

ON THE SUM OF A PRIME AND A SQUARE

By

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1. Introduction.

In 1923 G. H. Hardy and J. E. Littlewood [3] conjectured that every large integer, not being a square, may be expressed as the sum of a prime and a square. Let $\nu(n)$ be the number of representations of an integer n in this manner. They further stated the hypothetical asymptotic formula; As $n(\neq k^2) \rightarrow \infty$,

$$\nu(n) \sim \mathfrak{S}(n) \frac{\sqrt{n}}{\log n}$$

with

$$\mathfrak{S}(n) = \prod_{p > 2} \left(1 - \frac{(n/p)}{p-1} \right)$$

where $(-)$ is the Legendre symbol.

Define $\mathfrak{S}(k^2) = 0$. In 1968 R. J. Mieh [5] proved that

$$(1) \quad \sum_{n \leq x} \left| \nu(n) - \mathfrak{S}(n) \frac{\sqrt{n}}{\log n} \left(1 + O\left(\frac{\log \log n}{\log n} \right) \right) \right|^2 \ll x^2 (\log x)^{-4}$$

for any $A > 0$, from which it follows that

$$(2) \quad E(x) \ll x (\log x)^{-A}$$

where $E(x)$ denotes the number of integers $n \leq x$ with $\nu(n) = 0$. It seems difficult to sharpen the right hand side of (1). However (2) may be improved, see [1, 9, 12].

A. I. Vinogradov [12; p. 35] remarked that, for any $\varepsilon > 0$,

$$E(x) \ll x^{2/3+\varepsilon}$$

under the extended Riemann hypothesis. First of all we shall show

PROPOSITION. *Assume the extended Riemann hypothesis. Then*

$$(3) \quad E(x) \ll x^{1/2} (\log x)^5.$$

It is the main aim of this paper to prove the following unconditional results.

THEOREM 1. *Let $1/2 < \Theta \leq 1$ and $A > 0$ be given. We have*

$$\sum_{x-x^\theta < n \leq x} \left| \nu(n) - \mathfrak{S}(n) \frac{\sqrt{n}}{\log n} \left(1 + O\left(\frac{\log \log n}{\log n} \right) \right) \right|^2 \ll x^{\Theta+1} (\log x)^{-A}$$

where the O -constant is absolute and the \ll -constant depends on Θ and A only.

THEOREM 2. *Let $7/24 < \theta \leq 1$ and $A > 0$ be given. We have*

$$E(x+x^\theta) - E(x) \ll x^\theta (\log x)^{-A}$$

where the implied constant depends on θ and A only.

Our assertion may be regarded as a refinement of Miech's work (1)(2), and must be compared with a conditional bound (3). Within the frame of Circle method, we appeal to the large sieve [7, 8] and R. C. Vaughan's method [11; Chap. 4] on Weyl sums.

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2. Singular series.

In this section we collect the facts of $\mathfrak{S}(n)$. For the proof, see [1, 5, 9, 12].

For integers q and n , let $\rho(q, n)$ be the number of solutions of the congruence $x^2 \equiv n \pmod{q}$, and ρ_1 be the convolution inverse of ρ with respect to q . Define, for $Q \geq 3$,

$$(2.1) \quad \mathfrak{S}(n, Q) = \sum_{q \leq Q} \frac{\mu(q)}{\varphi(q)} \rho_1(q, n).$$

Then, uniformly for n ,

$$(2.2) \quad \mathfrak{S}(n, Q) \ll \log Q.$$

Let \mathcal{D} be the set of fundamental discriminants. An integer n may be uniquely written as $n = n_1 n_2^2$ with a square-free n_1 . Put

$$\delta(n) = \begin{cases} n_1 & \text{if } n_1 \equiv 1 \pmod{4} \\ 4n_1 & \text{otherwise.} \end{cases}$$

Thus, if $n \neq k^2$ then $\delta(n) \in \mathcal{D}$. For $d \in \mathcal{D}$, the Kronecker symbol (d/\cdot) is a primitive character to modulus d . Let $\mathcal{L} = \mathcal{L}(T)$, $T \geq 3$, denote the set of $d \in \mathcal{D}$ for which $L(s, (d/\cdot))$ the Dirichlet L -function has no zero in the region:

$\text{Re}(s) \geq 29/30$ and $|\text{Im}(s)| \leq T$. Suppose $x \asymp T$. If $\delta(n) \in \mathcal{L}$ then there exists a constant $\eta > 0$ such that

$$(2.3) \quad \mathfrak{S}(n, Q) = \mathfrak{S}(n) + O(Q^{-\eta} \exp(\sqrt{\log x}))$$

uniformly for $n \leq x$. Moreover,

$$(2.4) \quad \#\{d : d \in \mathcal{D} \setminus \mathcal{L}, d \leq 4x\} \ll x^{1/4} (\log x)^{14}.$$

Finally, for $n \neq k^2$,

$$(2.5) \quad \frac{\varphi(n)}{n} \ll \mathfrak{S}(n) L\left(1, \left(\frac{\delta(n)}{\cdot}\right)\right) \ll \frac{n}{\varphi(n)}.$$

3. A conditional estimate.

In this section we illustrate our device with the proof of Proposition. We employ the Circle method [11].

