

ON THE EXCEPTIONAL SET IN GOLDBACH'S PROBLEM

By

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1. Introduction.

Let $E(x)$ denote the number of even integers not exceeding x that are not representable as a sum of two primes. The Goldbach conjecture asserts $E(x) \ll 1$. Unfortunately this is far from our reach. In 1923 G.H. Hardy and J.E. Littlewood [4] showed, on assuming the extended Riemann hypothesis, that

$$(1) \quad E(x) \ll x^{1/2+\varepsilon}$$

for any $\varepsilon > 0$. After the fundamental work of I.M. Vinogradov [18], several authors have unconditionally given the non-trivial bounds for $E(x)$. The best one of these is due to H.L. Montgomery and R.C. Vaughan. In 1975 they [13] showed that there exists a positive constant δ such that

$$E(x) \ll x^{1-\delta}.$$

Chen J.-r. [2] gave an explicit value of δ , which is very small.

In 1973 K. Ramachandra [16] proved that, for any $A > 0$,

$$(2) \quad E(x+x^\theta) - E(x) \ll x^\theta (\log x)^{-A}$$

providing

$$(3) \quad \frac{7}{12} < \theta \leq 1.$$

This bound $7/12$ comes from a zero density estimate for the Dirichlet L -series. In 1981 Lou S.-t. and Yao Q. [9] attempted to sharpen the inequality (2). Later Yao [20] replaced, in the same range of θ as (3), the right hand side of (2) by $x^{\theta(1-\delta)}$ with some $\delta > 0$.

It is of some interest, from the point of view of (1), to demonstrate the formula (2) for θ less than $1/2$. We shall present such a result.

THEOREM. *Let $A > 0$ and $7/48 < \theta \leq 1$ be given. Then we have*

$$E(x+x^\theta) - E(x) \ll x^\theta (\log x)^{-A}$$

where the implied constant depends only on A and θ .

This work is inspired by the article of D. Wolke [19]. On combining the argument of [16] with the result of [19], one may prove our assertion for $\theta > (7/12) \cdot (5/8)$, which is less than $1/2$ surely. We appeal to the device of H. Iwaniec and M. Jutila [7]. See also [14]. Using the method in our previous papers [10, 11], we then conclude the bound $\theta > (7/12) \cdot (1/4)$.

Our notation is standard or self-explanatory. For a real number t , we write $\phi(x) = [x] - x + 1/2$, $e(x) = e^{2\pi i x}$ and $\|x\| = \min_{n \in \mathbb{Z}} |x - n|$. The convention $n \sim N$ means that $N < n \leq N' \leq 2N$. ρ in $\sum_{\rho(\chi)}$ runs through the set of non-trivial zeros of $L(s, \chi)$. c_i 's denote certain positive absolute constants. For simplicity, we write $\mathcal{L} = \log x$. F in \mathcal{L}^F stands for a positive numerical constant, which is not the same at each occurrence.

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2. Lemmas.

LEMMA 1. *Let $1 \leq a \leq q$, $(a, q) = 1$. The Hurwitz zeta-function $\zeta(s, a/q)$ is regular, except for a simple pole at $s=1$, of residue 1. Also, it satisfies the growth conditions:*

$$\zeta\left(s, \frac{a}{q}\right) \ll \begin{cases} q + \exp(c_1(\log \log x)^3); & 1 - c_2(\log x)^{-4/5} < \operatorname{Re}(s) \leq 1 + (\log x)^{-1} \\ & |\operatorname{Im}(s)| \leq x \\ & |s-1| \geq (\log x)^{-1} \\ q^{1/2} + x^{1/6}(\log x); & \operatorname{Re}(s) = 1/2 \\ & |\operatorname{Im}(s)| \leq x \\ qx^{(1-\operatorname{Re}(s))/3}(\log x); & 1/2 \leq \operatorname{Re}(s) \leq 1 + (\log x)^{-1} \\ & c_3 \leq |\operatorname{Im}(s)| \leq x. \end{cases}$$

