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GRADED COALGEBRAS

By

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1. Introduction

The notion of graded coalgebra does not appear very frequently in the literature of coalgebras, neither in papers nor in books (e.g. the well known references [1], [10]). In the mentioned books one can find considered the case when the group is $\mathbf{Z}$ and the components of negative degree are zero. The aim of this paper is to investigate the general concept of graded coalgebra over arbitrary groups and expound their more important properties. A very remarkable point of this paper is the study of the so-called "strongly graded coalgebras" (see § 5). The principal ideas, that we use to obtain the main results of this paper, come from the theory of graded rings (see [7]); these become useful by a clever interpretation of the codual methods. We finally remark that a graded coalgebra is a comodule coalgebra over the Hopf algebra $k[G]$, but we will not apply this idea here.

After the introduction and a section where we fix the notation and preliminaries, we give the general properties of graded coalgebras. Proposition 3.1 is the main tool in the computations of the rest of the paper. We show that the functor $U: gr^C \rightarrow \boldsymbol{M}^C$ admits a right adjoint functor F; moreover if the group G is finite, then F is also a left adjoint functor.

In Section 4, using the cotensor product introduced by Takeuchi in [11] we define the induced functor and we study the more important properties of this functor.

In Section 5 we consider the strongly graded coalgebras. We obtain some nice results about the relation between the categories $\boldsymbol{M}^{C_1}$ and gr^C.

In the next section, we associate a graded ring R to any graded coalgebra C and we study the conexion between the categories $R-gr$ and gr^C. We conclude the paper with examples of graded coalgebras, some of them are very well-known.

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2. Notation and Preliminaries

Let k be a field. A coalgebra over k is a k-space C together with two k-linear maps $\Delta: C \to C \otimes C$ (the unadorned tensor product is understood to be over k) and $\varepsilon: C \to k$ such that $(1 \otimes \Delta)\Delta = (\Delta \otimes 1)\Delta$ and $(1 \otimes \varepsilon)\Delta = (\varepsilon \otimes 1)\Delta = 1$. We shall use the so-called "sigma notation" (see Sweedler's book [10] or Abe's book [1]), that is $\Delta(c) = \sum_{(c)} c_1 \otimes c_2$ if $c \in C$.

If C is a coalgebra, a right C-comodule is a k-space M with a k-map $\rho_M: M \to M \otimes C$ such that $(\rho_M \otimes 1)\rho_M = (1 \otimes \Delta)\rho_M$ and $(1 \otimes \varepsilon)\rho_M = 1$. We also use the sigma notation for C-comodules, i. e. $\rho(m) = \sum_{(m)} m_0 \otimes m_1$, $m_0 \in M$, $m_1 \in C$. If M and N are C-comodules, a comodule map from M to N is a k-map $f: M \to N$ such that $(f \otimes 1)\rho_M = \rho_N f$. The k-space of all comodule maps from M to N is denoted by $Com_C(M, N)$ and $\boldsymbol{M}^C$ denote the category of right C-comodules. In the same way we can construct the category of left C-comodules ${}^C\boldsymbol{M}$.

It is well known that $\boldsymbol{M}^C$ is an abelian category (see [10] [1], etc.). In fact $\boldsymbol{M}^C$ is a Grothendieck category.

3. Graded coalgebras and graded comodules

Let G be a group with $1 \in G$ the identity element of G. Let C be a coalgebra. C is called G-graded coalgebra if C admits a decomposition as a direct sum of k-spaces $C = \bigoplus_{\sigma \in G} C_\sigma$ such that

i) $\Delta(C_\sigma) \subseteq \sum_{\lambda\mu=\sigma} C_\lambda \otimes C_\mu$ for any $\sigma \in G$;

ii) $\varepsilon(C_\sigma) = 0$ for any $\sigma \neq 1$.

If M is a right C-comodule then M is called G-graded comodule over C if M admits a decomposition as a direct sum of k-spaces $M = \bigoplus_{\sigma \in G} M_\sigma$ such that $\rho_M(M_\sigma) \subseteq \sum_{\lambda\mu=\sigma} M_\lambda \otimes C_\mu$ for any $\sigma \in G$. For any element $m \in M$ we have the decomposition $m = \sum_{\sigma \in G} m_\sigma$, $m_\sigma \in M_\sigma$ (the sum has only a finite number of nonzero elements). The nonzero elements m_σ, $\sigma \in G$ are called the homogeneous component of m; m_σ is the homogeneous element of degree σ and we write $\deg(m_\sigma) = \sigma$.

Associated to any G-graded coalgebra $C = \bigoplus_{\sigma \in G} C_\sigma$ we have the category gr^C of all right graded C-comodules. In this category if $M = \bigoplus_{\sigma \in G} M_\sigma$ and $N = \bigoplus_{\sigma \in G} N_\sigma$ are two objects, then the morphism from M to N is the set

$$Com_{gr^C}(M, N) = \{f \in Com_C(M, N) \mid f(M_\sigma) \subseteq N_\sigma \text{ for all } \sigma \in G\}.$$

It is easy to verify that gr^C is an abelian category. (In fact gr^C is also a

Grothendieck category (see § 4)). Analogously we can define ${}^{C}gr$ the category of all left G-graded C-comodules.

If $M=\bigoplus_{\sigma\in G}M_\sigma$ is a graded right C-comodule for any $\sigma\in G$ we denote by $\pi_\sigma^M : M\rightarrow M_\sigma$ the canonical projection. The following result is fundamental in the study of graded coalgebras and graded comodules.

Proposition 3.1. *Let $M=\bigoplus_{\sigma\in G}M_\sigma$ a right G-graded C-comodule. If $\sigma, \tau\in G$ there exists a unique k-morphism $u^M_{\sigma,\tau}: M_{\sigma\tau}\rightarrow M_\sigma\otimes C_\tau$ such that the following diagram is conmutative*

$$\begin{array}{ccc} M & \xrightarrow{\rho_M} & M\otimes C \\ {\scriptstyle \pi^M_{\sigma\tau}}\downarrow & & \downarrow{\scriptstyle \pi^M_\sigma\otimes\pi^C_\tau} \\ M_{\sigma\tau} & \xrightarrow[u^M_{\sigma\tau}]{} & M_\sigma\otimes C_\tau \end{array}$$

Moreover the morphisms $u^M_{\sigma,\tau}$ have the following properties:

1) *For any $\sigma, \tau, \lambda\in G$ the diagram*

$$\begin{array}{ccccc} & & M_\sigma\otimes C_{\tau\lambda} & & \\ & {\scriptstyle u^M_{\sigma,\tau\lambda}}\nearrow & & \searrow{\scriptstyle 1\otimes u^C_{\tau,\lambda}} & \\ M_{\sigma\tau\lambda} & & & & M_\sigma\otimes C_\tau\otimes C_\lambda \\ & {\scriptstyle u^M_{\sigma\tau,\lambda}}\searrow & & \nearrow{\scriptstyle u^M_{\sigma,\tau}\otimes 1} & \\ & & M_{\sigma\tau}\otimes C_\lambda & & \end{array}$$

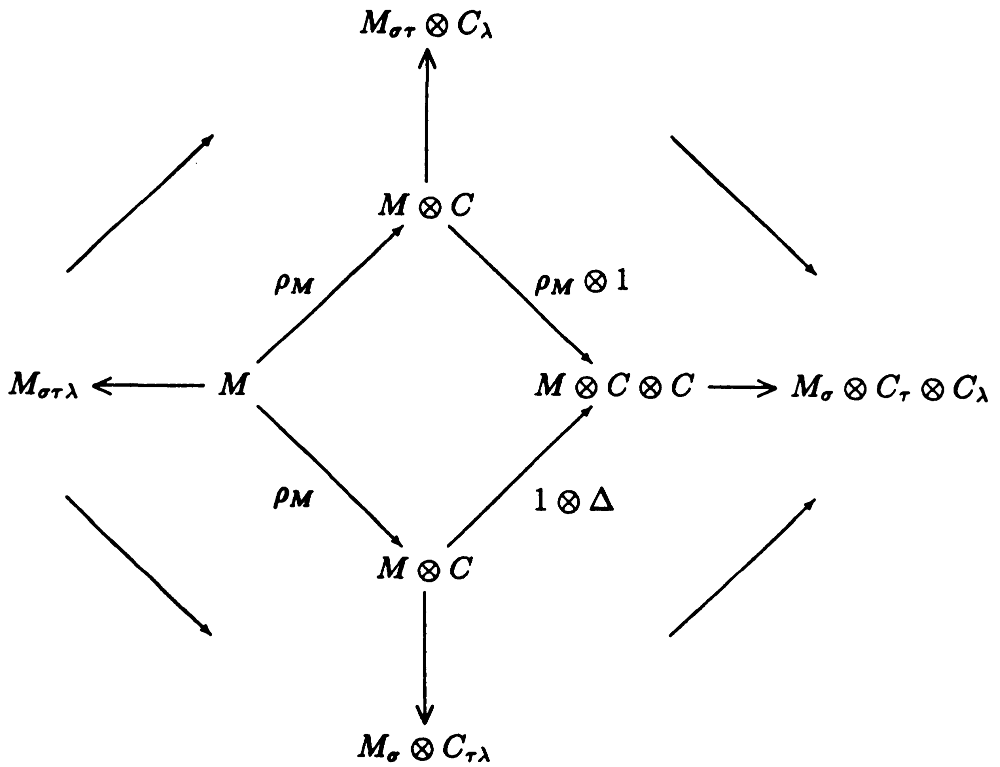
is commutative.