Let x be a large parameter. We divide the unit interval by the Farey dissections of order

$$Q = x^{1/2} (\log x)^2.$$

For $(a, q) = 1$, write

$$I_{q, a} = \left[\frac{a}{q} - \frac{1}{qQ}, \frac{a}{q} + \frac{1}{qQ} \right].$$

Put

$$M = \bigcup_{q \leq P} \bigcup_{\substack{0 < a \leq q \\ (a, q) = 1}} I_{q, a}, \quad P = x/100Q,$$

$$m = [Q^{-1}, 1 + Q^{-1}] \setminus M.$$

We define the exponential sum

$$W(\alpha) = \sum_{m^2 \leq x} e(\alpha m^2)$$

where $e(t) = e^{2\pi i t}$. By Weyl's inequality, we see that

$$(3.1) \quad |W(\alpha)|^2 \ll \left(\frac{x}{q} + x^{1/2} + q \right) \log qx,$$

for $|\alpha - (a/q)| \leq q^{-2}$ with $(a, q) = 1$. When $\alpha \in I_{q, a}$, $W(\alpha)$ is approximated by

$$V(\alpha) = q^{-1} g(a, q) v\left(\alpha - \frac{a}{q}\right)$$

where

$$g(a, q) = \sum_{m(q)} e\left(\frac{a}{q} m^2\right) \quad \text{and} \quad v(\beta) = \sum_{m \leq x} \frac{e(\beta m)}{2\sqrt{m}}.$$

Actually it follows from [11; Theorem 4.1] and [12; p. 38] that

$$(3.2) \quad |W(\alpha) - V(\alpha)|^2 \ll q(\log q)^2$$

for $|\alpha - (a/q)| \leq (4q\sqrt{x})^{-1}$. In addition, we note that

$$(3.3) \quad |g(a, q)|^2 \ll q,$$

$$(3.4) \quad |v(\beta)|^2 \ll \min(x, \|\beta\|^{-1})$$

where $\|t\| = \min_{n \in \mathbb{Z}} |t - n|$.

Put

$$S(\alpha) = \sum_{n \leq x} \Lambda(n) e(\alpha n)$$

where Λ is the von Mangoldt function. It is expected that, for $\alpha \in I_{q,a}$, $S(\alpha)$ is nearly equal to

$$T(\alpha) = \frac{\mu(q)}{\varphi(q)} t\left(\alpha - \frac{a}{q}\right)$$

where

$$(3.5) \quad t(\beta) = \sum_{n \leq x} e(\beta n) \ll \min(x, \|\beta\|^{-1}).$$

In order to show this, define

$$(3.6) \quad J(q) = \sum_{a=1}^{q^*} \int_{I_{q,a}} |S(\alpha) - T(\alpha)|^2 d\alpha$$

where $*$ in \sum_a^* stands for $(a, q) = 1$. If $\alpha \in I_{q,a}$,

$$S(\alpha) - T(\alpha) = \varphi(q)^{-1} \sum_{\chi(q)} \chi(a) \tau(\bar{\chi}) \sum_{n \leq x}^{\#} \chi(n) \Lambda(n) e\left(\left(\alpha - \frac{a}{q}\right)n\right) + O((\log x)^2).$$

Here $\#$ in $\sum_n^{\#}$ means that if χ is principal then $\chi(n)\Lambda(n)$ should be replaced by $\Lambda(n) - 1$. When $q \leq P$, by [2; Lemma 1], we have

$$\begin{aligned} J(q) &\ll \varphi(q)^{-1} \sum_{\chi(q)} |\tau(\bar{\chi})|^2 \int_{|\beta| \leq 1/qQ} \left| \sum_{n \leq x}^{\#} \chi(n) \Lambda(n) e(\beta n) \right|^2 d\beta + Q^{-1}(\log x)^4 \\ &\ll \varphi(q)^{-1} \sum_{\chi(q)} q(qQ)^{-2} \int_{-\infty}^{+\infty} \left| \sum_{y < n \leq \frac{x}{y} + qQ/2}^{\#} \chi(n) \Lambda(n) \right|^2 dy + Q^{-1}(\log x)^4. \end{aligned}$$

