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THE CYCLIC EXTENSIBILITY OF ESSENTIAL COMPONENTS OF THE FIXED POINT SET

By

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1. Introduction.

All spaces considered in this paper are separable metric and every mapping is continuous unless otherwise stated. Let X be a continuum[1]. If every continuous mapping $f: X\rightarrow X$ has at least one fixed point, X is called to have the *fixed point property* $(f.p.p.)$. In this paper we investigate the existence of essential components of the fixed point sets and the property f.*p.p., which are defined as follows: a component C of the fixed point set of f is called *essential*, if for any $\varepsilon>0$ there exists $\delta>0$ such that every continuous mapping $f': X\rightarrow X$ with $|f'-f|<\delta$ has a fixed point in the ε-neighborhood $U_{\varepsilon}(C)$ of C, and if otherwise it is called *non-essential*; and X has *f.*p.p.*, if X has f.p.p., and the fixed point set of every continuous mapping $f: X\rightarrow X$ has at least one essential component (see [2], [7]). Note that there exists a space which has f.p.p., but does not have f.*p.p. (see [6]).

The Hilbert cube I^{ω} has f.*p.p. and the property f.*p.p. is invariant under retractions. Hence every compact absolute retract has f.*p.p. (see [2]). Further, if X and Y are two continua with f.*p.p. and $X\cap Y$ is a single point, then $X\cup Y$ has f.*p.p. (see [1], [4], [5]). The last statement has been extended to the special case where the number of continua is countably infinite (see [5]). The purpose of this paper is to extend the above property to a more general setting; we prove that a continuum X has f.*p.p. whenever it can be expressed as the union of a null sequence of subcontinua X_{α}'s with f.*p.p. such that any pair of X_{α} and X_{β} $(\alpha\neq\beta)$ has at most one point in common and that the boundary of each component of $X-X_{\alpha}$ consists of a single point for every α (see the Main Theorem). When X is locally connected, it means the cyclic extensibility of f.*p.p. (see [3], [4] and the Corollary).

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1) A continuum means a compact, connected metric space.

Notation

$|f'-f|=\sup_{x\in X} d(f'(x), f(x))$.

$\bar{A}$: the closure of A.

Bdry A: the boundary of A.

Int A: the interior of A.

diam(A): the diameter of A.

2. Cyclic Extensibility and the Main Theorem.

The cyclic extensibility of f. p. p. was proved by K. Borsuk [1]. We will generalize it to our setting in Lemma 3.

DEFINITION 1. A point s of a connected topological space X is called a *separating point* of X if $X-s$ is the union of two disjoint sets and neither of them contains a limit point of the other.

DEFINITION 2. A point $p\in X$ of order one in a continuum X is called an *endpoint* of X, i. e., p is an *endpoint* of X provided there exist arbitrarily small open neighborhoods $V(p)$'s each boundary of which consists of a single point (see [4], p. 64).

DEFINITION 3. In a metric space X we shall call a subset A of X an *A-set* provided that $X-A=\bigcup_\alpha G_\alpha$, where (1) G_α is open, (2) $G_\alpha\cap G_\beta=\phi$ for $\alpha\neq\beta$, (3) Bdry G_α contains at most one point, and (4) diam$(G_i)\to 0$ $(i\to\infty)$ for any infinite sequence $\langle G_i\rangle$ of G_α, i. e., $X-A$ is the union of a finite number of or a null sequence of disjoint open sets each having at most one boundary point (see [4], p. 67).

DEFINITION 4. An A-set is a *true A-set* if either (1) it is non-degenerate, or (2) it is a separating point or an endpoint of X (see [4], p. 68).

MAIN THEOREM. *Let X be a continuum and $\{X_\alpha\}$ a null sequence of true A-sets of X which satisfy the following conditions*:

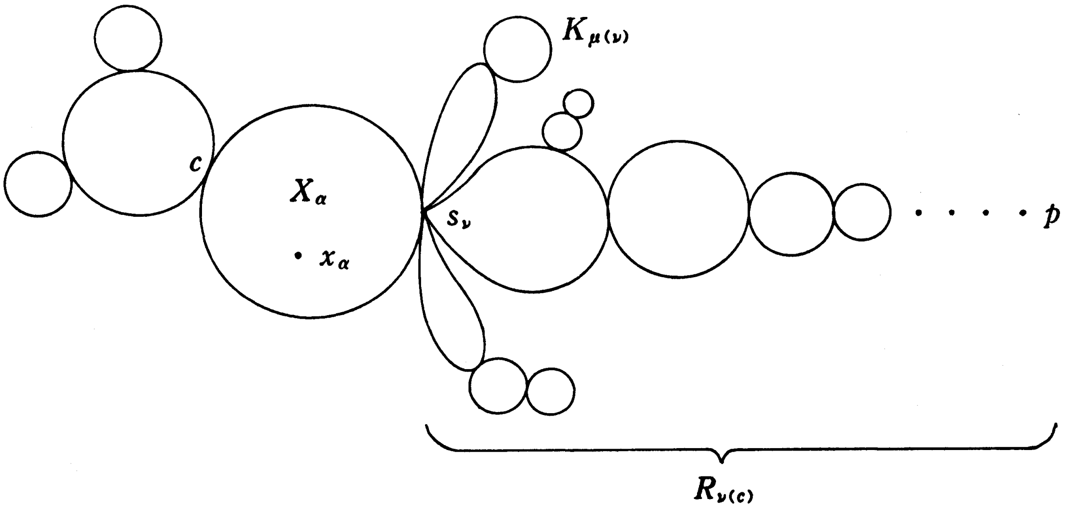
(1) $X=\bigcup_\alpha X_\alpha$,

(2) *whenever $X_\alpha\cap X_\beta\neq\phi$ $(\alpha\neq\beta)$, $X_\alpha\cap X_\beta$ is a separating point of X, and*

(3) *X_α has $f^*_.p.p.$ for every α.*

Then, X has $f^*_.p.p.$

REMARK 1. Note that Int X_α may contain a separating point of X.



X

Figure 1.

DEFINITION 5. Let X be a locally connected continuum and $\{X_\alpha\}$ a null sequence of true A-sets of X which satisfy the following conditions:

(1) Int X_α contains no separating point for every α,

(2) $X=\bigcup_\alpha X_\alpha$, and

(3) whenever $X_\alpha \cap X_\beta \neq \phi (\alpha \neq \beta)$, $X_\alpha \cap X_\beta$ is a separating point of X.

Then, each X_α, together with each separating point and endpoint, is called a *cyclic element* of X. A topological property P is said to be *cyclicly extensible*, if X has the property P whenever each cyclic element has the property P(see [3], [4]).

COROLLARY. *$f^*_\cdot p.p.$ is cyclicly extensible.*

3. Some Preliminaries to the proof of Main Theorem.

In the following discussions, we assume that X contains at least two X_α's. We always mean by s_ν a separating point of X not contained in the interior of any X_α.

DEFINITION 6. Let z be a point of X. For two points x, $y \in X$, define the partial order with base point $z \in X$ as follows:

(1) let $x \underset{z}{=} y$, if x and y are contained in the interior of the same X_α, or x and y are the same separating point or the same endpoint of X.

(2) let $x \underset{z}{>} y$, if x and y satisfy

(i) $x \underset{z}{\neq} y$,

(ii) X is the union of two subcontinua A and B with $A \cap B = s_\nu$ where A contains x and B contains both y and z, and

(iii) whenever $y \underset{z}{\neq} z$, X is not the union of two subcontinua A' and B' with $A' \cap B' = s_\nu$ where A' contains both x and z, and B' contains y.

Now, for the convenience of the proofs, we assign some special points of X for base points of the above partial order. Let c be a point of s_ν's and x_α a point of Int X_α for each non-degenerate X_α. Then, we will use the partial order with the four kinds of base points listed below:

s_ν: a separating point of X not contained in the interior of any X_α.

c: the pre-assigned s_ν of X.

p: an endpoint of X.

x_α: the pre-assigned point of Int X_α for each non-degenerate X_α.

DEFINITION 7. We define the subspaces $R_{\nu(c)}$ and $K_{\mu(\nu)}$ of X as follows: Let $R_{\nu(c)} = \{x \mid x \underset{c}{\geqq} s_\nu\}$, and $K_{\mu(\nu)}$ be the closure of one of the components of $X - s_\nu$.