2) *If $\sigma\in G$ we have the commutative diagram*

$$\begin{array}{ccc} M_\sigma & \xrightarrow{u^M_{\sigma,1}} & M_\sigma\otimes C_1 \\ & {\scriptstyle 1}\searrow & \downarrow{\scriptstyle 1\otimes\epsilon} \\ & & M_\sigma \end{array}$$

PROOF. If $i^M_{\sigma\tau}: M_{\sigma\tau}\to M$ is the inclusion map then we define $u^M_{\sigma,\tau}=(\pi^M_\sigma\otimes\pi^C_\tau)\rho_M i^M_{\sigma\tau}$. If $m\in M$ we can write $m=m_{\sigma\tau}+\sum_{\lambda\neq\sigma\tau} m_\lambda$ where $m_{\sigma\tau}\in M_{\sigma\tau}$ and $m_\lambda\in M_\lambda$. Since $\pi^M_{\sigma\tau}(m)=m_{\sigma\tau}$ then we can show that for any $\lambda\neq\sigma\tau$ we have $((\pi^M_\sigma\otimes\pi^C_\tau)\rho_M)(m_\lambda)=0$. Indeed, since $\rho_M(m_\lambda)\in\sum_{xy=\lambda} M_x\otimes C_y$ then from $xy\neq\sigma\tau$ result $x\neq\sigma$ or $y\neq\tau$ and therefore $(\pi^M_\sigma\otimes\pi^C_\tau)(M_x\otimes C_y)=0$. Hence $((\pi^M_\sigma\otimes\pi^C_\tau)\rho_M(m_\lambda)$ $=0$ for any $\lambda\neq\sigma\tau$. Now, since $\pi^M_{\sigma\tau}$ is surjective, then $u^M_{\sigma\tau}$ is unique.

Since M is a right C-comodule, we have the commutative diagram



Now, if $m\in M_{\sigma\tau\lambda}$ we can write $\rho_M(m)=\sum m_0\otimes m_1+\sum m'_0\otimes m'_1$ where m_0, m_1, m'_0, m'_1 are homogeneous with $\deg(m_0)=\sigma\tau$, and $\deg(m_1)=\lambda$. Thus, by the definition of $u^M_{\sigma\tau,\lambda}$, we have that $u^M_{\sigma\tau,\lambda}(m)=\sum m_0\otimes m_1$. If we denote $((\rho_M\otimes 1)\rho_M)(m)$ $=((1\otimes\Delta)\rho_M)(m)=\sum m_0\otimes m_1\otimes m_2$ (sigma notation) then it is easily seen that $((u^M_{\sigma,\tau}\otimes 1)u^M_{\sigma\tau,\lambda})(m)=\sum m_0\otimes m_1\otimes m_2$ where $\deg(m_0)=\sigma$, $\deg(m_1)=\tau$, $\deg(m_2)=\lambda$. On the other hand, since $((1\otimes\Delta)\rho_M)(m)=\sum m_0\otimes m_1\otimes m_2$, then $((1\otimes u^C_{\tau,\lambda})u^M_{\sigma,\tau\lambda})(m)$ $=\sum m_0\otimes m_1\otimes m_2$ where $\deg(m_0)=\sigma$, $\deg(m_1)=\tau$, $\deg(m_2)=\lambda$. Hence we have the equality $(u^M_{\sigma,\tau}\otimes 1)u^M_{\sigma\tau,\lambda}=(1\otimes u^C_{\tau,\lambda})u^M_{\sigma,\tau\lambda}$.

The commutativity of the diagram in the assertion 2) follows from the fact that $(1\otimes\varepsilon)\rho_M=1$.

COROLLARY 3.2. *Let C be a G-graded coalgebra. If we denote $\Delta_1=u^C_{1,1}$: $C_1\to C_1\otimes C_1$, then $(C_1, \Delta_1, \varepsilon)$ is a coalgebra and $\pi_1: C\to C_1$ is a morphism of co-*

algebras. Moreover, if $M=\bigoplus_{\sigma\in G}M_\sigma$ *is a right graded* C*-comodule, then for any* $\sigma\in G$, M_σ *is a right* C_1*-module via the canonical map* $u^M_{\sigma,1}: M_\sigma\to M_\sigma\otimes C_1$.

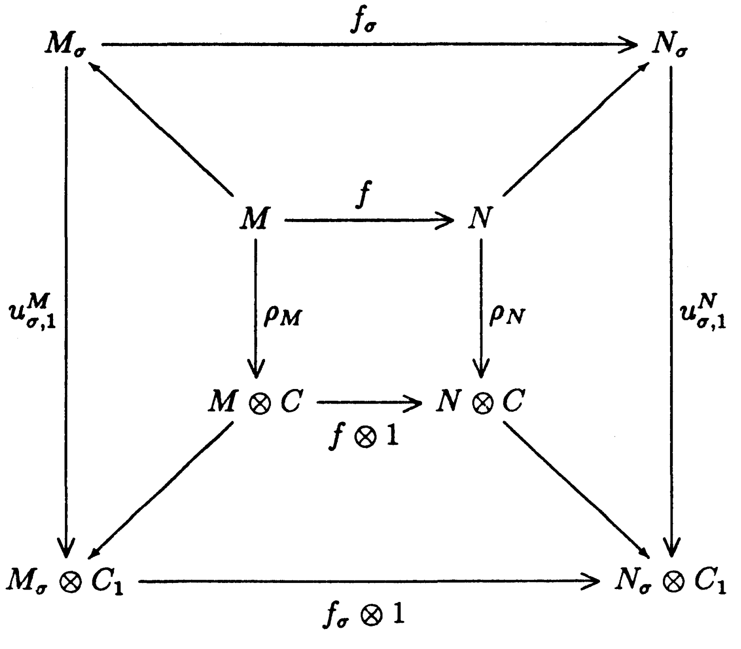
PROOF. If we consider $M=C$ as right C-comodule and $\sigma=\tau=\lambda=1$, the Proposition 3.1 yields that $(C_1, \Delta_1, \varepsilon)$ is a coalgebra.

The same Proposition 1.1 with $\tau=\lambda=1$ tell us that M_σ is a right C_1-comodule for any $\sigma\in G$.

Let $M=\bigoplus_{\sigma\in G}M_\sigma$, $N=\bigoplus_{\sigma\in G}N_\sigma$ two objects in gr^C. Let $f\in Com_{gr^C}(M, N)$. Since $f(M_\sigma)\subseteq N_\sigma$ for any $\sigma\in G$, we denote by $f_\sigma: M_\sigma\to N_\sigma$ the restriction of f, for $\sigma\in G$.

COROLLARY 3.3. *With the above notations for any* $\sigma\in G$, f_σ *is a morphism in the category* $\boldsymbol{M}^{C_1}$.

PROOF. We have to show that the following diagram is commutative



Let $m\in M_\sigma$. Since $\rho_M(M_\sigma)\subseteq\sum_{\lambda\mu=\sigma}M_\lambda\otimes M_\mu$, then we can write

$$\rho_M(m)=\sum_{\substack{\deg m_0=\sigma\\ \deg m_1=1}} m_0\otimes m_1+\sum m_0'\otimes m_1'$$

and $\deg m_0'\neq\sigma$ or $\deg m_1'\neq 1$. Then $u^M_{\sigma,1}(m)=\sum_{\substack{\deg m_0=\sigma\\ \deg m_1=1}} m_0\otimes m_1$ and therefore $((f_\sigma\otimes 1)u^M_{\sigma,1})(m)=\sum_{\substack{\deg m_0=\sigma\\ \deg m_1=1}} f(m_0)\otimes m_1$. Since f is a morphism of comodules, then $\rho_N f=(f\otimes 1)\rho_M$. Hence $(u^N_{\sigma,1}f_\sigma)(m)=u^N_{\sigma,1}(f(m))=((\pi^N_\sigma\otimes\pi^N_1)(\rho_N))(f(m))=((\pi^N_\sigma\otimes\pi^N_1)$

$(f\otimes 1)\rho_M)(m)=((\pi_\sigma^N\otimes\pi_1^N)\sum_{\substack{\deg m_0=\sigma\\ \deg m_1=1}} f(m_0)\otimes m_1+\sum f(m_0')\otimes m_1')=\sum_{\substack{\deg m_0=\sigma\\ \deg m_1=1}} f)m_0)\otimes m_1$.
Hence $u_{\sigma,1}^N f_\sigma=(f_\sigma\otimes 1)u_{\sigma,1}^M$.