On noting $qQ \leq PQ = x/100$, it is easy to show that the above integral is

$$\ll qQx(\log x)^4,$$

under the extended Riemann hypothesis. Therefore, uniformly for $q \leq P$,

$$(3.7) \quad J(q) \ll Q^{-1}x(\log x)^4.$$

Now, for $n \leq x$,

$$(3.8) \quad \sum_{l+m^2=n} \Lambda(n) = \int_{q^{-1}}^{1+q^{-1}} S(\alpha) W(\alpha) e(-n\alpha) d\alpha = \int_M + \int_m.$$

By Bessel's inequality and (3.1),

$$\begin{aligned}
 (3.9) \quad \sum_{n \leq x} \left| \int_{\mathcal{M}} S(\alpha)W(\alpha)e(-n\alpha)d\alpha \right|^2 &\leq \int_{\mathcal{M}} |S(\alpha)W(\alpha)|^2 d\alpha \\
 &\leq \sup_{\alpha \in \mathcal{M}} |W(\alpha)|^2 \int_{\mathcal{M}} |S(\gamma)|^2 d\gamma \\
 &\ll Qx(\log x)^2.
 \end{aligned}$$

On $\alpha \in \mathcal{M}$ we first exchange $S(\alpha)$ for $T(\alpha)$. Thus, by (3.1) and (3.7),

$$\begin{aligned}
 (3.10) \quad \sum_{n \leq x} \left| \int_{\mathcal{M}} (S(\alpha)-T(\alpha))W(\alpha)e(-n\alpha)d\alpha \right|^2 \\
 \leq \sum_{q \leq P} \sum_{a=1}^{q^*} \int_{I_{q,a}} |S(\alpha)-T(\alpha)|^2 |W(\alpha)|^2 d\alpha \\
 \ll \sum_{q \leq P} \frac{x}{q} (\log x) J(q) \\
 \ll x^2 Q^{-1} (\log x)^6.
 \end{aligned}$$

Next we replace $W(\alpha)$ by $V(\alpha)$. On using (3.2) and (3.5),

$$\begin{aligned}
 (3.11) \quad \sum_{n \leq x} \left| \int_{\mathcal{M}} T(\alpha)(W(\alpha)-V(\alpha))e(-n\alpha)d\alpha \right|^2 \\
 \ll \sup_{\alpha \in \mathcal{M}} |W(\alpha)-V(\alpha)|^2 \int_{\mathcal{M}} |T(\gamma)|^2 d\gamma \\
 \ll P(\log P)^2 \sum_{q \leq P} \sum_{a=1}^{q^*} \int_{|\beta| \leq 1/qQ} \frac{\mu^2(q)}{\varphi^2(q)} |t(\beta)|^2 d\beta \\
 \ll Px(\log x)^3.
 \end{aligned}$$

Finally we extend the Farey arc $I_{q,a}$ to $I_{q,a} = [(a/q) - (1/2), (a/q) + (1/2)]$. The resulting remainder is then equal to

$$\begin{aligned}
 (3.12) \quad r_n &= \sum_{q \leq P} \sum_{a=1}^{q^*} \int_{I_{q,a} \setminus I_{q,a}} T(\alpha)V(\alpha)e(-n\alpha)d\alpha \\
 &= \sum_{q \leq P} \sum_{a=1}^{q^*} \int_{1/qQ < |\beta| \leq 1/2} \frac{\mu(q)}{\varphi(q)} t(\beta)q^{-1}g(a,q)v(\beta)e\left(-n\left(\frac{a}{q} + \beta\right)\right) d\beta \\
 &= \int_{1/qQ < |\beta| \leq 1/2} t(\beta)v(\beta)e(-\beta n) \sum_{\substack{q \leq P \\ |\beta|qQ > 1}} \sum_{a=1}^{q^*} \frac{\mu(q)}{q\varphi(q)} g(a,q)e\left(-\frac{a}{q}n\right) d\beta.
 \end{aligned}$$

On using Cauchy's inequality and (3.4), we have

$$\begin{aligned}
 \sum_{n \leq x} |r_n|^2 &\leq \sum_{n \leq x} \int_{|\gamma| \leq 1/2} |v(\gamma)|^2 d\gamma \int_{1/qQ < |\beta| \leq 1/2} |t(\beta)|^2 |\sum_{\beta} |^2 d\beta \\
 &\ll (\log x) \int_{1/qQ < |\beta| \leq 1/2} |t(\beta)|^2 \left(\sum_{n \leq x} \left| \sum_{\substack{q \leq P \\ |\beta|qQ > 1}} \sum_{a=1}^{q^*} \frac{\mu(q)}{q\varphi(q)} g(a,q)e\left(-\frac{a}{q}n\right) \right|^2 \right) d\beta.
 \end{aligned}$$

