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GEODESIC HYPERSPHERES IN COMPLEX PROJECTIVE SPACE

By

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1. Introduction

Let $P^{n}C$ be an $n(\geqq 2)$-dimensional complex projective space with the Fubini-Study metric of constant holomorphic sectional curvature 4. A first interesting progress in the theory of real hypersurfaces in complex projective space is R. Takagi's work on homogeneous real hypersurfaces. In [T1], he classified all the homogeneous real hypersurfaces in $P^{n}C$ into six types, A_1, A_2, B, C, D and E. A real hypersurface of type A_1 is also called a geodesic hypersphere, which can be characterized as a real hypersurface with two constant principal curvatures [T2]. Furthermore he characterized real hypersurfaces of type A_2 and B as those with three constant principal curvatures [T3]. Next important studies are found in [C-R]. In thier paper [C-R], T. E. Cecil and P. J. Ryan investigated a real hypersurface which lies in a tube over a submanifold in $P^{n}C$. Especially, they found that every homogeneous real hypersurface in Takagi's classification can be realized as a tube of a constant radius over a compact Hermitian symmetric space of rank 1 or rank 2: Every homogeneous real hypersuface in $P^{n}C$ is locally congruent to a tube of radius r over one of the following;

(A_1) hyperplane $P^{n-1}C$, where $0<r<\pi/2$,

(A_2) totally geodesic $P^{k}C(1\leqq k\leqq n-1)$, where $0<r<\pi/2$,

(B) complex quadric Q^{n-1}, where $0<r<\pi/4$,

(C) $P^{1}C\times P^{(n-1)/2}C$, where $0<r<\pi/4$ and n is odd,

(D) complex Grassmann $G_{2,5}C$, where $0<r<\pi/4$ and $n=9$,

(E) Hermitian symmetric space $SO(10)/U(5)$, where $0<r<\pi/4$ and $n=15$.

On the other hand, many differential geometers have studied real hypersurfaces in $P^{n}C$ by making use of the almost contact structure induced from $P^{n}C$. For example, M. Okumura [Ok] proved that a real hypersurface is of type A_1 or A_2 if and only if the almost contact structure commutes with the second fundamental form of it.

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In this paper, we characterize a geodesic hypersphere by a certain condition on the second fundamental form (Theorem 4.1 and Theorem 4.2.).

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2. Preliminaries

Let M be a real hypersurface in P^nC. The Riemannian metrics of P^nC and M are denoted by the same letter g, while the Riemannian conections of them are denoted by ∇^P and ∇ respectively. Let ν be a (local) field of unit normal vector of M. Then Gauss's and Weingarten's formulas are given as

$$\nabla_X^P Y = \nabla_X Y + g(AX, Y), \tag{2.1}$$

$$\nabla_X^P \nu = -AX, \tag{2.2}$$

for any vector fields X and Y. Here A is an endomorphism of the tangent bundle TM of M which is defined by (2.2) and called the shape operator in the direction ν. Let J denote the complex structure of P^nC. Then we define ϕ of type (1, 1), a vector field ξ and a 1-form η on M as follows:

$$\phi X = (JX)^{\mathrm{T}}, \qquad \xi = -J\nu, \quad \text{and} \quad \eta(X) = g(X, \xi), \tag{2.3}$$

where $\cdot^{\mathrm{T}}$: $TP^nC \to TM$ indicates the orthogonal projection. From definitions above we obtain

$$\phi^2 = -I + \eta \otimes \xi, \qquad \phi\xi = 0, \qquad \eta(\xi) = 0, \tag{2.4}$$

where I denotes the identity transformation of TM. We also obtain

$$\nabla_X \phi(Y) = \eta(Y)AX - g(AX, Y)\xi, \tag{2.5}$$

$$\nabla_X \xi = \phi AX. \tag{2.6}$$

Let R^P and R denote the curvature tensor of P^nC and M respectively. Then since R^P is given by

$$R^P(X, Y)Z = g(Y, Z)X - g(X, Z)Y + g(JY, Z)JX - g(JX, Z)JY + 2g(X, JY)JZ,$$

the equations of Gauss and Codazzi are respectively given as follows:

$$R(X, Y)Z = g(Y, Z)X - g(X, Z)Y + g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y - 2g(\phi X, Y)\phi Z + g(AY, Z)AX - g(AX, Z)AY, \tag{2.6}$$

$$\nabla_X A(Y) - \nabla_Y A(X) = \eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi. \tag{2.7}$$