We also define the retractions $r_{\nu(c)}: X \to R_{\nu(c)}$, $r_{\mu(\nu)}: X \to K_{\mu(\nu)}$ and $r_\alpha: X \to X_\alpha$ by

$$r_{\nu(c)}(x) = \begin{cases} x & \text{for } x \in R_{\nu(c)}, \\ s_\nu & \text{for } x \in X - R_{\nu(c)}, \end{cases}$$

$$r_{\mu(\nu)}(x) = \begin{cases} x & \text{for } x \in K_{\mu(\nu)}, \\ s_\nu & \text{for } x \in X - K_{\mu(\nu)}, \text{ and} \end{cases}$$

$$r_\alpha(x) = \begin{cases} x & \text{for } x \in X_\alpha, \\ s_\nu & \text{for } x \in R_{\nu(x_\alpha)}, \text{ where } s_\nu \in \text{Bdry } X_\alpha. \end{cases}$$

Note that $\overline{X - X_\alpha} = \bigcup_\nu R_{\nu(x_\alpha)}$.

From above definitions, we have immediately the following two Lemmas.

LEMMA 1. *Any open neighborhood $U(s_\nu)$ of s_ν contains almost all $K_{\mu(\nu)}$ but a finite number of μ's.*

LEMMA 2. *If the boundary s_ν of $K_{\mu(\nu)}$ is not contained in any non-degenerate $X_\alpha \subset K_{\mu(\nu)}$, the point s_ν is an endpoint of $K_{\mu(\nu)}$* (see [4], p. 64).

First, we generalize the Borsuk's theorem of cyclic extensibility of f. p. p. to our setting (see [4], p. 242).

LEMMA 3. *Let X be a continuum and $\{X_\alpha\}$ a null sequence of true A-sets of X which satisfy the following conditions*:

(1) $X=\bigcup_\alpha X_\alpha$,

(2) *whenever* $X_\alpha\cap X_\beta\neq\phi(\alpha\neq\beta)$, $X_\alpha\cap X_\beta$ *is a separating point of* X, *and*

(3) X_α *has f.p.p. for every* α.

Then, X *has f.p.p.*

PROOF. Assume on the contrary that there exists a mapping $f: X\to X$ which has no fixed point. If there exists non-degenerate X_α such that every $s_\nu\in \mathrm{Bdry}\, X_\alpha$ satisfies $f(s_\nu)\underset{x_\alpha}{\not\geq} s_\nu$, then $r_\alpha f|_{X_\alpha}: X_\alpha\to X_\alpha$ has no fixed point, which is a contradiction. Hence, we consider the case where for every non-degenerate X_α there exists $s_\nu\in \mathrm{Bdry}\, X_\alpha$ with $f(s_\nu)\underset{x_\alpha}{>} s_\nu$.

Letting c be the initial point, we construct the ordered set $\langle s_\lambda\rangle$ (λ is a countable ordinal) of s_ν by the following procedure. Let $K_{m(\lambda)}$ be such that $f(s_\lambda)\in K_{m(\lambda)}$.

1. Define the immediate successor of s_λ as follows:

Case 1. s_λ is a boundary point of non-degenerate X_λ contained in $K_{m(\lambda)}$. In this case there exists $s_\nu\in \mathrm{Bdry}\, X_\lambda$ with $f(s_\nu)\underset{x_\lambda}{>} s_\nu$. Let s_ν be the immediate successor of s_λ.

Case 2. s_λ is an endpoint of $K_{m(\lambda)}$. Then, by the continuity of f, there exists $s_\nu(\neq s_\lambda)$ in a neighborhood of s_λ in $K_{m(\lambda)}$ such that $f(s_\nu)\underset{s_\lambda}{>} s_\nu$. Let s_ν be the immediate successor of s_λ.

2. When λ converges to ν, let s_ν be the limit point of $\langle s_\lambda\rangle$ if it is not an endpoint of X. We add s_ν to $\langle s_\lambda\rangle$. Note that s_ν satisfies $f(s_\nu)\underset{c}{>}s_\nu$.

By the construction of this ordered set, it is easy to see that $\langle s_\lambda\rangle$ and $\langle K_{m(\lambda)}\rangle$ satisfy the following conditions:

(1) $f(s_\lambda)\underset{c}{>}s_\lambda$ for every λ,

(2) $K_{m(\lambda)}\supset K'_{m(\lambda')}$ $(\lambda<\lambda')$, and

(3) either $\langle s_\lambda\rangle$ ends in s_e which is the single boundary point of X_α, or $\langle s_\lambda\rangle$ converges to an endpoint p of X.

Applying the above ordered set, we now prove the Lemma.