The large sieve inequality [7, 8] yields that

$$\begin{aligned}
 \sum_{n \leq x} |r_n|^2 &\ll (\log x) \int_{1/Q < |\beta| \leq 1/2} |t(\beta)|^2 \left(\sum_{\substack{q \leq P \\ |\beta|qQ > 1}}^{q^*} \sum_{a=1}^{q^*} (x+qP) \left| \frac{\mu(q)}{q\varphi(q)} g(a, q) \right|^2 \right) d\beta \\
 &\ll (\log x) \sum_{q \leq P} (x+qP) \frac{\mu^2(q)}{q\varphi(q)} \int_{1/Q < |\beta| \leq 1/2} |t(\beta)|^2 d\beta \\
 &\ll (\log x)(x+P^2)Q \sum_{q \leq P} \frac{\mu^2(q)}{\varphi(q)} \\
 (3.13) \quad &\ll Qx(\log x)^2.
 \end{aligned}$$

Here we used the bounds (3.3) and (3.5).

It remains to calculate

$$\begin{aligned}
 (3.14) \quad &\sum_{q \leq P} \sum_{a=1}^{q^*} \int_{I_{q,a}} T(\alpha) V(\alpha) e(-n\alpha) d\alpha \\
 &= \sum_{q \leq P} \sum_{a=1}^{q^*} \frac{\mu(q)}{q\varphi(q)} g(a, q) e\left(-\frac{a}{q}n\right) \int_{-1/2}^{+1/2} \sum_{l, m \leq x} \frac{e(\beta(l+m-n))}{2\sqrt{m}} d\beta.
 \end{aligned}$$

The above sum is $\mathfrak{S}(n, P)$ with the definition (2.1). The integral is equal to

$$\sum_{l < n} \frac{1}{2\sqrt{n-l}} = \sqrt{n} + O(1).$$

Hence, by (2.2), (3.14) becomes

$$(3.15) \quad \mathfrak{S}(n, P) \sqrt{n} + O(\log x).$$

On summing up the above argument (3.9)-(3.15), we obtain

$$\begin{aligned}
 \sum_{n \leq x} \left| \sum_{p+m^2=n} \log p - \mathfrak{S}(n, P) \sqrt{n} \right|^2 &\ll Qx(\log x)^2 + x^2 Q^{-1}(\log x)^6 + Px(\log x)^3 \\
 (3.16) \quad &\ll x^{3/2}(\log x)^4.
 \end{aligned}$$

Now, the extended Riemann hypothesis implies that $\mathcal{L} = \mathcal{D}$ the sets introduced in section 2, and that $\mathfrak{S}(n) \gg (\log \log n)^{-2}$ for $n \neq k^2$ and $n \gg 1$. By (2.3) we then have that, for $n \neq k^2$ ($\gg 1$),

$$\mathfrak{S}(n, P) \gg (\log \log n)^{-2}.$$

Consequently (3.16) leads that

$$\begin{aligned}
 x^{3/2}(\log x)^4 &\gg \sum_{n \leq x} \left| \sum_{p+m^2=n} \log p - \mathfrak{S}(n, P) \sqrt{n} \right|^2 \\
 &\geq \sum_{\substack{x/2 < n \leq x \\ \nu(n) \neq 0 \\ n \neq k^2}} |\mathfrak{S}(n, P)|^2 n \\
 &\gg x(\log \log x)^{-4} \sum_{\substack{x/2 < n \leq x \\ \nu(n) \neq 0 \\ n \neq k^2}} 1
 \end{aligned}$$

or

$$E(x) \leq \sum_{\substack{n \leq x \\ \nu(n) = 0 \\ n \neq k^2}} 1 + \sum_{k^2 \leq x} 1 \ll \sum_t \left(\frac{x}{2^t}\right)^{1/2} (\log x \log \log x)^4 + x^{1/2} \\ \ll x^{1/2} (\log x)^5,$$

as required.

4. Proof of Theorems.

In this section we derive Theorems from the known results mentioned in section 2 and our main lemma below. Lemma will be verified in the next section.

LEMMA. Let x be a large parameter. For given $1/2 < \theta \leq 3/4$ and $7/12 < \varepsilon \leq 1$, put $\Delta = y^\theta$ and $y = x^\varepsilon$. Write

$$\nu(n, y) = \#\{(p, m) : x - y < p \leq x, m^2 \leq y, p + m^2 = n\},$$

and

$$K(n, y) = \int_1^{n-(x-y)-3} \frac{dt}{2\sqrt{t} \log(n-t)}.$$

Then, for any $A > 0$, we have

$$\sum_{x-A < n \leq x} |\nu(n, y) - \mathfrak{S}(n, \sqrt{y})K(n, y)|^2 \ll \Delta y (\log x)^{-A}$$

where the implied constant depends on θ, ε and A only.