Let $M=\oplus_{\sigma\in G}M_\sigma$ be an object in gr^C and $\sigma\in G$. We can define another G-graded comodule denoted by $M(\sigma)$: as C-comodule $M(\sigma)$ coincides with M, i. e. $M(\sigma)=M$ and $\rho_{M(\sigma)}=\rho_M$, but the grading of $M(\sigma)$ is given by the equality $M(\sigma)_\lambda=M_{\sigma\lambda}$ for any $\lambda\in G$. Since $\rho_{M(\sigma)}(M(\sigma(_\tau)=\rho_M(M_{\sigma\tau})\subseteq\sum_{xy=\sigma\tau}M_x\otimes C_y=\sum_{xy=\tau}M(\sigma)_x\otimes C_y$ results that $M(\sigma)$ is a graded C-comodule. $M(\sigma)$ is called the σ-suspension of M. It is clear that $M\to M(\sigma)$ defines an isomorphism of categories from gr^C to gr^C.

We denote by $U: gr^C\to M^C$ the forgetful functor. Clearly U is an exact functor.

PROPOSITION 3.4. *With the above notation U has an exact right adjoint functor $F: \boldsymbol{M}^C\to gr^C$. Moreover if the group G is finite, then F is also a left adjoint of U.*

PROOF. Let $M\in\boldsymbol{M}^C$. We construct $F(M)=\oplus_{\sigma\in G}M^\sigma$ where $M^\sigma=M$ for $\sigma\in G$. We have to define $\rho_{F(M)}: F(M)\to F(M)\otimes C$, but it is enough to define it for the elements of the form $x=(\cdots, 0, m^\sigma, 0, \cdots)$, where $m^\sigma=m\in M$ and the other components of x are zero. Since $\rho_M(m)\in M\otimes C=\oplus_\lambda M\otimes C_\lambda$, then $\rho_M(m)=\sum_{(m)}m_0\otimes m_1^\lambda$ where $m_0\in M$ and $m_1^\lambda\in C_\lambda$. Now, we define $\rho_{F(M)}(x)=\sum_{\tau\in G}(\cdots, m_0\otimes m_1^\tau, \cdots)$ where here m_0 is considered as element in $M^{\sigma\tau^{-1}}$. It is easy to see that since M is a right C-comodule, the map $\rho_{F(M)}$ defines on $F(M)$ a structure of right C-comodule. Clearly $F(M)$ is a graded C-comodule where the grading is given by $F(M)_\sigma=M^\sigma$ for any $\sigma\in G$. The action of F on the morphism is defined in the natural way. Then F is a functor from M^C to gr^C. Clearly F is exact.

If $M\in gr^C$ and $N\in\boldsymbol{M}^C$ we define

$$Com_C(U(M), N)\underset{\psi}{\overset{\phi}{\rightleftarrows}}Com_{gr^C}(M, F(N))$$

For any $u\in Com_C(U(M), N)$, we define $\phi(u)(m_\sigma)=u(m_\sigma)^\sigma$ where $m_\sigma\in M_\sigma$ and $u(m)^\sigma$ is considered as element in $F(N)_\sigma=N^\sigma$. It is easily seen that $\phi(u)$ is a morphism of C-comodules.

Now, if $v\in Com_{gr^C}(M, F(N))$, we put $\psi(v)=\beta v$ where $\beta: F(N)\to N$ is the canonical map, i. e. $\beta(\cdots, n^\sigma, \cdots)=\sum_{\sigma\in G}n^\sigma$. Clearly β is a morphism of C-comodules. It is easy to show that ϕ and ψ are inverse maps.

Suppose now that G is a finite group. Let $M \in \boldsymbol{M}^C$. We define $\alpha: M \to F(M)$, by $\alpha(m)=(m, \cdots, m, \cdots, m)$, i. e. $\alpha(m)$ is an element of $F(M)$ which has the element m in every position $\sigma \in G$. α is a morphism of C-comodules, i. e. the following diagram is commutative

$$\begin{array}{ccc} M & \xrightarrow{\ \alpha\ } & F(M) \\ \Big\downarrow{\scriptstyle \rho_M} & & \Big\downarrow{\scriptstyle \rho_{F(M)}} \\ M \otimes C & \xrightarrow[\alpha \otimes 1]{} & F(M) \otimes C \end{array}$$

Indeed, if $m \in M$ then $(\rho_{F(M)}\alpha)(m)=\rho_{F(M)}(\sum_{\sigma \in G}(0, \cdots, m^\sigma, 0, \cdots))$ (here $m^\sigma = m$, for any $\sigma \in G) = \sum_{\sigma \in G} \rho_{F(M)}(0, \cdots, m^\sigma, 0, \cdots)$. If $\rho_M(m)=\sum_{(m)} m_0 \otimes m_1^\tau$ where $m_1^\tau \in C_t$ then $(\rho_{F(M)}\alpha)(m) = \sum_{\substack{\sigma \in G \\ \tau \in G}} (0, \cdots, m_0^{\sigma\tau^{-1}-1} \otimes m_1^\tau, 0, \cdots)$ (here $m_0^{\sigma\tau^{-1}} = m_0$) $=$ $\sum_{\tau \in G} \sum_{\sigma \in G} (0, \cdots, m_0^{\sigma^{-1}} \otimes m_1^\tau, 0, \cdots) = \sum_{\tau \in G} (0, \cdots, m_0 \otimes m_1^\tau, 0, \cdots) = ((\alpha \otimes 1)\rho_M)(m)$. Now, let $M \in M^C$ and $N \in gr^C$. We define the maps

$$Com_{gr^C}(F(M), N) \underset{\psi'}{\overset{\phi'}{\rightleftarrows}} Com_C(M, U(N))$$

by $\phi'(u)=u\alpha$, for any $u \in Com_{gr^C}(F(M), N)$ and $\psi'(v)(0, \cdots, 0, m^\sigma, 0, \cdots, 0) = v(m^\sigma)_\sigma$, where $v(m^\sigma)_\sigma$ is the homogeneous component of degree σ of the element $v(m^\sigma) \in N$. It is easy to see that ϕ' and ψ' are inverse.

PROPOSITION 3.5. *Let* $M \in gr^C$. *Then* $(FU)(M) \cong \bigoplus_{\sigma \in G} M(\sigma)$.

PROOF. We have $(FU)(M) = \bigoplus_{\lambda \in G} M^\lambda$. We consider in $(FU)(M)$ an element x of degree τ $(\tau \in G)$, i. e. $x=(0, \cdots, 0, m^\tau, 0, \cdots, 0)$ where $m^\tau = m \in M$. But $m = \sum_{y \in G} m_y$, where $m_y \in M_y$. Since $(\bigoplus_{\sigma \in G} M(\sigma))_\tau = \bigoplus_{\sigma \in G} M(\sigma)_\tau = \bigoplus_{\sigma \in G} M_{\sigma\tau}$ then we define the canonical morphism $\alpha: (FU)(M) \to \bigoplus_{\sigma \in G} M(\sigma)$ by $\alpha(x) = (\cdots, m_y, \cdots)$ where $(\cdots, m_y, \cdots)$ is the system that contains all the homogeneous components of x (rest is zero) but m_y is considered as an element in $M(y\tau^{-1})$. It is easily verified that α is an isomorphism of comodules.

COROLLARY 3.6. 1) *If the category* gr^C *has enough projective modules, then* $\boldsymbol{M}^C$ *has enough projectives.*

2) *If the group is finite and* $M \in gr^C$ *is an injective object, then* $U(M)$ *is also injective in* $\boldsymbol{M}^C$.

PROOF. 1) Let $M\in \boldsymbol{M}^C$, then there exists $P\in gr^C$ and an epimorphism $P\to F(M)\to 0$ in the category gr^C. Since U is exact then we have in $\boldsymbol{M}^C$ the exact sequence $U(P)\to UF(M)\to 0$. But we have in $\boldsymbol{M}^C$ the canonical epimorphism $\beta:(UF)(M)\to M$ (see the proof of Proposition 3.4). On the other hand since F is an exact right adjoint functor of U, then $U(P)$ is projective. Hence we have an epimorphism in $\boldsymbol{M}^C$, $U(P)\to M\to 0$, where $U(P)$ is projective. 2) Follows from general properties of adjoint functors.

4. Cotensor products and induced functor

Let C be an arbitrary coalgebra, M be a right C-comodule and N be a left C-comodule, the **cotensor product** $M\square_C N$ is the kernel of the k-map $\rho_M\otimes 1-1\otimes\rho_N: M\otimes N\to M\otimes C\otimes N$. Following [3], the cotensor product is a left exact functor $\boldsymbol{M}^C\times {}^C\boldsymbol{M}\to \boldsymbol{M}_k$ ($\boldsymbol{M}_k$ is the category of k-spaces). Moreover the mapping $m\otimes c\to\varepsilon(c)m$ and $c\otimes n\to\varepsilon(c)n$ yield a natural isomorphism $M\square_C C\cong M$ and $C\square_C N\cong N$.

