_n with $\tau_A^{-1} X_i \notin C_n$, numbered in such a way that $i < j$ whenever $X_i \leq X_j$. Define $D_0 = C_n, D_{i+1}$ as the full subquiver of Γ_A consisting of D_i and the predecessors of $\tau_A^{-1} X_{i+1}$ and $C_{n+1} = D_t$. It is enough to show inductively that D_i satisfies condition (i) above. Consider the Auslander-Reiten sequence $0 \to X_{i+1} \to X \to \tau_A^{-1} X_{i+1} \to 0$ and assume that D_i satisfies (i). We shall prove that each indecomposable direct summand Y of X has only finitely many predecessors, all of them directing.

We first show the following: let $(V, N, \gamma: V \to \operatorname{Hom}_B(M,N))$ be an indecomposable module in D_i, then every indecomposable direct summand N' of N belongs to $\mathcal{P}$ and is M-finite. We proceed by induction on the path order in D_i (which satisfies (i)). As a first case, assume that $V = 0$. If $N = P_y$ is projective, then every direct summand R_i^y of $\operatorname{rad} P_y$ belongs to $\mathcal{P}$ and is M-finite. Therefore $N \in \mathcal{P}$. Moreover, since the canonical inclusion $R_i^y \to N$ is not in $\operatorname{rad}_A^\infty(R_i^y, N)$, then N is M-finite. If N is not projective, consider the Auslander-Reiten sequence $0 \to \tau_B N \xrightarrow{\sigma} E \to N \to 0$ in mod_B and the corresponding sequence $0 \to \overline{\tau_B N} \to \overline{E} \to N \to 0$ in mod_A, where $\overline{E} = (\operatorname{Hom}_B(M, \tau_B N), E, \operatorname{Hom}_B(M, \sigma))$. Since the indecomposable direct summands of $\overline{E}$ belong to D_i by induction hypothesis we get that the indecomposable direct summands of E belong to $\mathcal{P}$ and are M-finite. Hence $N \in \mathcal{P}$. Moreover, since N is in D_i, it has only finitely many predecessors and therefore any irreducible map $E_i \to N$ in $\mathcal{P}$ is M-finite. For the second case, assume that $V \neq 0$ and take an indecomposable direct summand N' of N. Hence $\operatorname{Hom}_B(M, N') \neq 0$. Suppose that N' is not in $\mathcal{P}$, then $\operatorname{rad}_B^\infty(M, N') \neq 0$

and N' has infinitely many predecessors. The same happens to (V,N,γ) which contains $(0,N',0)$. A contradiction showing that $N' \in \mathcal{P}$. In the same way N' is M-finite.

Now we continue the main line of the proof. Let Y be an indecomposable direct summand of X. If Y is not projective, then Y belongs to D_i and we are done. Assume that Y is projective. Consider first the case $Y = P_a$. By (a), P_a is directing and therefore the predecessors of P_a in mod_A are B-modules and are predecessors of some direct summand M_i of $M = \text{rad}\, P_a$ in mod_B. Since every M_i belongs to $\mathcal{P}$, then $Y = P_a$ has only finitely many (all directing) predecessors. Finally assume that $Y = P_y$ for some $y \neq a$. Let R_1^y be a direct summand of rad P_y belonging to D_i. By the claim shown above, $R_1^y \in \mathcal{P}$ and R_1^y is M-finite. Therefore $P_y \in \mathcal{P}$ and it is also M-finite. By hypothesis (c), $Y = P_y$ has only finitely many (all directing) predecessors in Γ_A. This finishes our proof. □

2.4. We consider again the *example* (1.7). With the notation introduced there $A = C[N]$ where $N = P_1$ is simple projective. We sketch part of the postprojective component $\mathcal{P}$ of Γ_C where N lies.



The walk $\alpha_5^{-1}\alpha_4\alpha_3^{-1}\alpha_2\alpha_1$ from N to Y is formed by N-finite irreducible maps, therefore Y is N-finite. On the other hand, $\dim_k \text{Hom}_C(N,X) = 2$ and $\dim_k \text{Hom}_C(N,Y) = 1$, therefore by (2.2), h is not N-finite. By (2.3), P_2 does not belong to a postprojective component in Γ_A.

3. Some quadratic conditions.

3.1. In this section we consider again the situation of section 2 and we find some necessary conditions for the existence of a postprojective component in Γ_A containing the projective module corresponding to the extension vertex. These conditions are expressed by the values of certain quadratic forms.

Let $A = B[M]$ be a one-point extension of the algebra B by the module $M = \operatorname{rad} P_a$. Let $M = \oplus_{i=1}^{s} M_i$ be the indecomposable decomposition of M. Consider the Euler form associated with B:

$$\langle \underline{\dim} X, \underline{\dim} Y \rangle_B = \sum_{j=0}^{\infty} (-1)^j \dim_k \operatorname{Ext}_B^j(X,Y),$$

where $\underline{\dim} X$ is the element of the Grothendieck group $K_0(B)$ corresponding to X. See [3].