Finally we recall the Ricci formula. For each tensor field T of type (r, s), its covariant derivative ∇T, a tensor field of type $(r, s+1)$, is defined by

$$\nabla T(X_1, \cdots, X_s; X)=\nabla_X T(X_1, \cdots, X_s).$$

Then the second covariant derivative $\nabla^2 T=\nabla\nabla T$ is computed as

$$\nabla^2 T(X_1, \cdots, X_s; X; Y)=\nabla_Y\nabla_X T(X_1, \cdots, X_s)-\nabla_{\nabla_Y X}T(X_1, \cdots, X_s). \tag{2.8}$$

From (2.8) we have the following which is known as the Ricci formula:

$$\begin{aligned}\nabla^2 T(X_1, \cdots, X_s; X; Y)-\nabla^2 T(X_1, \cdots, X_s; Y; X)\\ =-(R(X, Y)T)(X_1, \cdots, X_s),\end{aligned} \tag{2.9}$$

where $R(X, Y)$ acts on T as a derivation.

3. Key lemma

In the study of real hypersurfaces of $P^n C$, it is a crucial condition that the structure vector ξ is principal. In fact in proofs of many known results, it seems that the most difficult part is to show that ξ is principal under a certain condition. For this reason, this section is devoted to show the following lemma:

LEMMA 3.1. *Assume $n\geqq 3$ and the shape operator A satisfies*

$$(R(Y, Z)A)X=0$$

for each vector X, Y, Z perpendicular to ξ. Then ξ is principal.

PROOF. We denote by $\xi^\perp$ the subbundle of TM consisting of vectors perpendicular to ξ. In what follows $e_1, \cdots, e_{2n-2}$ stand for an orthonormal basis of $\xi^\perp$ at a point in M, and the index j runs from 1 to $2n-2$.

On account of (2.6) and the condition, the following holds:

$$\begin{aligned} &g(Z, AX)Y-g(Y, AX)Z+g(\phi Z, AX)\phi Y-g(\phi Y, AX)\phi Z\\ &\quad -2g(\phi Y, Z)\phi AX+g(AZ, AX)AY-g(AY, AX)AZ\\ &\quad -g(Z, X)AY+g(Y, X)AZ-g(\phi Z, X)A\phi Y+g(\phi Y, X)A\phi Z\\ &\quad +2g(\phi Y, Z)A\phi X-g(AZ, X)A^2Y+g(AY, X)A^2Z\\ &=0,\end{aligned} \tag{3.2}$$

where X, Y, Z are tangent vectors perpendicular to ξ. Putting $X=e_j$ and $Z=\phi e_j$ in (3.2), and taking summation on j, we obtain

(3.3)
$$-\{TrA-\eta(A\xi)\}\phi Y-3\phi AY+(2n+1)A\phi Y$$
$$-A\phi A^2Y+A^2\phi AY-\eta(A\phi Y)\xi=0.$$

Taking ξ- and Y-component of (3.3) to get

(3.4)
$$2n\eta(A\phi Y)-\eta(A\phi A^2Y)+\eta(A^2\phi AY)=0$$

and

(3.5)
$$(2n+4)g(A\phi Y, Y)+2g(A^2\phi AY, Y)=0.$$

Note that $TrA\phi=TrA^2\phi A=0$ because A is symmetric and ϕ is skew-symmetric. Therefore putting $Y=e_j$ in (3.5) and taking summation on j,

(3.6)
$$g(A^2\phi A\xi, \xi)=0.$$

Now define a cross section U of $\xi^\perp$ and a smooth function α on M by
$$A\xi=U+\alpha\xi.$$

Then $\phi A\xi=\phi U$ and $A^2\xi=AU+\alpha U+\alpha^2\xi$, so (3.6) implies

(3.7)
$$g(\phi U, AU)=\eta(A^2\phi U)=0.$$

Using (3.7), we also have

(3.8)
$$g(A^2U, \phi U)=0$$

by putting $Y=U$ in (3.4). We also note

(3.9)
$$g(\phi U, A\xi)=\eta(A\phi U)=0.$$

Thus from (3.7) and (3.9), we get the following by putting $Z=U$ and $X=\phi U$ in (3.2):