Now if C and D are two coalgebras and M is (C, D)-bicomodule, i. e. $(1\otimes\rho^+)\rho^-=(\rho^-\otimes 1)\rho^+$, where $\rho^-: M\to C\otimes M$ and $\rho^+: M\to M\otimes D$ are the structure maps of M, and N is a left D-comodule, then the map $\rho^-\otimes 1: M\otimes N\to C\otimes M\otimes N$ define over $M\otimes N$ a structure of left C-comodule. In this case $M\square_D N$ is a C-subcomodule of $M\otimes N$ (cf. [3]).

Now we consider the graded case. Let C be a G graded coalgebra. By Corollary 3.1 C_1 is a coalgebra and the canonical map $\pi_1: C\to C_1$ is a morphism of coalgebras. Let $M\in M^{C_1}$ be a right C_1 comodule. Since C has in the natural way (via the morphism π_1) a structure of (C_1, C_1)-bicomodule we can consider $M\square_{C_1}C\in M^C$. Since $C=\oplus_{\sigma\in G}C_\sigma$ and C_σ is a left C_1-comodule (see Corollary 3.1) (in fact C_σ is a (C_1, C_1)-bicomodule), for any $\sigma\in G$, then $M\square_{C_1}C=\oplus_{\sigma\in G}(M\square_{C_1}C_\sigma)$ (the functor cotensor product commute with direct sums). Thus, $M\square_{C_1}C$ has the natural graduation if we put $(M\square_{C_1}C)_\sigma=M\square_{C_1}C_\sigma$. In fact $M\square_{C_1}C=Ker(\rho_M\otimes 1-1\otimes\rho_1)$ where $\rho_1=(\pi_1\otimes 1)\Delta$. Indeed the canonical morphism which defines the structure of right C-comodule of $M\square_{C_1}C$ is the restriction of $1\otimes\Delta: M\otimes C\to M\otimes C\otimes C$ to $M\square_{C_1}C$. Since $\Delta(C_\sigma)\subseteq\sum_{\lambda\mu=\sigma}C_\lambda\otimes C_\mu$, then $(1\otimes\Delta)((M\square_{C_1}C)_\sigma)\subseteq\sum_{\lambda\mu=\sigma}(M\square_{C_1}C_\lambda)\otimes C_\mu$. Hence $M\square_{C_1}C$ is a right G-graded C-comodule.

Clearly, in this way, we obtain the functor $-\square_{C_1}C: M^{C_1}\to gr^C$, $M\to M\square_{C_1}C$, $M\in M^{C_1}$. This functor is called induced functor and it is left exact.

On the other hand we have the functor $(-)_1: gr^C\to M^{C_1}$ defined by $M\to M_1$, for any $M\in gr^C$. This functor is exact. More general for any $\sigma\in G$ we have the exact functor $(-)_\sigma: gr^C\to M^{C_1}$, defined by $M\to M_\sigma$, where $M=\bigoplus_{\lambda\in G}M_\lambda$.

Let $M=\bigoplus_{\lambda\in G}M_\lambda$ an object in gr^C, we will show that there exists a natural morphism $\alpha(M): M\to M_1\square_{C_1}C$. Indeed we consider the diagram

$$\begin{array}{ccccc} M_1\square_{C_1}C & \longrightarrow & M_1\otimes C & \overset{\rho_{M_1}}{\underset{1\otimes((\pi_1^C\otimes 1)\Delta)}{\rightrightarrows}} & M_1\otimes C_1\otimes C \\ & & \uparrow & & \uparrow \\ M & \underset{\rho_M}{\longrightarrow} & M\otimes C & \overset{\rho_M}{\underset{1\otimes\Delta}{\rightrightarrows}} & M\otimes C\otimes C \end{array}$$

Since $\rho_{M_1}=(\pi_1^M\otimes\pi_1^C)\rho_M i_1^M$ then $(\rho_{M_1}\otimes 1)(\pi_1^M\otimes 1)\rho_M=((\rho_{M_1}\pi_1^M)\otimes 1)\rho_M=((\pi_1^M\otimes\pi_1^C)\rho_M)\otimes 1\rho_M=(\pi_1^M\otimes\pi_1^C\otimes 1)(\rho_M\otimes 1)\rho_M=((\pi_1^M\otimes\pi_1^C\otimes 1)(1\otimes\Delta)\rho_M=(\pi_1^M\otimes((\pi_1^C\otimes 1)\Delta)\rho_M=(1\otimes(\pi_1^C\otimes 1)\Delta)(\pi_1^M\otimes 1)\rho_M$, then the image of the morphism $(\pi_1^M\otimes 1)\rho_M$ is contained in $M_1\square_{C_1}C$. Hence we define $\alpha(M): M\to M_1\square_{C_1}C$ by $\alpha(M)=(\pi_1^M\otimes 1)\rho_M$. Since M is graded in gr^C, then $\alpha(M)$ is a morphism in gr^C. Next we observe that $\alpha(M)_1: M_1\to M_1\square_{C_1}C_1$ is an isomorphism. Indeed, since $(1\otimes\varepsilon)(\pi_1^M\otimes 1)\rho_M=(\pi_1^M\otimes 1)(1\otimes\varepsilon)\rho_M=\pi_1^M$ then the restriction of $1\otimes\varepsilon$ to $M_1\square_{C_1}C_1$ is the inverse of $\alpha(M)_1$.

Now, by using the natural morphism $\alpha(M): M\to M_1\square_{C_1}C$ we can prove the following result.

PROPOSITION 4.1. *The functor $-\square_{C_1}C: M^{C_1}\to gr^C$ is a right adjoint of the functor $(-)_1: gr^C\to M^{C_1}$. Moreover we have the composition $(-)_1(-\square_{C_1}C)\cong 1_{M^{C_1}}$.*

PROOF. Let $M=\bigoplus_{\sigma\in G}M_\sigma\in gr^C$ and $N\in M^{C_1}$. We define the maps

$$Com_{C_1}(M_1, N)\underset{\phi}{\overset{\varphi}{\rightleftarrows}} Com_{gr^C}(M, N\square_{C_1}C)$$

Indeed, if $u\in Com_{C_1}(M_1, N)$ then we put $\varphi(u)=(u\square_{C_1}1)\alpha(M)$ and if $v\in Com_{gr^C}(M, N\square_{C_1}C)$, $\phi(v)=(1\otimes\varepsilon)v_1$ where $v_1: M_1\to N\square_{C_1}C_1$ is the morphism induced by v. Now $(\phi\varphi)(u)=\phi(\varphi(u))=(1\otimes\varepsilon)(u\square_{C_1}1)\varphi(M)_1$. Since the following diagram is commutative

$$\begin{array}{ccc} M_1\square_{C_1}C_1 & \xrightarrow{u\square_{C_1}1} & N\square_{C_1}C_1 \\ {\scriptstyle 1\otimes\epsilon}\downarrow & & \downarrow{\scriptstyle 1\otimes\epsilon} \\ M_1 & \underset{u}{\longrightarrow} & N \end{array}$$

then $(\psi\varphi)(u)=u(1\otimes\varepsilon)\alpha(M)_1=u$. On the other hand, $\varphi(\psi(v))=\varphi((1\otimes\varepsilon v_1)=(((1\otimes\varepsilon)v_1)\square_{C_1}1)\alpha(M)=((1\otimes\varepsilon)\square_{C_1}1)\alpha(N\square_{C_1}C)v=v$ (since α is a functorial morphism.)

5. Strongly graded coalgebras

Let $C=\bigoplus_{\sigma\in G}C_\sigma$ be a G-graded coalgebra. By Proposition 3.1 for any σ, $\tau\in G$ we have the canonical morphisms $u^C_{\sigma,\tau}: C_{\sigma\tau}\to C_\sigma\otimes C_\tau$. If $u^C_{\sigma,\tau}$ are monomorphisms for any σ, $\tau\in G$ then C is called a **strongly graded coalgebra**. Since $u^C_{\sigma,\sigma^{-1}}: C_1\to C_\sigma\otimes C_{\sigma^{-1}}$ are monomorphisms, it follows that if $C_1\neq 0$, then $C_\sigma\neq 0$ for any $\sigma\in C$ in a strongly graded coalgebra.

The following result characterizes the strongly graded coalgebras.