For different $i, j \in \{1, \cdots, s\}$, we define the quadratic form

$$q_{ij}(\omega) = \langle \omega, \underline{\dim} M_i \rangle_B \langle \omega, \underline{\dim} M_j \rangle_B .$$

3.2. PROPOSITION. *Let $\mathcal{P}_1, \cdots, \mathcal{P}_m$ be the postprojective components of Γ_B and assume that $m \geq 1$. Suppose that Γ_A has a postprojective component, then there exists a component $\mathcal{P}_t$ such that for every two different $i, j, \in \{1, \cdots, s\}$ and every $X \in \mathcal{P}_t$ with* $\operatorname{projdim}_B X \leq 1$, *we have*

$$q_{ij}(\underline{\dim} X) \geq 0$$

PROOF. First assume that for some $t \in \{1, \cdots, m\}$, there is no M_i belonging to $\mathcal{P}_t$. Take $X \in \mathcal{P}_t$ with proj $\dim_B X \leq 1$, then

$$\langle \underline{\dim} X, \underline{\dim} M_i \rangle_B = \dim_k \operatorname{Hom}_B(X, M_i) - \dim_k \operatorname{Ext}_B^1(X, M_i).$$

Since $M_i \notin \mathcal{P}_t$, then $\operatorname{Ext}_B^1(X, M_i) = 0$ and $\langle \underline{\dim} X, \underline{\dim} M_i \rangle_B \geq 0$. This shows that $q_{ij}(\underline{\dim} X) \geq 0$ for any two $i, j \in \{1, \cdots, s\}$.

In the other case, choose $t = 1$. Take $i, j \in \{1, \cdots, s\}$ different and $X \in \mathcal{P}_1$ with proj $\dim_B X \leq 1$. Assume that

$$\langle \underline{\dim} X, \underline{\dim} M_i \rangle_B < 0 < \langle \underline{\dim} X, \underline{\dim} M_j \rangle_B .$$

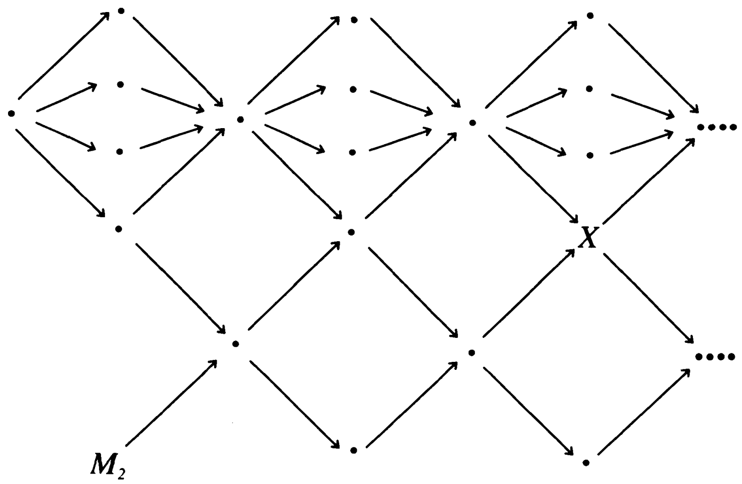
Since proj $\dim_B X \leq 1$, this implies that $\operatorname{Ext}_B^1(X, M_i) \neq 0 \neq \operatorname{Hom}_B(X, M_j)$. The Auslander-Reitern formula gives $0 \neq D\operatorname{Ext}_B^1(X, M_i) \cong \overline{\operatorname{Hom}}_B(M_i, \tau_B X)$ (see [3]). Therefore there is a path in $\Gamma_B, M_i \leq \tau_B X \leq X \leq M_j$. By (1.2), P_a is not directing. Let $\mathcal{P}$ be a postprojective component of Γ_A. Since each $\mathcal{P}_\ell$ for $1 \leq \ell \leq m$, contains a summand of M, then $\mathcal{P} \neq \mathcal{P}_\ell$. Therefore $\mathcal{P}$ is not a component of Γ_B. Hence it contains a module $Y \in \mathcal{P}$ with $0 \neq Y(a) = \operatorname{Hom}_A(P_a, Y)$. This implies that $P_a \in \mathcal{P}$. But then P_a should be directing, a contradiction. We are done. □

3.3. We come back to our example (1.7) now considering $A = B[M]$ where M

$= \text{rad}\, P_7$. Thus $M = M_1 \oplus M_2$, where

$$\underline{\dim}\, M_1 = (1,0,0,1,1,0,0) \text{ and } \underline{\dim}\, M_2 = (0,0,0,0,0,1,0)$$

in $K_0(B)$. There is a unique postprojective component $\mathcal{P}_1$ of Γ_B which has the shape



where $\underline{\dim} X = (6,2,2,2,3,0,1) \in K_0(B)$ and clearly proj $\dim_B X \leq 1$.

We have

$$\langle x, \underline{\dim} M_1 \rangle_B = x_1 - x_2 - x_3 - x_7 \text{ and } \langle x, \underline{\dim} M_2 \rangle_B = x_6 - x_7.$$

Hence $q_{12}(\underline{\dim} X) = -1$. The quiver Γ_A has no postprojective component (as we already knew).

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