(3.10)
$$-g(Y, A\phi U)U-g(\phi Y, A\phi U)\phi U+g(\phi U, A\phi U)\phi Y-2g(\phi Y, U)\phi A\phi U$$
$$-g(AY, A\phi U)AU+3g(Y, \phi U)AU-\|U\|^2A\phi Y+g(Y, U)A\phi U$$
$$+g(AY, \phi U)A^2U=0,$$

where $\|U\|^2=g(U, U)$. Taking ϕU-component of (3.10),
$$g(g(A\phi U, \phi U)U+\|U\|^2\phi A\phi U, Y)=0.$$

Since this equation holds for all Y perpendicular to ξ, we obtain

(3.11)
$$-\|U\|^2A\phi U=g(\phi A\phi U, U)\phi U.$$

Now suppose $\|U\|^2\neq 0$ at a point, say x. Then a contradiction is derived as follows. In this case, by virture of (3.11), there exists a certain real number λ such that

(3.12) $$A\phi U=\lambda\phi U .$$

That is, ϕU is principal curvature vector with principal curvature λ. Then (3.10) is reduced to

(3.13) $$-3\lambda g(Y, \phi U)U+\lambda\|U\|^{2}\phi Y+(3-\lambda^{2})g(Y, \phi U)AU$$
$$-\|U\|^{2}A\phi Y+\lambda g(Y, \phi U)A^{2}U=0 .$$

Therefore if Y is perpendicular to all of U, ϕU and ξ,

$$\lambda\|U\|^{2}\phi Y-\|U\|^{2}A\phi Y=0 ,$$

so that

$$A\phi Y=\lambda\phi Y .$$

Now let $T_{x}M=V\oplus\mathrm{span}\{U, \xi\}$ be the orthogonal decomposition. Then the above argument implies

(3.14) $$A|V=\lambda I_{V} ,$$

where I_{V} stands for the identity transformation of V. Further we decompose V orthogonally as $V=V'\oplus\mathrm{span}\{\phi U\}$. Note that $\dim V'\geqq 1$ by the assumption $n\geqq 3$. Since V' is invariant by ϕ, (3.3) reduces to

$$-\{TrA-\alpha\}\phi Y-3\lambda\phi Y+(2n+1)\lambda\phi Y=0 ,$$

for each $Y\in V'$. So we have

(3.15) $$TrA-(2n-2)\lambda-\alpha=0 .$$

On the other hand, (3.14) implies

(3.16) $$TrA=(2n-3)\lambda+g(AU, U)+\alpha .$$

Thus $g(AU, U)=\lambda$, which implies

(3.17) $$AU=\lambda U+\|U\|^{2}\xi ,$$

and

(3.18) $$A^{2}U=(\lambda^{2}+\|U\|^{2})U+(\alpha+\lambda)\|U\|^{2}\xi .$$

Putting $Y=\phi U$ in (3.13) and substituting (3.17), (3.18) into it, we get

$$\lambda\|U\|^{4}U+(\alpha\lambda+4)\|U\|^{4}\xi=0 ,$$

which contradicts to $\|U\|^{2}\neq 0$. Consequently $U=0$ and ξ is principal. ■

Next lemma is contained in previous Lemma (3.1) in the case $n\geqq 3$, but is verified even in the case $n=2$:

LEMMA 3.19. *Assume the shape operator A satisfies*

$$(\nabla^2 A)(X ; Y ; Z)=f\{g(X, \phi Y)\phi Z+g(X, \phi Z)\phi Y\}$$

for all X, Y, Z perpendicular to ξ, where f in a C^∞-function on M. Then ξ is principal.

PROOF. By making use of the equation of Codazzi (2.7), we find the following formula in general:

$$(3.20) \qquad (\nabla^2 A)(X ; Y ; Z)-(\nabla^2 A)(Y ; X ; Z)$$
$$=g(Y, \phi AZ)\phi X-g(X, \phi AZ)\phi Y-2g(X, \phi Y)\phi AZ$$
$$+3\{\eta(X)g(AY, Z)-\eta(Y)g(AX, Z)\}\xi,$$

for arbitrary tangent vectors X, Y, Z.

Therefore the condition and (3.20) implies

$$(3.21) \qquad -f\{g(Y, \phi Z)\phi X-g(X, \phi Z)\phi Y-2g(X, \phi Y)\}$$
$$=g(Y, \phi AZ)\phi X-g(X, \phi AZ)\phi Y-2g(X, \phi Y)\phi AZ .$$

Putting $Y=\phi X$ in (3.21) and taking ϕX-component, we obtain

$$(3.22) \qquad AZ=-fZ+\eta(AZ)\xi,$$

for all Z perpendicular to ξ.