LEMMA 2. *Let $N(\sigma, T, \chi)$ denote the number of zeros of $L(s, \chi)$ in the rectangle; $\operatorname{Re}(s) \geq \sigma$, $|\operatorname{Im}(s)| \leq T$. Let $A > 0$ be given. If $q \leq (\log T)^A$ and $\sigma \geq 1 - c_4(\log T)^{-4/5}$ then*

$$\sum_{\chi \pmod{q}} N(\sigma, T, \chi) = 0.$$

For Lemmas 1 and 2, see [15, Kap. VIII Satz 5.3. Kap. IV Satz 5.3. Kap. IX Satz 4.2. Anhang Satz 9.1. Kap. VIII Satz 6.2.]. The following Lemma 3 is due to H.L. Montgomery [12, Theorem 12.1] and M.N. Huxley [5]. Lemma

4 follows from [12, Theorem 7.6] immediately.

LEMMA 3. *Let $1/2 \leq \sigma \leq 1$. For any $\varepsilon > 0$,*

$$\sum_{\chi \pmod{q}} N(\sigma, T, \chi) \ll (qT)^{(12/5+\varepsilon)(1-\sigma)} (\log qT)^{14}$$

where the implied constant depends only on ε .

LEMMA 4. *For $k=1, 2$,*

$$\sum_{\chi \pmod{q}} \sum_{\substack{\rho(n) \\ \operatorname{Re}(\rho) \geq \sigma \\ |\operatorname{Im}(\rho)| \leq T}} \left| \sum_{n \sim N} \chi(n) \Lambda(n) n^{-\rho} \right|^{2k} \ll (qT + N^k) N^{k(1-2\sigma)} (\log qN)^{12}.$$

Let Ω be a non-negative arithmetical function with a compact support, and \mathcal{P} a set of primes. For $z > 2$, put

$$S(\Omega, \mathcal{P}, z) = \sum_{(a, P(z))=1} \Omega(a)$$

where

$$P(z) = \prod_{\substack{p \leq z \\ p \in \mathcal{P}}} p.$$

Suppose that, for a square-free d composed by primes $\in \mathcal{P}$,

$$\sum_{a \equiv 0 \pmod{d}} \Omega(a) = \frac{\omega(d)}{d} X + r(\Omega, d)$$

where X is an approximation of $\sum \Omega(a)$ and $\omega(d)$ is a multiplicative function which satisfies some conditions of regularity. Write

$$V(z) = \prod_{\substack{p \leq z \\ p \in \mathcal{P}}} \left(1 - \frac{\omega(p)}{p} \right).$$

LEMMA 5. *Let $z, D > 2$ and $s = \log D / \log z$. For $2 \leq s \leq 4$ we have*

$$S(\Omega, \mathcal{P}, z) \geq V(z) X \left\{ \frac{2e^\gamma \log(s-1)}{s} + O((\log \log D)^{-1}) \right\} + R,$$

where γ is the Euler constant and the remainder term R has the form

$$R = \sum_{d | P(z)} \lambda_d(D, z) r(\Omega, d).$$

Here, the weights $(\lambda_d) = (\lambda_d(D, z))$ satisfy that

$$\lambda_d = 0 \quad \text{if } d \geq D,$$

$$|\lambda_d| \leq \mu^2(d).$$

Moreover, on writing $z_i = \exp((\log D)(\log \log D)^{-i/10})$, if $\lambda_d \neq 0$ for $d > z_1 z$ then λ_d is decomposed into the shape

$$(2.1) \quad \lambda_d = \sum_{h \leq \log D} \sum_{\substack{m \\ d=m p}} \sum_{P_h \leq p < P_{h'}} \alpha(h, m)$$

where

$$z_2 \leq P_h < D^{1/4}, \quad P_{h'} = P_h z_2, \quad \text{and} \quad |\alpha(h, m)| \leq 1.$$