Proof of Theorem 1. Let $\mathcal{L} = \mathcal{L}(x)$ in section 2. Choose $\varepsilon = 1$ in Lemma. Then, because of $\nu(n, x) = \nu(n)$,

$$\sum_{x-x^\theta < n \leq x} |\nu(n) - \mathfrak{S}(n, \sqrt{x})K(n, x)|^2 \ll x^{\theta+1} (\log x)^{-A}$$

for any $A > 0$. We note that

$$K(n, x) = \frac{\sqrt{n}}{\log n} \left(1 + O\left(\frac{\log \log n}{\log n}\right)\right)$$

with an absolute O -constant. Combining the above with (2.3) and (2.5) we have

$$S = \sum_{x-x^\theta < n \leq x} |\nu(n) - \mathfrak{S}(n)K(n, x)|^2 \\ = \sum_{\substack{\delta(n) \in \mathcal{L} \\ n \neq k^2}} + \sum_{\substack{\delta(n) \notin \mathcal{L} \\ n \neq k^2}} + \sum_{n=k^2} \\ \ll \sum_{\substack{x-x^\theta < n \leq x \\ \delta(n) \in \mathcal{L}}} |\nu(n) - \mathfrak{S}(n, \sqrt{x})K(n, x)|^2 + \sum_{\substack{x-x^\theta < n \leq x \\ \delta(n) \notin \mathcal{L}}} |\mathfrak{S}(n) - \mathfrak{S}(n, \sqrt{x})|^2 \frac{n}{(\log n)^2} \\ + \sum_{\substack{x-x^\theta < n \leq x \\ \delta(n) \notin \mathcal{L} \\ n \neq k^2}} \left(\nu(n)^2 + \left(\frac{\mathfrak{S}(n)}{\log n}\right)^2 n\right) + \sum_{x-x^\theta < k^2 \leq x} \nu(k^2)^2$$

$$\begin{aligned} &\ll_{x-x^\theta < n \leq x} |\nu(n) - \mathfrak{S}(n, \sqrt{x})K(n, x)|^2 + x^{\theta+1} \sup_{\substack{n \leq x \\ \delta(n) \in \mathcal{L}}} |\mathfrak{S}(n) - \mathfrak{S}(n, \sqrt{x})|^2 \\ &\quad + x \left(1 + \sup_{\substack{n \leq x \\ \delta(n) \notin \mathcal{L} \\ n \neq k^2}} \left(\frac{\mathfrak{S}(n)}{\log n} \right)^2 \right) \sum_{\substack{d \leq 4x \\ d \in \mathcal{D} \setminus \mathcal{L} \\ \text{or } d=1}} \sum_{\substack{x-x^\theta < n \leq x \\ \delta(n)=d}} 1 \\ &\ll x^{\theta+1} (\log x)^{-A} + x \left(1 + \sup_{\substack{d \leq 4x \\ d \in \mathcal{D}}} \left| L\left(1, \left(\frac{d}{\cdot}\right)\right) \right|^{-2} \right) \left(1 + \sum_{\substack{d \leq 4x \\ d \in \mathcal{D} \setminus \mathcal{L}}} 1 \right) x^{\theta-1/2}. \end{aligned}$$

By (2.4) and Siegel’s theorem [10; Kap. IV, § 8], S becomes

$$\begin{aligned} &\ll x^{\theta+1} (\log x)^{-A} + x^{\theta+1/2} (x^\varepsilon) (1 + x^{1/4} (\log x)^{14}) \\ &\ll x^{\theta+1} (\log x)^{-A}. \end{aligned}$$

Hence we obtain Theorem 1 in case $1/2 < \theta \leq 3/4$. If $3/4 < \theta \leq 1$, Theorem 1 follows from the case of $\theta = 2/3$, by splitting up the interval $(x - x^\theta, x]$ into the sum of smaller intervals of type $(u - u^{2/3}, u]$.