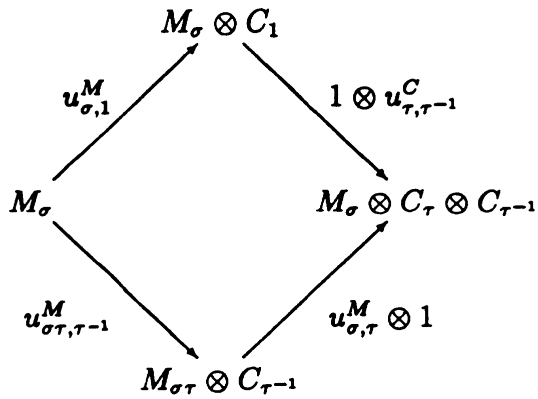
PROPOSITION 5.1. *Let $C=\bigoplus_{\sigma\in G}C_\sigma$ be a graded coalgebra. The following assertions are equivalent*

i) *C is strongly graded;*

ii) *for any $\sigma\in G$ the morphism $u^C_{\sigma,\sigma^{-1}}: C_1\to C_\sigma\otimes C_{\sigma^{-1}}$ is monomorphism;*

iii) *If $M=\bigoplus_{\sigma\in G}M_\sigma$ is a right graded C-comodule, then for any σ, $\tau\in G$ the canonical morphisms $u^M_{\sigma,\tau}: M_{\sigma\tau}\to M_\sigma\otimes C_\tau$ are monomorphisms.*

PROOF. i)$\Rightarrow$ii) and iii)$\Rightarrow$i) are clear. We prove only that ii)$\Rightarrow$iii). From the diagram in the first part of Proposition 3.1 with $\lambda=\tau^{-1}$ we have the commutative diagram



Since $u^C_{\tau,\tau^{-1}}$ is monomorphism by hypothesis then $1\otimes u^C_{\tau,\tau^{-1}}$ is a monomorphism. Since $(1\otimes\varepsilon)u^M_{\sigma,1}=1_{M_1}$, then also $u^M_{\sigma,1}$ is a monomorphism. Hence from the equality $(1\otimes u^C_{\tau,\tau^{-1}})u^M_{\sigma,1}=(u^M_{\sigma,\tau}\otimes 1)u^M_{\sigma\tau,\tau^{-1}}$ result that $u^M_{\sigma\tau,\tau^{-1}}: M_\sigma\to M_{\sigma\tau}\otimes C_{\tau^{-1}}$ is monomorphism. Now if we change $\sigma\tau$ by σ and τ^{-1} by τ, then we obtain that $u^M_{\sigma\tau}: M_{\sigma\tau}\to M_\sigma\otimes C_\tau$ is a monomorphism.

COROLLARY 5.2. *Let $C=\bigoplus_{\sigma\in G}C_\sigma$ be a strongly graded coalgebra. If $M=\bigoplus_{\sigma\in G}M_\sigma\in gr^C$ then the following assertions are equivalent*:

1) $M=0$,
2) $M_1=0$,
3) $M_0=0$ *for some* $\sigma\in G$.

PROOF. 1)$\Rightarrow$2) is clear.

2)$\Rightarrow$3) Follows from the fact that $u^M_{1,\sigma}: M_\sigma\to M_1\otimes C_\sigma$ is monomorphism.

3)$\Rightarrow$1) Since $u^M_{\sigma\tau}: M_{\sigma\tau}\to M_\sigma\otimes C_\tau$ is monomorphism, then $M_{\sigma\tau}=0$ for the $\tau\in G$. Therefore $M=0$.

COROLLARY 5.3. *Let $C=\bigoplus_{\sigma\in G}C_\sigma$ be a strongly graded coalgebra. Let $M=\bigoplus_{\sigma\in G}M_\sigma$ and $N=\bigoplus_{\sigma\in G}N_\sigma$ two objects in gr^C and $f\in Com_{gr^C}(M, N)$. The following assertions hold*:

1) *f is injective $\Leftrightarrow$ $f_1: M_1\to N_1$ is injective.*
2) *f is surjective $\Leftrightarrow$ $f_1: M_1\to N_1$ is surjective.*
3) *f is an isomorphism $\Leftrightarrow$ $f_1: M_1\to N_1$ is an isomorphism.*

PROOF. We apply Corollary 5.2.

Now, we are in position to show the main result of this section.

THEOREM 5.4. *Let $C=\bigoplus_{\sigma\in G}C_\sigma$ be a G-graded coalgebra. The following statements are equivalent*:

1) *C is strongly graded coalgebra.*
2) *The induction functor $-\square_{C_1}C: \boldsymbol{M}^{C_1}\to gr^C$ is an equivalence of categories.*
3) *The functor $(-)_1: gr^C\to\boldsymbol{M}^{C_1}$ is an equivalence of categories*

PROOF. From Proposition 4.1 it is enough to show the equivalence 1)$\Rightarrow$2).

Assume 1). By Proposition 4.1, we know that $(-)_1(-\square_{C_1}C)\cong 1_{M^{C_1}}$. Hence we can show only that $(-\square_{C_1}C)(-)_1\cong 1_{gr^C}$. For this we consider $M=\bigoplus_{\sigma\in G}M_\sigma\in gr^C$. We have the canonical morphism $\alpha(M): M\to M_1\square_{C_1}C$ such that $\alpha(M)_1: M_1\to(M_1\square_{C_1}C)_1$ is an isomorphism. By Corollary 5.2 it follows that $\alpha(M)$ is an isomorphism. Hence $(-\square_{C_1}C)(-)_1\cong 1_{gr^C}$.

2)$\Rightarrow$1) Since $(-)\square_{C_1}C: \boldsymbol{M}^{C_1}\to gr^C$ is an equivalence of categories, then Proposition 4.1 yields that the functor $(-)_1: gr^C\to\boldsymbol{M}^{C_1}$ is its inverse. Thus, if $M\in gr^C$, then the canonical morphism $\alpha(M): M\to M_1\square_{C_1}C$ is an isomorphism. Let $\sigma\in G$ and $M=C(\sigma)$ (the σ-suspension of C, where C is considered as right graded C-comodule). Hence $\alpha(C(\sigma)): C(\sigma)\to C(\sigma)_1\square_{C_1}C$ is an isomorphism. Since $C(\sigma)_1=C_\sigma$, then for any $\tau\in G$ we have the canonical isomorphism $\alpha(C(\sigma))_\tau$:

$C(\sigma\tau)\to C(\sigma)_1\square_{C_1}C_\tau$. Since $C(\sigma)_1\square_{C_1}C_\tau\subseteqq C_\sigma\otimes C_\tau$, then $\alpha(C(\sigma))_\tau: C(\sigma\tau)\to C(\sigma)_1\otimes C_\tau$ is a monomorphism. We observe that $\alpha(C(\sigma))_\tau$ is exactly $u_{\sigma,\tau}$ and the result follows.

COROLLARY 5.5. *If* $C=\bigoplus_{\sigma\in G}C_\sigma$ *is a strongly graded coalgebra and* $M=\bigoplus_{\sigma\in G}M_\sigma\in gr^C$ *then for any* $\sigma, \tau\in G$, $M_\sigma\square_{C_1}C_\tau\cong M_{\sigma\tau}$. *In particular for any* $\sigma\in G$, $C_\sigma\square_{C_1}C_{\sigma^{-1}}\cong C_1$, *i.e.* C_σ *are coinvertible* (C_1, C_1)*-bicomodules.*

If C is a coalgebra, following [8], C is called **right semiperfect** if the category $\boldsymbol{M}^C$ has enough projectives.

COROLLARY 5.6. *Let* $C=\bigoplus_{\sigma\in G}C_\sigma$ *be a strongly graded coalgebra. If* C_1 *is right semiperfect coalgebra, then* C *is right semiperfect coalgebra.*

PROOF. By Theorem 3.4, the category gr^C has enough projectives. Now the Corollary 3.2 tell us that $\boldsymbol{M}^C$ has enough projectives.

COROLLARY 5.7. *If* $C=\bigoplus_{\sigma\in G}C_\sigma$ *is a strongly graded coalgebra and* C_1 *is a coalgebra of finite dimension, then* C *is a left and right semiperfect coalgebra.*

PROOF. Since C_1 has finite dimension, then $\boldsymbol{M}^{C_1}$ and ${}^{C_1}\boldsymbol{M}$ are isomorphic to the category of left and right modules over the ring C_1^*, respectively. Hence $\boldsymbol{M}^{C_1}$ (resp. ${}^{C_1}\boldsymbol{M}$) has enough projectives. By Corollary 5.4 the result follows.

6. The graded ring associated to a graded coalgebra

Let $C=\bigoplus_{\sigma\in G}C_\sigma$ be a coalgebra. Following [1] or [10], $C^*=Hom_k(C, k)$ has a natural structure of ring. Indeed if $f, g\in C^*$ then $fg=(f\otimes g)\Delta$ (here $k\otimes k\cong k$). Hence if $c\in C$ and $\Delta(c)=\sum_{(c)}c_1\otimes c_2$ then $(fg)(c)=\sum_{(c)}f(c_1)g(c_2)$.

If V is a k-vector space, then $V^*=Hom_k(V, k)$. If S is any subset of V, we set $S^\perp=\{f\in V^*\mid f(x)=0$, for any $x\in S\}$ and if X is a subset of V^* we set $X^\perp=\{x\in V\mid f(x)=0$, for any $f\in X\}$. Given $f\in V^*$, by letting the family of subsets of V^*, $\{f+S^\perp\mid S\subseteqq V$ is a finite set$\}$ be a base for a system of neighborhoods of f, V^* becomes a linear topological space. This topology is called the finite topology. If $X\subseteqq V^*$ is an arbitrary subspace then the closure $\bar{X}$ of X in the finite topology of V^* is exactly $(X^\perp)^\perp$. In particular any finite dimensional subespace of V^* is closed (e.g. [10] page 68). When $V=C$, then C^* is a linear topological space (In fact C^* is a topological ring). Let $\rho_M: M\to M\otimes C$ be a right C-comodule, then M has a natural structure of left C^*-module. Indeed, if $m\in M$ and $c^*\in C^*$, then $c^*m=\sum_{(m)}m_0c^*(m_1)$ where $\rho_M(m)=\sum_{(m)}m_0\otimes m_1$.