This is the Rosser-Iwaniec linear sieve [6, 14]. We explain the decomposition (2.1). It follows from [6, 14] that if $\lambda_d \neq 0$ then d is the form

$$d = \nu \text{ or } \nu p_1 \cdots p_r.$$

Here,

$$\begin{aligned} \nu &\leq z_1 & D_i &\leq p_i < D_i z_2 \\ \nu &| P(z_2) & p_i &| P(z)/P(z_2), \end{aligned}$$

(D_1, \dots, D_r) is a subsequence of $(z_2 z_2^n)_{n \geq 0}$ such that

$$D_1 \geq \dots \geq D_r,$$

$$D_1 \cdots D_{2l-1} D_{2l}^3 < D \quad \text{for all } 1 \leq l \leq r/2.$$

Moreover, the coefficient of d depends only on ν and (D_1, \dots, D_r) . And, the number of (D_1, \dots, D_r) 's is at most $\log D$.

Now, if $d > z_1 z$ then $z_1 z < d = \nu p_1 \cdots p_r \leq z_1 z^r$, whence $r \geq 2$. Thus, $D_2^4 \leq D_1 D_2^3 < D$. Since $D_i \geq z_2$, we obtain (2.1).

Next lemma is the combinatorial identity of R.C. Vaughan [17].

LEMMA 6. *If $Y < l \leq XY$ then*

$$A(l) = \sum_{\substack{l=m n \\ n > Y}} \left(\sum_{\substack{d | m \\ d \leq X}} \mu(d) \right) A(n).$$

LEMMA 7. *Let $1 < \Delta < x/2$. For arbitrary complex numbers a_n , put*

$$\mathcal{F} = \left(\frac{\Delta}{2} \right)^2 \int_{-1/2\Delta}^{1/2\Delta} \left| \sum_{x < n \leq 2x} a_n e(\beta n) \right|^2 d\beta.$$

Then we have

$$\mathcal{F} \ll \int_x^{2x} \left| \sum_{t < n \leq t+\Delta} a_n \right|^2 dt + \Delta^3 \left(\sup_{x < n \leq 2x} |a_n| \right)^2.$$

LEMMA 8. *Under the assumption of Lemma 7, we have*

$$\begin{aligned} \mathcal{F} &\leq \Delta \sum_{x < n \leq 2x} |a_n|^2 + 2 \operatorname{Re} \sum_{0 < r \leq \Delta} (\Delta - r) \sum_{\substack{x < m, n \leq 2x \\ m-n=r}} a_m \bar{a}_n \\ &\ll \Delta \sum_{x < n \leq 2x} |a_n|^2 + \Delta \sup_{t \leq \Delta} \left| \sum_{0 < r \leq t} \sum_{x < n \leq 2x} \bar{a}_n a_{n+r} \right| + \Delta^3 \left(\sup_{x < n \leq 2x} |a_n| \right)^2. \end{aligned}$$

Lemma 7 is [3, Lemma 1] and Lemma 8 follows from a familiar Fourier

integral. We, however, reproduce the proof. It is well known that

$$\max(0, \Delta - |x|) = \int_{-\infty}^{\infty} K(y, \Delta) e(xy) dy$$

where

$$K(y, \Delta) = \left(\frac{\sin \pi \Delta y}{\pi y} \right)^2.$$

Put

$$\mathcal{D} = \sum_{x < m, n \leq 2x} a_m \bar{a}_n \max(0, \Delta - |m - n|),$$

$$\mathcal{S} = \int_x^{2x} \left| \sum_{t < n \leq t + \Delta} a_n \right|^2 dt,$$

and

$$R = \Delta^3 \left(\sup_{x < n \leq 2x} |a_n| \right)^2.$$

We shall show $\mathcal{S} \geq \mathcal{D} + O(R)$ and $\mathcal{D} \geq \mathcal{F}$, from which Lemmas follow immediately. Now,

$$\mathcal{S} \geq \int_x^{2x - \Delta} = \sum_{x < m, n \leq 2x} a_m \bar{a}_n \text{meas.} \left\{ t : \begin{array}{l} x \leq t \leq 2x - \Delta \\ m - \Delta \leq t < m \\ n - \Delta \leq t < n \end{array} \right\}.$$