Case 1. $\langle s_\lambda\rangle$ ends in s_e which is the single boundary point of X_α. In this case, $r_\alpha f|_{X_\alpha}: X_\alpha\to X_\alpha$ has no fixed point, which contradicts to the assumption that X_α has f.p.p.

Case 2. $\langle s_\lambda\rangle$ converges to an endpoint p of X. It is easy to see that p is fixed by f, which contradicts to our assumption.

Next, we state some lemmas on essential components of the fixed point set of a mapping $f: X \to X$.

LEMMA 4. *Let X and Y be compact metric spaces such that $X \supset Y$. Assume that there exists a retraction $r: X \to Y$. Then, if a mapping $f: X \to X$ is continuous, for every $\varepsilon > 0$ there exists $\delta > 0$ such that every continuous mapping f': $X \to X$ with $|f' - f| < \delta$ satisfies $|rf' - rf| < \varepsilon$.*

PROOF. By the uniform continuity of r, for given $\varepsilon > 0$ there exists $\delta > 0$ such that $|rf'(x) - rf(x)| < \varepsilon$ for any pair of $f'(x)$ and $f(x)$ with $|f'(x) - f(x)| < \delta$. Then, if $|f'(x) - f(x)| < \delta$ for every $x \in X$, we have $|rf'(x) - rf(x)| < \varepsilon$ for every $x \in X$.

LEMMA 5. *Let C_γ be a component of the fixed point set of a mapping f: $X \to X$ such that $C_\gamma \subset \mathrm{Int}\, X_\alpha$ for a non-degenerate X_α. If C_γ is a non-essential component of the fixed point set of f, then C_γ is a non-essential component of the fixed point set of $r_\alpha f|_{X_\alpha}: X_\alpha \to X_\alpha$.*

PROOF. Since C_γ is a non-essential component of the fixed point set of f, C_γ has an open neighborhood U such that for each n there exists a mapping $f_n: X \to X$ which satisfies

(i) $|f_n - f| < 1/n$, and

(ii) f_n has no fixed point in U_γ.

Since C_γ is contained in Int X_α, there exists a neighborhood U' of C_γ such that $U' \subset U \cap \mathrm{Int}\, X_\alpha$. Then for each n' there exists $r_\alpha f_n|_{X_\alpha}: X_\alpha \to X_\alpha$ which satisfies

(i′) $|r_\alpha f_n|_{X_\alpha} - r_\alpha f|_{X_\alpha}| < 1/n'$, and

(ii′) $r_\alpha f_n|_{X_\alpha}$ has no fixed point in U',

where condition (i′) follows from Lemma 4.

LEMMA 6. *Let C_γ be a component of the fixed point set of a mapping f: $X \to X$ such that $C_\gamma \cap \mathrm{Bdry}\, X_\alpha = \{s_\nu\}$ for a non-degenerate X_α. Assume that C_γ has an open neighborhood U such that for each n there exists a mapping f_n: $X \to X$ which satisfies*

(i) $|f_n - f| < 1/n$,

(ii) *f_n has no fixed point in U, and*

(iii) *$f_n(s_\nu) \underset{X_\alpha}{\ngeq} s_\nu$ for every $s_\nu \in U \cap \mathrm{Bdry}\, X_\alpha$.*

Then, $C_\gamma \cap X_\alpha$ is a non-essential component of the fixed point set of $r_\alpha f|_{X_\alpha}$: $X_\alpha \to X_\alpha$.

PROOF. Note that any $s_\nu \in U \cap \mathrm{Bdry}\, X_\alpha$ is not fixed by $r_\alpha f_n|_{X_\alpha}: X_\alpha \to X_\alpha$. Hence for each n' there exists $r_\alpha f_n|_{X_\alpha}: X_\alpha \to X_\alpha$ which satisfies

(i) $|r_\alpha f_n|_{X_\alpha} - r_\alpha f|_{X_\alpha}| < 1/n'$, and

(ii) $r_\alpha f_n|_{X_\alpha}$ has no fixed point in $U \cap X_\alpha$.

LEMMA 7. *Let $f: X \to X$ be a mapping and p an endpoint of X. Assume that there exist arbitrarily small open neighborhoods $V(p)$'s $(V(p) \not\ni c)$ such that* Bdry $V(p)$ *is a single point s_ν which satisfies $f(s_\nu) \underset{c}{>} s_\nu$. Then, p is an essential component of the fixed point set of f.*

PROOF. Assume on the contrary that p is a non-essential component of the fixed point set of f. Then, p has an open neighborhood U such that for every $\delta > 0$ there exists a mapping $f': X \to X$ which satisfies

(i) $|f' - f| < \delta$, and

(ii) f' has no fixed point in U.