Proof of Theorem 2. Put $\theta = \theta E$ in Lemma. Then, $7/24 < \theta \leq 3/4$. It is sufficient to prove Theorem 2 for θ in the above range only. Since $\nu(n) = 0$ implies $\nu(n, y) = 0$, Lemma yields that

$$\sum_{\substack{x-x^\theta < n \leq x \\ \nu(n)=0}} |\mathfrak{S}(n, \sqrt{y})K(n, y)|^2 \ll y x^\theta (\log x)^{-A-5}.$$

Here, $K(n, y)^2 \asymp y (\log x)^{-2}$. Thus,

$$\sum_{\substack{x-x^\theta < n \leq x \\ \nu(n)=0 \\ \delta(n) \in \mathcal{L}}} 1 \ll x^\theta (\log x)^{-A-3} (\log x \log x)^2$$

by (2.3) and (2.5) with $\mathcal{L} = \mathcal{L}(x)$. Hence, by (2.4), we obtain

$$\begin{aligned} E(x) - E(x - x^\theta) &\leq \sum_{\substack{x-x^\theta < n \leq x \\ \nu(n)=0 \\ \delta(n) \in \mathcal{L}}} 1 + \sum_{\substack{x-x^\theta < n \leq x \\ \delta(n) \notin \mathcal{L}}} 1 \\ &\ll x^\theta (\log x)^{-A} + \sum_{\substack{d \leq 4x \\ d \in \mathcal{D} \setminus \mathcal{L} \\ \text{or } d=1}} (x^{\theta-1/2} + 1) \\ &\ll x^\theta (\log x)^{-A} + x^{1/4} (\log x)^{14} \\ &\ll x^\theta (\log x)^{-A}, \end{aligned}$$

as required.

5. Proof of Lemma.

Put

$$S(\alpha) = \sum_{x-y < p \leq x} e(\alpha p).$$

And we define the exponential sums $W(\alpha)$ and $V(\alpha)$ by the similar way in section 3, except for changing the parameter x in section 3 by y . The Farey arcs are determined as follows:

$$\begin{aligned}
 Q &= y^{1/2}(\log x), \\
 M &= \bigcup_{q \leq P} \bigcup_{\substack{0 < a \leq q \\ (a, q) = 1}} I_{q, a}, \quad L = \bigcup_{q \leq P} \bigcup_{\substack{0 < a \leq q \\ (a, q) = 1}} I_{q, a}, \quad P = (\log x)^4 \\
 m &= \bigcup_{P < q \leq R} \bigcup_{\substack{0 < a < q \\ (a, q) = 1}} I_{q, a}, \quad R = y/Q \\
 n &= [Q^{-1}, 1 + Q^{-1}] \setminus (M \cup m).
 \end{aligned}$$

Here $I_{q, a}$ and $I_{q, a}$ are similar to that in section 3. We then have

$$\begin{aligned}
 \nu(n, y) &= \int_{Q^{-1}}^{1+Q^{-1}} S(\alpha)W(\alpha)e(-n\alpha)d\alpha \\
 &= \int_L SV - \int_{L \setminus M} SV + \int_m SV + \int_{M \cup m} S(W - V) + \int_n SW \\
 &= J_1 - J_2 + J_3 + J_4 + J_5, \quad \text{say.}
 \end{aligned}$$

First we evaluate J_1 . An elementary calculation leads that

$$\begin{aligned}
 J_1(n) &= \int_L S(\alpha)V(\alpha)e(-n\alpha)d\alpha \\
 &= \sum_{q \leq P} \sum_{a=1}^{q^*} \int_{-1/2}^{+1/2} S\left(\frac{a}{q} + \beta\right)q^{-1}g(a, q)v(\beta)e\left(-n\left(\frac{a}{q} + \beta\right)\right)d\beta \\
 &= \sum_{q \leq P} \sum_{a=1}^{q^*} q^{-1}g(a, q)e\left(-\frac{a}{q}n\right) \sum_{\substack{x-y < p \leq x \\ m \leq y}} \sum_{n \leq y} \frac{e((a/q)p)}{2\sqrt{m}} \int_{-1/2}^{+1/2} e(\beta(p+m-n))d\beta \\
 &= \sum_{q \leq P} \sum_{a=1}^{q^*} q^{-1}g(a, q)e\left(-\frac{a}{q}n\right) \sum_{x-y < p < n} \frac{e((a/q)p)}{2\sqrt{n-p}} \\
 &= \sum_{q \leq P} q^{-1} \sum_{\substack{x-y < p < n \\ m \leq y}} \frac{c_q(p+m^2-n)}{2\sqrt{n-p}} \\
 (5.2) \quad &= \sum_{q \leq P} q^{-1} \sum_{m(q)} \sum_{\substack{d|q \\ (d, m^2-n)=1}} \mu\left(\frac{q}{d}\right)d \sum_{\substack{x-y < p < n \\ p \equiv n-m^2(d)}} \frac{1}{2\sqrt{n-p}} + O\left(\sum_{q \leq P} \tau(q)q\right).
 \end{aligned}$$

On using partial summation, the innermost sum is equal to

$$(5.3) \quad \frac{K(n, y)}{\varphi(d)} + O\left(1 + \sup_{(b, d)=1} \sup_{1 \leq t \leq y} t^{-1/2} \left| \sum_{\substack{n-t \leq p < n \\ p \equiv b(d)}} \log p - \frac{t}{\varphi(d)} \right| \right).$$