On the other hand, the condition and the Ricci formula (2.9) implies

$$(R(Y, Z)A)X=0$$

for all vectors X, Y, Z perpendicular to ξ. In what follows we use notation in the proof of lemma (3.1). Suppose $U\neq 0$ at a point. Then from (3.12) and (3.22), $-f=\lambda$ at the point, so that

$$AU=\lambda U+\|U\|^2\xi .$$

This derives a contradiction by a similar argument in the proof of lemma (3.1). ■

Type number at $x\in M$ is, by definition, the rank of linear transformation A, and denoted by $t(x)$. As a result of this proof, we obtain

PROPOSITION 3.23. *There exist no real hypersurfaces in P^nC satisfing*

$$(\nabla^2 A)(X ; Y ; Z)=0$$

for all X, Y, Z perpendicular to ξ.

PROOF. Since ξ is principal under the condition as $f=0$ on M in Lemma (3.9), (3.22) reduces to $AZ=0$. Thus $t(x)\leqq 1$ at each $x\in M$. However it is known that any real hypersurface has a point x with $t(x)>1$ (cf. p. 156 [Y-K], see all so [T1]). This contradiction shows the assertion. ■

4. Theorems

In this section we will prove the following two Theorems:

THEOREM 4.1. *Let M be a real hypersurface in P^nC, $n\geqq 3$. If the shape operator A satisfies*

$$(R(Y, Z)A)X=0$$

for all tangent vectors X, Y, Z perpendicular to ξ, then M is locally congruent to a geodesic hypersphere.

THEOREM 4.2. *Let M be a real hypersurface in P^nC, $n\geqq 2$. If the shape operator A satisfies*

$$(\nabla^2 A)(X\,;\,Y\,;\,Z)=f\{g(X, \phi Y)\phi Z+g(X, \phi Z)\phi Y\}$$

for all tangent vectors X, Y, Z perpendicular to ξ, where f is a C^∞-function on M, then f is non-zero constant and M is locally congruent to a geodesic hypersphere.

For proof we need the following results:

FACT 4.3. ([K-Ms]) *Let M be a real hypersurface in P^nC, $n\geqq 2$. Suppose that M satisfies*

$$\mathfrak{S}_{X,Y,Z}(R(Y, Z)A)X=0$$

for all X, Y, $Z\in TM$. Here $\mathfrak{S}_{X,Y,Z}$ indicates cyclic sum with respect to X, Y, Z. Then M is locally congruent to one of the following:

(i) *a geodesic hypersphere, $n\geqq 3$,*

(ii) *a real hypersurface in P^2C on which ξ is a principal curvature vector.*

FACT 4.4. ([T2]) *If M is a connected complete real hypersurface in P^nC with two constant principal curvatures, then M is a geodesic hypersphere. If we do not assume the completeness of M, M is locally congruent to a geodesic hypersphere.*

PROOF OF THEOREM 4.1. We have seen in Lemma (3.1) that the structure vector ξ is principal under the condition. Then it is easy to verify $\mathfrak{S}_{X,Y,Z}$

$(R(Y, Z)A)X=0$ for all tangent vectors X, Y, Z. Therefore our assertion comes from Fact (4.3). ■

REMARK 4.5. Maeda [Ms] proved that there exist no real hypersurfaces in P^nC, $n\geqq 3$, satisfying $RA\equiv 0$.

PROOF OF THEOREM 4.2. Theorem 4.2 is contained in Theorem 4.1 in the case $n\geqq 3$, but we proceed independently.

Since ξ is principal by Lemma (3.19), let Y be a (local) vector field orthogonal to ξ such that $AY=\lambda Y$. Then it is known ([My]) that

$$A\phi Y=\frac{\alpha\lambda+2}{2\lambda-\alpha}\phi Y.$$

Putting $X=Z=\phi Y$ in (3.2) to get

$$-2\alpha\lambda^4+(2\alpha^2-20)\lambda^3+30\alpha\lambda^2+(20-8\alpha^2)\lambda-8\alpha=0.$$

It is also known ([My]) that α is locally constant. Thus λ is constant. On the other hand, from (3.22)

$$A|\xi^{\perp}=-fI_{\xi^{\perp}},$$

and so $f=-\lambda$ is constant. Consequently M has two constant principal curvatures. Therefore Fact (4.4) implies the assertion. Moreover this constant f is not zero by Proposition (3.23). ■

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