By the assumption of the lemma, we can choose s_ν such that $R_{\nu(c)} \subset U$ and $f(s_\nu) \underset{c}{>} s_\nu$. Let $d(f(s_\nu), s_\nu) = a$ and $\delta = a/2$. By condition (i), we have $f'(s_\nu) \underset{c}{>} s_\nu$. Since f' has no fixed point in $R_{\nu(c)} \subset U$, $r_{\nu(c)} f'|_{R_{\nu(c)}}: R_{\nu(c)} \to R_{\nu(c)}$ has no fixed point. Note that $R_{\nu(c)}$ has f. p. p. by Lemma 3. Hence we have a contradiction.

LEMMA 8. *Let $f: X \to X$ be a mapping and p an endpoint of X such that $f(p) = p$. Assume that p belongs to a non-essential component C of the fixed point set of f; i.e., C has an open neighborhood $U(C)$ such that for each n there exists a mapping $f_n: X \to X$ with $|f_n - f| < 1/n$ which has no fixed point in $U(C)$. Then, there exists an open neighborhood $V(p)$ such that every $s_\nu \in V(p)$ satisfies either*

(a) *$f(s_\nu) \underset{p}{>} s_\nu$, or*

(b) *$f(s_\nu) \underset{p}{=} s_\nu$, and $f_n: X \to X$ which has no fixed point in $U(C)$ satisfies $f_n(s_\nu) \underset{p}{>} s_\nu$.*

PROOF. Since p is an endpoint of X, we can choose s_{ν_0} such that $R_{\nu_0(c)} \subset U(C)$. Then, our statement follows from the fact that $R_{\nu_0(c)}$ has f. p. p.

REMARK 2. Above Lemmas 7 and 8 can be applied to the endpoint s_ν of $K_{\mu(\nu)}$ and $r_{\mu(\nu)} f|_{K_{\mu(\nu)}}: K_{\mu(\nu)} \to K_{\mu(\nu)}$.

LEMMA 9. *Let $f: X \to X$ be a mapping, s_ν an endpoint of $K_{\mu(\nu)}$ $(\ni c)$ and C the component, containing s_ν, of the fixed point set of f. Assume that there exist arbitrarily small open neighborhoods $V(s_\nu)$'s $(V(s_\nu) \not\ni c)$ whose boundary in $K_{\mu(\nu)}$ is* *a single point* $s_{\nu'}$ *which satisfies*

(1) $f(s_{\nu'})=s_{\nu'}$, *and*

(2) C *has an open neighborhood* $U(C)$ *such that for each* n *there exists a mapping* $f_n: X\to X$ *which satisfies*

(i) $|f_n-f|<1/n$,

(ii) f_n *has no fixed point in* $U(C)$, *and*

(iii) $f_n(s_{\nu'})\underset{c}{>}s_{\nu'}$.

Then, the component $C\cap R_{\nu(c)}$ *of the fixed point set of* $r_{\nu(c)}f|_{R_{\nu(c)}}: R_{\nu(c)}\to R_{\nu(c)}$ *is non-essential.*

PROOF. Choose $V(s_\nu)$ such that $V(s_\nu)\subset U(C)$ in the assumption. Then each f_n satisfies $f_n(s_\nu)\underset{c}{>}s_\nu$ because $\overline{R_{\nu'(c)}-R_{\nu(c)}}$ has f. p. p. Hence our conclusion follows immediately.

LEMMA 10. *Let* C_γ *be a non-essential component of the fixed point set of a mapping* $f: X\to X$ *such that* $C_\gamma\cap s_\nu\neq\phi$. *Then, there exist* $K_{m(\nu)}$ *and an open neighborhood* $U_{m(\nu)}$ *of* $C_\gamma\cap K_{m(\nu)}$ *in* $K_{m(\nu)}$ *such that for each n there exists a mapping* $f_n: X\to X$ *which satisfies*

(i) $|f_n-f|<1/n$,

(ii) f_n *has no fixed point in* $U_{m(\nu)}$, *and*

(iii) $f_n(s_\nu)\in K_{m(\nu)}-s_\nu$,

i.e., $C_\gamma\cap K_{m(\nu)}$ *is a non-essential component of the fixed point set of* $r_{m(\nu)}f|_{K_{m(\nu)}}: K_{m(\nu)}\to K_{m(\nu)}$.