We now appeal to the well known result on primes in arithmetical progressions [10; Kap. IX. § 3]. It follows from [10; Kap. VIII. Satz 6.2, Kap. IV. Satz 8.1] zero free region and [4, 6; Theorem 12.1] zero density estimates for the Dirichlet L -functions that, for given positive constants ε, E and F ,

$$\sum_{\substack{X < p \leq X+Y \\ p \equiv l(k)}} \log p = \frac{Y}{\varphi(k)} + O(Y(\log X)^{-\varepsilon})$$

uniformly for $(l, k)=1, k \leq (\log X)^F$ and $X^{7/12+\varepsilon} \leq Y \leq X$. Hence the O -term in (5.3) is at most

$$y^{1/2}(\log x)^{-3A-1},$$

and contributes to (5.2)

$$\begin{aligned} &\ll y^{1/2}(\log x)^{-3A-1} \sum_{q \leq P} \tau(q)q \\ &\ll y^{1/2}(\log x)^{-3A-1} P^2(\log P) \\ &\ll y^{1/2} P^{-1}. \end{aligned}$$

On combining this with (5.2) and (5.3) we have

$$(5.4) \quad J_1(n) = K(n, y) \sum_{q \leq P} q^{-1} \sum_{m(q)} \sum_{\substack{d|q \\ (d, m^2-n)=1}} \mu\left(\frac{q}{d}\right) \frac{d}{\varphi(d)} + O(y^{1/2} P^{-1}).$$

Notice that the above sum is

$$(5.5) \quad \sum_{q \leq P} \frac{\mu(q)}{q\varphi(q)} \sum_{m(q)} c_q(m^2-n) = \mathfrak{S}(n, P) = \sum_{q \leq P} \sum_{a=1}^{q^*} \frac{\mu(q)}{q\varphi(q)} g(a, q) e\left(-\frac{a}{q}n\right).$$

We widen the range of q up to \sqrt{y} . Let $J_{11}(n)$ be the resulting cost. On employing the large sieve inequality [7, 8] and (3.3),

$$\begin{aligned} (5.6) \quad \sum_{x-J < n \leq x} |J_{11}(n)|^2 &\ll K(n, y)^2 \sum_{x-J < n \leq x} \left| \sum_{P < q \leq \sqrt{y}} \sum_{a=1}^{q^*} \frac{\mu(q)}{q\varphi(q)} g(a, n) e\left(-\frac{a}{q}n\right) \right|^2 \\ &\ll y(\log x)^{-2} \sum_{P < q \leq \sqrt{y}} \sum_{a=1}^{q^*} (\Delta + q\sqrt{y}) \left| \frac{\mu(q)}{q\varphi(q)} g(a, q) \right|^2 \\ &\ll y(\log x)^{-2} \left(\frac{\Delta}{P} + \sqrt{y}\right) \sum_{q \leq \sqrt{y}} \frac{\mu^2(q)}{\varphi(q)} \\ &\ll y(\Delta P^{-1} + \sqrt{y}). \end{aligned}$$

In conjunction with (5.4), (5.5) and (5.6) we obtain

$$(5.7) \quad \sum_{x-J < n \leq x} |J_1(n) - \mathfrak{S}(n, \sqrt{y})K(n, y)|^2 \ll \Delta y P^{-1} + y^{3/2}.$$

We proceed to J_2 . On using Cauchy's inequality and (3.3),

$$\begin{aligned} J_2(n) &= \int_{L \setminus M} S(\alpha) V(\alpha) e(-n\alpha) d\alpha \\ &= \sum_{q \leq P} \sum_{a=1}^{q^*} q^{-1} g(a, q) e\left(-\frac{a}{q}n\right) \int_{1/qQ < \beta \leq 1/2} S\left(\frac{a}{q} + \beta\right) v(\beta) e(-n\beta) d\beta \\ &\ll P \left(\sum_{q \leq P} q^{-1} \sum_{a=1}^{q^*} \left| \int_{1/qQ < \beta \leq 1/2} S\left(\frac{a}{q} + \beta\right) v(\beta) e(-n\beta) d\beta \right|^2 \right)^{1/2}. \end{aligned}$$