We note that $\text{meas.} \{t\} = 0$ for $|m - n| \geq \Delta$. If $x + \Delta < m, n \leq 2x - \Delta$ then $\text{meas.} \{t\} = \Delta - |m - n|$. Otherwise, the number of (m, n) is $O(\Delta^2)$. Also, $\text{meas.} \{t\} \leq \Delta$ trivially. We thus have

$$\begin{aligned} \mathcal{S} &\geq \sum_{\substack{x + \Delta < m, n \leq 2x - \Delta \\ |m - n| \leq \Delta}} a_m \bar{a}_n (\Delta - |m - n|) + O(R) \\ &= \sum_{x < m, n \leq 2x} a_m \bar{a}_n \max(0, \Delta - |m - n|) + O(R) \\ &= \mathcal{D} + O(R). \end{aligned}$$

Moreover,

$$\begin{aligned} \mathcal{D} &= \int_{-\infty}^{\infty} K(\beta, \Delta) \left| \sum_{x < n \leq 2x} a_n e(\beta n) \right|^2 d\beta \\ &\geq \int_{-1/2\Delta}^{1/2\Delta} \left(\frac{\Delta}{2} \right)^2 \left| \sum_{x < n \leq 2x} a_n e(\beta n) \right|^2 d\beta \\ &= \mathcal{F}, \end{aligned}$$

since $K(y, \Delta) \geq (\Delta/2)^2$ for $|y| \leq 1/2\Delta$.

\mathcal{F} reminds us of Circle method or Saddle point method. Also \mathcal{D} , Dispersion method or Kloostermania. \mathcal{S} , Large sieve or Dirichlet polynomial methods. This observation makes a feature of our argument below.

3. Proof of Theorem.

Let x be a sufficiently large parameter. We take $y=Y^{\theta_1}(1/4<\theta_1<1/2)$ and $Y=x^{\theta_2}(7/12<\theta_2<2/3)$, so that $y=x^\theta$, $7/48<\theta<1/3$. Put $\mathcal{K}=(x, x+y]$, $\mathcal{A}=(2Y, 4Y]$ and $\mathcal{B}=(x-3Y, x-2Y]$. Our aim is to give a lower bound of the sum

$$g=g(2k)=\sum_{\substack{2k=p+p' \\ p \in \mathcal{B}}} \log p,$$

for $2k \in \mathcal{K}$. Moreover, define

$$S=S(2k, z)=\sum_{\substack{2k=p+n \\ p \in \mathcal{B} \\ (n, \mathcal{P}(2k, z))=1}} \log p$$

where

$$\mathcal{P}(2k, z)=\prod_{\substack{p < z \\ p \nmid 2k}} p.$$

Suppose that

$$(3.1) \quad (4Y)^{1/3} < z < Y^{1/2}.$$

Since $2k \in \mathcal{K}$ and $p \in \mathcal{B}$ imply $n=2k-p \in \mathcal{A}$, n counted by S has at most two prime factors $\geq z$. We thus have

$$S-g=\sum_{\substack{2k=p+n \\ p \in \mathcal{B}}} (\log p) \rho'(n)$$

where

$$\rho'(n)=\sum_{\substack{n=p p' \\ z \leq p \leq p'}} 1.$$

For any $l \in \mathcal{A}$,

$$\begin{aligned} \rho'(l) &\leq \sum_{\substack{l=mn \\ z \leq m \leq n}} \frac{\Lambda(m)\Lambda(n)}{\log m \log n} \\ &\leq \sum_{\substack{l=mn \\ \sqrt{2Y} < n \leq 4Y/z}} \frac{\Lambda(m)\Lambda(n)}{\log m \log n} \\ &\leq \frac{1}{\log(Y/z) \log z} \sum_{\substack{l=mn \\ n \in \mathcal{G}}} \Lambda(m)\Lambda(n) \\ &= \mathcal{E} \rho(l), \quad \text{say.} \end{aligned}$$