PROOF. Since C_γ is non-essential, C_γ has an open neighborhood U such that for each n there exists a mapping $f_n: X\to X$ which satisfies

(i′) $|f_n-f|<1/n$, and

(ii′) f_n has no fixed point in U.

If there exists n such that $f_n(s_\nu)\in K_{\mu(\nu)}\subset U$ for a $K_{\mu(\nu)}$, then by Lemma 4, f_n has a fixed point in U, which contradicts to above condition (ii′). Hence, we are only to consider the case where, for each n, $f_n(s_\nu)$ belongs to some $K_{\mu(\nu)}$ not contained in U. By Lemma 1, the number of $K_{\mu(\nu)}$ not contained in U is finite. Then there exists $K_{m(\nu)}$ which contains $f_n(s_\nu)$ for infinitely many n. Let $U_{m(\nu)}=U\cap K_{m(\nu)}$. Then we have our conclusion.

4. Proof of Main Theorem.

Assume on the contrary that there exists a mapping $f: X\to X$ whose fixed point set has no essential component. Then, each component C_γ of the fixed point set of f has an open set U_γ such that for each n there exists a mapping $f_n: X \to X$ satisfying

(i) $|f_n - f| < 1/n$, and

(ii) f_n has no fixed point in U_γ.

Since the fixed point set of f is compact, we can choose a finite open covering $\{W_i\}$ of this fixed point set such that (1) $W_i \subset U_{\gamma_i}$ and (2) $W_i \cap W_j = \phi$ $(i \neq j)$. Let C_{γ_i} be a component of the fixed point set of f with $C_{\gamma_i} \subset W_i \subset U_{\gamma_i}$.

Let S be the set of all s_ν of X and define the correspondence $F: S \to X$ as follows:

Case 1. s_ν is not fixed by f. In this case, let $F(s_\nu) = f(s_\nu)$.

Case 2. s_ν is fixed by f. Then, by Lemma 10, for the neighborhood W_i containing s_ν, there exists $K_{m(\nu)}$ such that for each n there exists a mapping $f_n: X \to X$ satisfying

(i) $|f_n - f| < 1/n$,

(ii) f_n has no fixed point in W_i, and

(iii) $f_n(s_\nu) \in K_{m(\nu)} - s_\nu$.

Whenever there exists $K_{m(\nu)} \subset R_{\nu(c)}$ with the above conditions, we choose this $K_{m(\nu)}$, and let $F(s_\nu) = k_m$, where k_m is a point of $\mathrm{Int}\, K_{m(\nu)}$.

First, we assume that there exists a non-degenerate X_α such that every $s_\nu \in \mathrm{Bdry}\, X_\alpha$ satisfies $F(s_\nu) \underset{x_\alpha}{\ngeq} s_\nu$. It follows from Lemmas 5 and 6 that $C_\gamma \cap X_\alpha$ is a non-essential component of the fixed point set of $r_\alpha f|_{X_\alpha}: X_\alpha \to X_\alpha$ if $C_\gamma \cap X_\alpha \neq \phi$, which contradicts to our assumption that X_α has f*p. p. Then, we consider the case where for any non-degenerate X_α there exists $s_\nu \in \mathrm{Bdry}\, X_\alpha$ such that $F(s_\nu) \underset{x_\alpha}{>} s_\nu$.

Letting c be the initial point, we construct the ordered set $\langle s_\lambda \rangle$ (λ is a countable ordinal) of s_ν by the following procedure. Let $K_{m(\lambda)}$ be such that $F(s_\lambda) \in K_{m(\lambda)}$.

1. Define the immediate successor of s_λ as follows:

Case 1. s_λ is a boundary point of non-degenerate X_λ contained in $K_{m(\lambda)}$. Then, there exists $s_\nu \in \mathrm{Bdry}\, X_\lambda$ with $F(s_\nu) \underset{x_\lambda}{>} s_\nu$. Let s_ν be the immediate successor of s_λ.

Case 2. s_λ is an endpoint of $K_{m(\lambda)}$.

Case (1). $f(s_\lambda) \underset{c}{>} s_\lambda$. By the continuity of f, there exists s_ν $(\neq s_\lambda)$ in a neighborhood of s_λ in $K_{m(\lambda)}$ such that $f(s_\nu) \underset{s_\lambda}{>} s_\nu$.