Following [10], if ${}_{C^*}M$ is a left C^*-module, we have the canonical morphisms

$$\begin{array}{ccc} M\otimes C & \xrightarrow{\ \alpha\ } & Hom_k(C^*, M) \\ & \nearrow_{\beta} & \\ M & & \end{array}$$

where $\alpha(m\otimes c)(c^*)=mc^*(c)$ for any $m\in M$, $c\in C$ and $c^*\in C^*$. Here α is an injective morphism of vector spaces. If $m\in M$, then we define $\beta(m)(c^*)=c^*m$. When β factorizes through α, i. e. for any $m\in M$ there exists a unique element $\sum_1^n m_i\otimes c_i\in M\otimes C$ such that $\beta(m)=\alpha(\sum_1^n m_i\otimes c_i)$, then M is called **rational module.** If we denote by $\mathbf{Rat}^{C^*}$ the class of all (left) rational C^*-modules then by [1], [10], the category $\boldsymbol{M}^C$ is isomorphic with the category $\mathbf{Rat}^{C^*}$.

Let now $\boldsymbol{A}$ be a Grothendieck category and $\boldsymbol{C}$ a full subcategory of $\boldsymbol{A}$. $\boldsymbol{C}$ is called *closed* ([4], page 395), if $\boldsymbol{C}$ is closed under subobjects, quotient objects and direct sums. If $\boldsymbol{C}$ is furthermore closed under extensions, then $\boldsymbol{C}$ is called a *localizing subcategory* $\boldsymbol{A}$. It may be easily seen that a closed subcategory of a Grothendieck category is also a Grothendieck category (indeed, if $U\in\boldsymbol{A}$ is the generator, then $\{U/K\mid U/K\in\boldsymbol{C}\}$ is a family of generators of $\boldsymbol{C}$ and the direct sum is the generator).

By [1], [10], it follows that $\mathbf{Rat}^{C^*}$ is a closed subcategory of the category ${}_{C^*}\boldsymbol{M}$ (the category of all left C^*-modules). Under certain hypotheses (cf. [8]) $\mathbf{Rat}^{C^*}$ is also a localizing subcategory of ${}_{C^*}\boldsymbol{M}$.

Let $\boldsymbol{A}$ be any Grothendieck category and $M\in\boldsymbol{A}$ and object. We denote by $\sigma_{\boldsymbol{A}}[M]$ (or shortly $\sigma[M]$) the class of all the objects of $\boldsymbol{A}$ subgenerated by M, i. e. the objects isomorphic to subjects of quotient objects of direct sums of copies of M. It is easily seen that $\sigma_{\boldsymbol{A}}[M]$ is a closed subcategory of $\boldsymbol{A}$ (in fact $\sigma_{\boldsymbol{A}}[M]$ is the smallest closed subcategory of $\boldsymbol{A}$ containing M). By [10], if C is an arbitrary coalgebra, then $\mathbf{Rat}\ C^*=\sigma[{}_{C^*}C]$. Moreover ${}_{C^*}C$ is a injective cogenerator in $\sigma[{}_{C^*}C]$.

Let $C=\bigoplus_{\sigma G\in}C_\sigma$ be a G-graded coalgebra. For any $\sigma\in G$ we put $R_\sigma=\{f\in C^*\mid f(C_\tau)=0 \text{ for all } \tau\neq\sigma\}$. R_σ is a k-subspace of C^*. In fact $R_\sigma=(\sum_{\tau\neq\sigma}C_\tau)^\perp$ and therefore $R_\sigma\cong C_\sigma^*$ as vector spaces. It is easy to see that the sum $\sum_{\sigma\in G}R_\sigma$ is direct. We define $R=\sum_{\sigma\in G}R_\sigma=\bigoplus_{\sigma\in G}R_\sigma$.

PROPOSITION 6.1. *With the above notations R is a G-graded ring with*

$\varepsilon: C \to k$ the identity element.

PROOF. We can show that $R_\sigma R_\tau \subseteq R_{\sigma\tau}$ for any $\sigma, \tau \in G$. Let $f \in R_\sigma$, $g \in R_\tau$, $c \in C_\lambda$ where $\lambda \neq \sigma\tau$, then $(fg)(c) = \sum f(c_1) g(c_2)$, where $\Delta(c) = \sum_{(c)} c_1 \otimes c_2$. Since $\Delta(C_\lambda) \subseteq_{xy=\lambda} C_x \otimes C_y$, then from $xy \neq \sigma\tau$ results either $x \neq \sigma$ or $y \neq \tau$. Hence $f(g)(c) = 0$ and $fg \in R_{\sigma\tau}$. The observation that $\varepsilon \in R_1$ and ε is the identity element of R is clear.

PROPOSITION 6.2. *Let $C = \bigoplus_{\sigma \in G} C_\sigma$ be a G-graded coalgebra such that C_σ has finite dimension for any $\sigma \in G$. Then the coalgebra C is strongly graded if and only if R is a strongly graded ring.*

PROOF. Assume that C is a strongly graded coalgebra. We know that the canonical maps $u_{\sigma,\tau}^C : C_{\sigma\tau} \to C_\sigma \otimes C_\tau$ are injective for any $\sigma\tau \in G$. Hence the canonical morphisms

$$(C_\sigma \otimes C_\tau)^* \xrightarrow{(u_{\sigma\tau}^C)^*} C_{\rho\tau}^* \longrightarrow 0$$

are surjective. Since $(C_\sigma \otimes C_\tau)^* \cong C_\sigma^* \otimes C_\tau^*$ via the canonical morphism

$$\alpha : C_\sigma^* \otimes C_\tau^* \longrightarrow (C_\sigma \otimes C_\tau^*)^*,$$

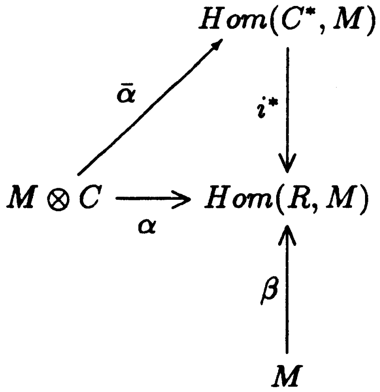
defined by $\alpha(u \otimes v)(x \otimes y) = u(x)v(y)$, where $u \in C_\sigma^*$, $v \in C_\tau^*$, $x \in C_\sigma$, $y \in C_\tau$. Now if $f \in C_{\sigma\tau}^*$, then there exists an element $\sum_{i=1}^n u_i \otimes v_i \in C_\sigma^* \otimes C_\tau^*$ such that $f = ((u_{\sigma\tau}^C)^* \alpha)(\sum_{i=1}^n u_i \otimes v_i)$. If we consider now, $\theta_{\sigma\tau} : R_\sigma \otimes R_\tau \to R_{\sigma,\tau}$ the canonical map, i. e. $\theta_{\sigma,\tau}(r_\sigma \otimes r_\tau) = r_\sigma r_\tau$ where $r_\sigma \in R_\sigma$, $r_\tau \in R_\tau$. We claim that $\theta_{\sigma,\tau}(\sum_{i=1}^n u_i \otimes v_i) = (\sum_{i=1}^n u_i v_i) = f$. Let $c \in C_{\sigma\tau}$. Since $\Delta(C_{\sigma\tau}) \subseteq \sum_{\lambda\mu=\sigma\tau} C_\lambda \otimes C_\mu$, we can write $\Delta(c) = \sum_{(c)} c_1 \otimes c_2 + \sum_{(c)} \bar{c}_1 \otimes \bar{c}_2$, where $\sum_{(c)} c_1 \otimes c_2 \in C_\sigma \otimes C_\tau$ and $\sum_{(c)} \nu_1 \otimes \bar{c}_2 \in \sum_{\lambda \neq \sigma \text{ or } \mu \neq \tau}^{\lambda\mu=\sigma\tau} C_\lambda \otimes C_\mu$. Clearly $(\sum_{i=1}^n u_i v_i)(c) = \sum_{i=1}^n \sum_{(c)} u_i(c_1) v_i(c_2)$. On the other hand $f = (\sum_{i=1}^n \alpha(u_i \otimes v_i) u_{\sigma,\tau}^C$. Hence $f(c) = (\sum_{i=1}^n \alpha(u_i \otimes v_i)(u_{\sigma,\tau}^C(c))$. But $u_{\sigma,\tau}^C(c) = \sum_{(c)} c_1 \otimes c_2$. Thus $f(c) = \sum_{i=1}^n \sum_{(c)} u_i(c_1) v_i(c_2)$. This proves the claim and it follows that R is a strongly graded ring.