By Bessel's inequality and (3.4), we have

$$\begin{aligned}
 \sum_{x-\Delta < n \leq x} |J_2(n)|^2 &\ll P^2 \sum_{q \leq P} q^{-1} \sum_{a=1}^{q^*} \int_{1/qQ < |\beta| \leq 1/2} \left| S\left(\frac{a}{q} + \beta\right) v(\beta) \right|^2 d\beta \\
 &\ll P^2 Q \sum_{q \leq P} \sum_{a=1}^{q^*} \int_{|\beta| \leq 1/2} \left| S\left(\frac{a}{q} + \beta\right) \right|^2 d\beta \\
 (5.8) \qquad &\ll P^4 Q y(\log x)^{-1}.
 \end{aligned}$$

Next we consider J_3 . Changing the order of summation and integration, we use Cauchy's inequality and (3.4). Thus,

$$\begin{aligned}
 J_3(n) &= \int_m S(\alpha) V(\alpha) e(-n\alpha) d\alpha \\
 &= \sum_{P < q \leq R} \sum_{a=1}^{q^*} \int_{|\beta| \leq 1/qQ} S\left(\frac{a}{q} + \beta\right) q^{-1} g(a, q) v(\beta) e\left(-n\left(\frac{a}{q} + \beta\right)\right) d\beta \\
 &= \int_{|\beta|, P Q \leq 1} v(\beta) e(-n\beta) \sum_{\substack{P < q \leq R \\ |\beta|, q Q \leq 1}} \sum_{a=1}^{q^*} q^{-1} g(a, q) S\left(\frac{a}{q} + \beta\right) e\left(-\frac{a}{q} n\right) d\beta
 \end{aligned}$$

or

$$\begin{aligned}
 \sum_{x-\Delta < n \leq x} |J_3(n)|^2 \\
 \ll (\log x) \int_{|\beta|, P Q \leq 1} \sum_{x-\Delta < n \leq x} \left| \sum_{\substack{P < q \leq R \\ |\beta|, q Q \leq 1}} \sum_{a=1}^{q^*} q^{-1} g(a, q) S\left(\frac{a}{q} + \beta\right) e\left(-\frac{a}{q} n\right) \right|^2 d\beta.
 \end{aligned}$$

The large sieve [7, 8] yields that

$$\begin{aligned}
 \sum_{x-\Delta < n \leq x} |J_3(n)|^2 &\ll (\log x) \int_{|\beta|, P Q \leq 1} \sum_{\substack{P < q \leq R \\ |\beta|, q Q \leq 1}} \sum_{a=1}^{q^*} (\Delta + qR) \left| q^{-1} g(a, q) S\left(\frac{a}{q} + \beta\right) \right|^2 d\beta \\
 &\ll (\log x) \sum_{P < q \leq R} \sum_{a=1}^{q^*} \int_{|\beta| \leq 1/qQ} \left(\frac{\Delta}{q} + R\right) \left| S\left(\frac{a}{q} + \beta\right) \right|^2 d\beta \\
 &\ll (\log x) \left(\frac{\Delta}{P} + R\right) \int_m |S(\alpha)|^2 d\alpha \\
 (5.9) \qquad &\ll (\Delta P^{-1} + R) y.
 \end{aligned}$$

We turn to J_4 .

$$\begin{aligned}
 \sum_{x-\Delta < n \leq x} |J_4(n)|^2 &= \sum_{x-\Delta < n \leq x} \left| \int_{M \cup m} S(\alpha) (W(\alpha) - V(\alpha)) e(-n\alpha) d\alpha \right|^2 \\
 &\leq \int_{M \cup m} |S(\alpha)|^2 |W(\alpha) - V(\alpha)|^2 d\alpha \\
 &\ll R(\log x)^2 \int_{M \cup m} |S(\alpha)|^2 d\alpha \\
 (5.10) \qquad &\ll y^{3/2},
 \end{aligned}$$

by Bessel's inequality and (3.2). Similarly, by (3.1),

$$\begin{aligned}
\sum_{x-d < n \leq x} |J_5(n)|^2 &= \sum_{x-d < n \leq x} \left| \int_n S(\alpha) W(\alpha) e(-n\alpha) d\alpha \right|^2 \\
&\leq \int_n |S(\alpha) W(\alpha)|^2 d\alpha \\
&\ll \left(\frac{y}{R} + Q \right) (\log x) \int_n |S(\alpha)|^2 d\alpha \\
(5.11) \quad &\ll y^{3/2} (\log x).
\end{aligned}$$

In conjunction with (5.7)-(5.11) and (5.1), we have that

$$\begin{aligned}
\sum_{x-d < n \leq x} |\nu(n, y) - \mathfrak{S}(n, \sqrt{y}) K(n, y)|^2 &\ll \Delta y P^{-1} + P^4 Q y (\log x)^{-1} + R y + y^{3/2} (\log x) \\
&\ll \Delta y P^{-1} + P^4 y^{3/2} \\
&\ll \Delta y (\log x)^{-4},
\end{aligned}$$

as required.

This completes our proof.

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