Case (2). $f(s_\lambda) \underset{c}{=} s_\lambda$. By Lemma 8, there exists $s_\nu (\neq s_\lambda)$ in a neighborhood of s_λ in $K_{m(\lambda)}$ such that $F(s_\nu) \underset{s_\lambda}{>} s_\nu$.

In the both cases, let each s_ν be the immediate successor of s_λ.

2. When λ converges to ν, let s_ν be the limit point of $\langle s_\lambda\rangle$ if it is not an endpoint of X. We add s_ν to $\langle s_\lambda\rangle$. Note that s_ν is an endpoint of $K_{\mu(\nu)}$ containing c and s_ν. Then, by Lemma 7 or 9, s_ν belongs to a non-essential component of the fixed point set of $r_{\nu(c)}f|_{R_\nu(c)}: R_{\nu(c)}\rightarrow R_{\nu(c)}$. Hence, in this case, s_ν satisfies $F(s_\nu)\underset{c}{>}s_\nu$.

From the construction of this ordered set, it is easy to see that $\langle s_\lambda\rangle$ and $\langle K_{m(\lambda)}\rangle$ satisfy the following conditions:

(1) $F(s_\lambda)\underset{c}{>}s_\lambda$ for every λ,

(2) $K_{m(\lambda)}\supset K_{m'(\lambda')}(\lambda<\lambda')$, and

(3) Either $\langle s_\lambda\rangle$ ends in s_e which is the single boundary point of X_α, or $\langle s_\lambda\rangle$ converges to an endpoint p of X.

Applying the above ordered set, we are going to complete our proof of the Main Theorem.

Case 1. $\langle s_\lambda\rangle$ ends in s_e which is the single boundary point of X_α. From Lemmas 5 and 6 it follows that X_α does not have f*p.p., which contradicts to the assumption of the Main Theorem.

Case 2. $\langle s_\lambda\rangle$ converges to an endpoint p of X. In this case, there exists s_λ in any neighborhood of p such that $F(s_\lambda)\underset{c}{>}s_\lambda$. On the other hand, by our assumtion, p belongs to a non-essential component of the fixed point set of f. Then, by Lemma 8 we have a contradiction. Thus our proof is complete.

EXAMPLE. By letting X_α be a disk with a spiral about its boundary, which is shown to have f*p.p. in [6], we obtain the following example of a not

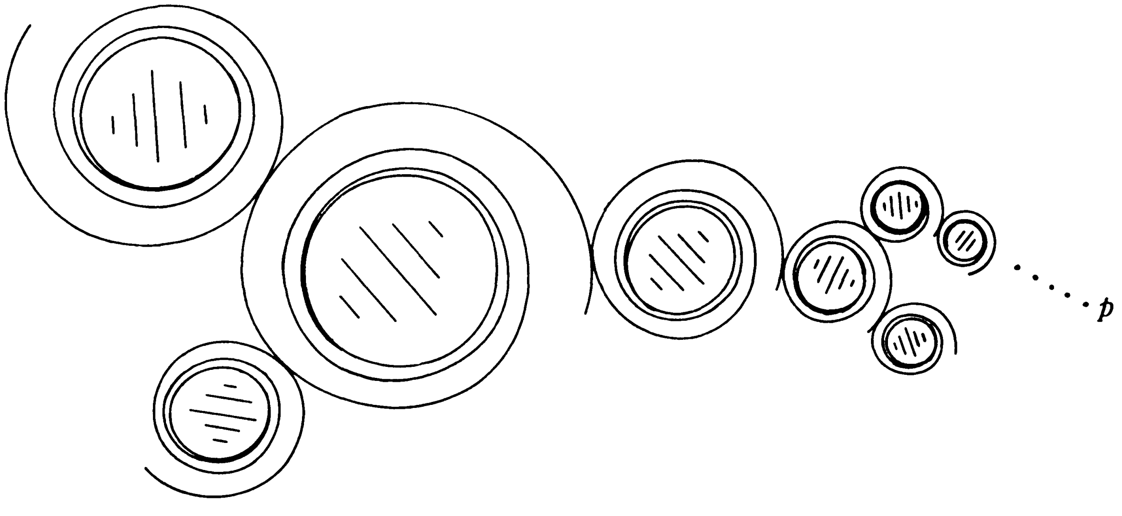


Figure 2.

locally connected continuum with f*p. p. in our Main Theorem.

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