Conversely, if R is strongly graded, then the canonical maps $\theta_{\sigma,\tau} : R_\sigma \otimes R_\tau \to R_{\sigma\tau}$ are surjective for any $\sigma, \tau \in G$. Hence the map $\theta_{\sigma,\tau}^* : R_{\sigma\tau}^* \to (R_\sigma \otimes R_\tau)^* \cong R_\sigma^* \otimes R_\tau^*$ is injective. The canonical isomorphism $R_\sigma^* \cong C_\sigma$, $R_\tau^* \cong C_\tau$, $R_{\sigma\tau}^* \cong C_{\sigma,\tau}$ yield that the natural map $u_{\sigma,\tau}^C : C_{\sigma,\tau} \to C_\sigma \otimes C_\tau$ is injective for any $\sigma, \tau \in G$. Hence C is a strongly graded coalgebra.

If $C = \bigoplus_{\sigma \in G} C_\sigma$ is a G-graded coalgebra and $M = \bigoplus_{\sigma \in G} M_\sigma$ together with $\rho_M : M \to M \otimes C$ is a right graded C-comodule, by [1] or [10], M has a natural structure of C^*-module. Indeed if $f \in C^*$, $m \in M$ then $fm = \sum_{(m)} m_0 f(m_1)$ where $\rho_M(m) = \sum_{(m)} m_0 \otimes m_1$. Since M is a graded comodule we define a left graded

R-module, $\overline{M}$, in the following way: $\overline{M}=M$ as k-space but for any $\sigma\in G$ we put $\overline{M}_\sigma=M_{\sigma^{-1}}$. Now, if $f\in R_\sigma$ and $m\in\overline{M}_\tau=M_{\tau^{-1}}$, then we have $fm=\sum_{(m)} m_0 f(m_1)$, where $\rho_M(m)=\sum_{(m)} m_0\otimes m_1$. Since $\Delta(M_{\tau^{-1}})\subseteq\sum_{xy=\tau^{-1}} M_x\otimes C_y$ and since $f\in R_\sigma$, then $fm=\sum_{(m)} m_0 f(m_1)$, where $m_1\in C_\sigma$. Hence $m_0\in M_{\tau^{-1}\tau^{-1}}=\overline{M}_{\sigma\tau}$. Thus $R_\sigma\overline{M}_\tau\subseteq\overline{M}_{\sigma\tau}$ and $\overline{M}$ is left graded R-module. If we denote by $R-gr$ the category of left graded R-module, we have obtained a functor $F: gr^C\to R-gr$, $F(M)=\overline{M}$.

Let $M=\bigoplus_{\sigma\in G}M_\sigma$ a left G-graded R-module. We consider the natural morphisms



where $\alpha(m\otimes c)(r)=mr(c)$, $r\in R$; $\bar{\alpha}(m\otimes c)(f)=mf(c)$, $f\in C^*$, and $i^*: Hom(C^*, M)\to Hom(R, M)$ is the canonical epimorphism given by the inclusion $i: R\hookrightarrow C^*$. Since $M\in R-gr$ we can define $\beta(m)(r)=rm$. Clearly α and $\bar{\alpha}$ are monomorphisms and $\alpha=i^*\bar{\alpha}$. We observe that $\alpha(M\otimes C_\sigma)\subseteq Hom(R_\sigma, M)$ for any $\sigma\in G$. (Here $Hom(R_\sigma, M)=\{f\in Hom(R, M)\mid f(R_\tau)=0$ for any $\tau\neq\sigma\}$). Now if β factorizes through α, i. e. there exists a (unique) morphism $\rho_M: M\to M\otimes C$ such that $\alpha\rho_M=\beta$, then M is called **gr-rational**. Thus, M is gr-rational if and only if there exists an element $\sum_{i=1}^n m_i\otimes c_i\in M\otimes C$ such that $rm=\sum_{i=1}^n m_i r(c_i)$ for any $r\in R$. In particular, if M is gr-rational and $m\in M$ then Rm has finite dimension over k.

Clearly if M is a right graded C-comodule, then $\overline{M}$ is gr-rational in $R-gr$. Assume $M\in R-gr$ is gr-rational, we define a right graded C-comodule $\overline{M}$ in the following way: $\overline{M}=M$ as vector space; for any $\sigma\in G$ we put $\overline{M}_\sigma=M_{\sigma^{-1}}$ and $\rho_M:\overline{M}\to\overline{M}\otimes C$ is the canonical map. We can prove that $\rho_M(\overline{M}_\sigma)\subseteq\sum_{xy=\sigma}\overline{M}_x\otimes C_y$. Indeed, if $m\in\overline{M}_\sigma=M_{\sigma^{-1}}$, then $\rho_M(m)=\sum_{i=1}^n m_i\otimes c_i$, where $\alpha(\sum_{i=1}^n m_i\otimes c_i)=\beta(m)$. Since $\overline{M}\otimes C=\bigoplus_{\substack{\lambda\in G\\ \mu\in G}} M_\lambda\otimes C_\mu$, we can assume that m, and c_i are homogeneous. Then $\rho(m)=\sum_{j=1}^n m_i\otimes c_i$ where $m_i\in M_{\lambda_i}$, $c_i\in C_{\mu_i}$. Also we can assume that c_i are linear independent over the field k. Since $c_i\in C_{\mu_i}$, there exists $r_i\in R_{\mu_i}$ such that $r_i(c_j)=\delta_{i,j}$. Now $\alpha(\sum_{j=1}^n m_j\otimes c_j(r_i))=\beta(m)(r_i)$, it follows that $m_i = r_i m$. Therefore $\lambda_i^{-1} = \mu_i \sigma^{-1}$ and $\lambda_i = \sigma \mu_i^{-1}$. Since $\overline{M}_{\lambda_i} = M_{\lambda_i^{-1}}$ it results clearly that $\rho_M(\overline{M}_\sigma) \subseteq \sum_{xy=\sigma} \overline{M}_x \otimes C_y$. By the same argument as the Sweedler's book ([10], page 37), we can prove that $\overline{M}$ with the map $\rho_M : \overline{M} \to \overline{M} \otimes C$ is a right C-comodule.

If we denote by $\mathbf{Rat}(R-gr)$ the full subcategory of $R-gr$ of all gr-rational modules, we obtain the functor $G : \mathbf{Rat}(R-gr) \to gr^C$, defined by $H(M) = \overline{M}$. It is easy to see that $\mathbf{Rat}(R-gr)$ is a closed subcategory of $R-gr$. Moreover $\mathbf{Rat}(R-gr)$ is rigid (or G-invariant) closed subcategory of $R-gr$. i.e. for any $M \in \mathbf{Rat}(R-gr)$ the σ-suspension $M(\sigma) \in \mathbf{Rat}(R-gr)$ for every $\sigma \in G$ (Here because M is left graded module, $M(\sigma)_\tau = M_{\tau\sigma}$ for all $\tau \in G$ (see [7])). Then with the above notation we have the following result.

THEOREM 6.3. *If $C = \bigoplus_{\sigma \in G} C_\sigma$ is a G-graded coalgebra, then the categories gr^C and $\mathbf{Rat}(R-gr)$ are isomorphic via the functors F and H. Moreover if for any $\sigma \in G$, C_σ has finite dimension, then gr^C is isomorphic with the subcategory $\{X \in R-gr \mid dim_k(Rx) < \infty$ for all $x \in X\}$. In particular if $M \in R-gr$ with $dim_k M < \infty$, then $M \in \mathbf{Rat}(R-gr)$.*

PROOF. For the first part we make the same proof as in the case non-graduate (see Sweedler [10] or Abe [1]). Assume now that C_σ has finite dimension for any $\sigma \in G$. Since $\alpha(M \otimes C_\sigma) \subseteq Ham(R_\sigma, M)$ and $R_\sigma \cong C_\sigma^*$ then we obtain that $\alpha(M \otimes C_\sigma) = Hom(R_\sigma, M)$. Hence every object in $R-gr$ is gr-rational.

COROLLARY 6.4. *If $C = \bigoplus_{\sigma \in G} C_\sigma$ is a G-strongly graded coalgebra, then the group G is finite.*

PROOF. Since $C_\sigma \neq 0$ we take $c \in C_\sigma$. We consider ${}_R C$ as left graded R-module. Since $M = Rc$ is a graded submodule of ${}_R C$, then M as C-comodule is a right graded C-comodule. But M has finite dimension over k and hence $M = \bigoplus_{\sigma \in G} M_\sigma$ has only a finite nonzero homogeneous components M_σ, $\sigma \in G$. If we now apply the Corollary 5.2, then it follows that G is a finite group.

REMARKS. 1) Let C be a G-graded coalgebra with R the graded ring associated and let $f \in C^*$. For any $\sigma \in G$ we define $f_\sigma \in R_\sigma$ such that $f_\sigma(c) = f(c)$ if $c \in C_\sigma$ and $f_\sigma(C_\tau) = 0$ for any $\tau \neq \sigma$. We obtain a family $(f_\sigma)_{\sigma \in G}$ where $f_\sigma \in R$ $(\sigma \in G)$. If we consider C^* with the finite topology then $f = \sum_{\sigma \in G} f_\sigma$ i.e. the family $(f_\sigma)_{\sigma \in G}$ is summable to f, i.e. for any neighbourhood $V(f)$ of f there exists a finite subset $F \subseteq G$ such that $\sum_{\sigma \in F'} f_\sigma \in V(f)$ for any finite subset F' of G such that $F \subseteq F'$. Indeed it is sufficient to consider the neighbourhood of the form $f+S^{\perp}$ where S is a finite subset of C. Since C is a graded coalgebra we can assume that S contains only homogeneous elements. Assume $S=\{x_1, \cdots, x_n\}$ where the x_i are homogeneous. If $deg(x_i)=\sigma_i$, then $F=\{\sigma_i, 1\leqq i\leqq n\}$. Clearly $\sum_{\sigma\in F'} f_\sigma\in f+S^{\perp}$ for any finite subset F' of G such that $F\subseteqq F'$.

2) Now if $M\in R-gr$ is gr-rational we can define over M an structure of C^*-module. Indeed we assume $m\in M$, m homogeneous. Since M is gr-rational then Rm is a graded submodule of M and has finite dimension over k. Hence for the twosided ideal $I=Ann_R(Rm)$, R/I has finite dimension over k. Since R/I is a graded ring with the grading $R/I=\oplus_{\sigma\in G}R_\sigma/I\cap R_\sigma$, and R/I has finite dimension, then $R_\sigma\subseteqq I$ for almost all $\sigma\in G$. Assume that $F=\{\sigma\in G\mid R_\sigma\not\subseteqq I\}$, then F is finite. If $f\in C^*$ is an arbitrary element of C^* then we define

$$(*)\quad fm=\sum_{\sigma\in F} f_\sigma m\,.$$

If $g\in C^*$ is another element, we observe that $g_\sigma f_\tau\in I$ if $\sigma\notin F$ or $\tau\notin I$ (I is twosided). Hence $(gf)m=\sum_{\substack{\sigma\in F\\ \tau\in F}}(g_\sigma f_\sigma)m$. On the other hand, since $f_\sigma m\in Rm$, then $I\subseteqq Ann(Rf_\sigma m)$. Hence $g(fm)=(\sum_{\sigma\in F} g_\sigma)fm=(\sum_{\sigma\in F} g_\sigma)(\sum_{\sigma\in F} f_\sigma m)$. Therefore $(gf)m=g(fm)$ and M with the multiplication given in (*) is a C^*-module. Clearly M is also a rational C^*-module.

3) If C^* is a G-graded coalgebra, then C is, in particular, a right graded C-comodule. Exactly as in ([3] Corollary 1) ${}_RC$, as graded R-module, is injective in **Rat**$(R-gr)$. Moreover the family $\{C(\sigma)\}_{\sigma\in G}$ is a family of injective cogenerators in **Rat**$(R-gr)$, then $\oplus_{\sigma\in G}C(\sigma)$ is an injective cogenerator of **Rat**$(R-gr)$ (or $\oplus_{\sigma\in G}C(\sigma)$ is an injective cogenerator in gr^C). In fact **Rat**$(R-gr)$ $=\sigma_{R-gr}[\oplus_{\sigma\in G}C(\sigma)]$.

4) Let $M=\oplus_{\sigma\in G}M_\sigma$ and $N=\oplus_{\sigma\in G}N_\sigma$ two objects in gr^C. Then for any $\sigma\in G$ we can define

$$COM(M, N)_\sigma=\{f\in Com_C(M, N)\mid f(M_\lambda)\subseteqq N_{\sigma\lambda},\ \text{for all } \lambda\in G\}.$$

Clearly $COM(M, N)_\sigma$ is a subgroup of $Com_C(M, N)$ and the sum

$$\sum_{\sigma\in G} COM(M, N)_\sigma$$

is direct. We denote by

$$COM(M, N)=\sum_{\sigma\in G} COM(M, N)_\sigma=\oplus_{\sigma\in G}COM(M, N)_\sigma\,.$$

$COM(M, N)$ is subgroup of $Com_C(M, N)$. We observe that $COM(M, N)_1=Com_{gr^C}(M, N)$. If M (resp. N) are the left graded R-modules associated to M and N, then it is easy to see that

$$COM(M, N)_\sigma = HOM_R(\overline{M}, \overline{N})_{\sigma^{-1}}.$$

Now by ([7]), we have the following result: If M has finite dimension over k or the group is finite then $COM(M, N) = Com_C(M, N)$.

7. Examples

1) If k is a field and G a finite group then we note by kG the free k-module generated by G. If we define the k-linear maps

$$\Delta: kG \longrightarrow kG \otimes kG, \qquad \varepsilon: kG \longrightarrow k$$

such that $\Delta(g) = \sum_{h \in G} gh^{-1} \otimes h$ and $\varepsilon(g) = \delta_{g,1}$ where $\delta_{g,1}$ is the Kronecker symbol, then $(kG, \Delta, \varepsilon)$ is G-graded coalgebra with the grading $(kG)_g = kg$, for all $g \in G$. Here the graded ring associated to the coalgebra $(kG, \Delta, \varepsilon)$ is exactly the group ring $k[G]$ with the natural grading. Here $(kG, \Delta, \varepsilon)$ is a strongly graded coalgebra.

2) If $S = \{c_0, c_1, c_2, \cdots\}$, we set $C = kS$, where kS is the free k-module generated by S. We define $\Delta: kS \to kS \otimes kS$ such that $\Delta(c_n) = \sum_{j=0}^{n} c_i \otimes c_{n-i}$, $\varepsilon(c_n) = \delta_{0,n}$. $C = (kS, \Delta, \varepsilon)$ is $\mathbb{Z}$-graded coalgebra where the grading is given by the equality $C_n = ks_n$ if $n \geqq 0$ and $C_n = 0$ if $n < 0$. It is well known (see [1], pag. 56) that the ring C^* is the ring of power series in one variable over k. On the other hand it is easy to see that the graded ring associated to a graded coalgebra C is exactly the ring of polynomes in one variable over k with the natural grading. Clearly $dim_k(C_n) = 1$ but gr^C is not isomorphic to $R-gr$, when $R = k[x]$ is the graded ring associated to the graded coalgebra C.

3) If V is a vector space, then the symmetric k-algebra $S(V)$ over the k-linear space V is a $\mathbb{Z}$-graded coalgebra (in fact is a $\mathbb{Z}$-graded Hopf algebra) (see Example 2.10 [1], page 92).

4) Let $R = \bigoplus_{\sigma \in G} R_\sigma$ be a G-graded k-algebra (assume here that the group G is finite). Then $R^* = \bigoplus_{\sigma \in G} R_\sigma^*$. Let R^o be the dual k-coalgebra

$$R^o = \{f \in R^* \mid Ker\ f \text{ contains a finite codimensional ideal of } R\}$$

$$= \{f \in R^* \mid \text{there exists an ideal } I \subseteqq R,\ I \subseteqq Ker\ f \text{ and } dim_k(R/I) < \infty\}$$

If I is an ideal of R, then $(I)_g = \sum_{\sigma \in G}(R_\sigma \cap I)$ is a graded ideal of R and $(I)_g$ is the greatest graded ideal of R contained in I. Since we have the monomorphism $0 \to R_\sigma / R_\sigma \cap I \to R/I$ for any $\sigma \in G$, then $dim_k(R_\sigma / R_\sigma \cap I) < \infty$. On the other hand $R/(I)_g \cong \bigoplus_{\sigma \in G} R_\sigma / (R_\sigma \cap I)$ and therefore $dim_k(R/I)_g) < \infty$. Hence

$$R^o = \{f \in R^* \mid Ker\ f \text{ contains a finite codimensional graded ideal of } R\}.$$

If for any $\sigma \in G$ we put

$$(R^o)_\sigma = \{f \in R^o \mid f(R_\tau) = 0 \forall \tau \neq \sigma\}$$

Clearly we have $R^o = \bigoplus_{\sigma \in G}(R^o)_\sigma$. It is easy to see that R^o is a G-graded coalgebra. Moreover if the ring R is strongly graded, then R^o is strongly gɹaded coalgebra.

References

[1] E. Abe, Hopf algebras, Cambridge Univ. Press, 1977.
[2] E.C. Dade, Group-Graded Rings and Modules, Math. Z. **174** (1980), 241-262.
[3] Y. Doi, Homological coalgebra, J. Math. Soc. Japan **33** (1981), 31-50.
[4] P. Gabriel, Des catégories abeliennes, Bull. Soc. Math. France **90** (1962), 323-448.
[5] R.G. Heyneman and D.E. Radford, Reflexivity and Coalgebras of Finite Type, J. of Algebra **28** (1974), 215-246.
[6] B. Mitchell, Theory of Categories, Academic Press, Pure and Appl. Math., 1965.
[7] Năstăsescu and F. Van Oystaeyen, Graded Ring Theory, North-Holland, Amsterdam, 1982.
[8] I-P. Lin, Semiperfect coalgebras, J. of Algebra **49** (1977), 357-373.
[9] D.E. Radford, Coreflexive coalgebras, J. of Algebra **26** (1973), 512-535.
[10] M.E. Sweedler, Hopf Algebras, Benjamin, New York, 1969.
[11] Yakeuchi, A note on geometrically reductive groups, J. Fac. Sci. Univ. Tokyo **20** (1973), 387-396.

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