Here, $\mathcal{G}=(\sqrt{2Y}, 4Y/z]$. We therefore have our fundamental inequality

$$(3.2) \quad \begin{aligned} S-g &\leq \mathcal{E} \sum_{\substack{2k=p+n \\ p \in \mathcal{B}}} (\log p) \rho(n) \\ &= \Omega, \quad \text{say.} \end{aligned}$$

We proceed to evaluate Ω by means of Circle method. Define

$$Q=yQ_1^{-2}$$

$$Q_1 = (\log Y)^D$$

$$M = \bigcup_{q \leq Q_1} \bigcup_{a=1}^q *I_{q,a} \quad I_{q,a} = \left[\frac{a}{q} - \frac{1}{qQ}, \frac{a}{q} + \frac{1}{qQ} \right]$$

$$m = [1/Q, 1+1/Q] \setminus M,$$

where D is a constant specified later, and $*$ stands for the restriction $(a, q) = 1$. Furthermore,

$$T(\alpha) = \mathfrak{E} \sum_{n \in \mathcal{A}} \rho(n) e(\alpha n),$$

$$W(\alpha) = \sum_{p \in \mathcal{B}} (\log p) e(\alpha p).$$

For $\alpha = a/q + \beta \in I_{q,a}$, we write

$$V(\alpha) = \mathfrak{E} \sum_{n \in \mathcal{A}} \left(\rho(n) e\left(\frac{a}{q} n\right) - \Gamma \frac{\mu(q)}{\varphi(q)} \right) e(\beta n)$$

where

$$\Gamma = \frac{1}{2} \log\left(\frac{8Y}{z^2}\right).$$

Then,

$$\begin{aligned} \Omega &= \int_{1/Q}^{1+1/Q} T(\alpha) W(\alpha) e(-2k\alpha) d\alpha \\ &= \int_M (T(\alpha) - V(\alpha)) W(\alpha) e(-2k\alpha) d\alpha \\ &\quad + \int_M V(\alpha) W(\alpha) e(-2k\alpha) d\alpha \\ &\quad + \int_m T(\alpha) W(\alpha) e(-2k\alpha) d\alpha \\ (3.3) \quad &= \Omega_0 - \mathfrak{R}_1 - \mathfrak{R}_2, \quad \text{say.} \end{aligned}$$

We first consider Ω_0 .

$$\begin{aligned} \Omega_0 &= \sum_{q \leq Q_1} \sum_{a=1}^q * \int_{-1/qQ}^{1/qQ} \mathfrak{E} \Gamma \frac{\mu(q)}{\varphi(q)} \left(\sum_{n \in \mathcal{A}} e(\beta n) \right) \left(\sum_{p \in \mathcal{B}} (\log p) e\left(\left(\frac{a}{q} + \beta\right)p\right) \right) \\ &\quad \times e\left(-2k\left(\frac{a}{q} + \beta\right)\right) d\beta \\ &= \mathfrak{E} \Gamma \sum_{q \leq Q_1} \frac{\mu(q)}{\varphi(q)} \sum_{p \in \mathcal{B}} (\log p) c_q(p-2k) \cdot \sum_{n \in \mathcal{A}} \int_{-1/2}^{1/2} e((n+p-2k)\beta) d\beta \\ &\quad + O\left(\mathfrak{E} \Gamma \sum_{q \leq Q_1} \frac{|\mu(q)|}{\varphi(q)} \sum_{a=1}^q * \int_{1/qQ}^{1/2} \left| \sum_{n \in \mathcal{A}} e(\beta n) \right| \left| W\left(\frac{a}{q} + \beta\right) \right| d\beta\right). \end{aligned}$$

The above O -term is

$$(3.4) \quad \begin{aligned} &\ll \mathcal{L}^{-1} \sum_{q \leq Q_1} (qQ \cdot Y \mathcal{L})^{1/2} \\ &\ll Y^{3/4}. \end{aligned}$$

$2k \in \mathcal{K}$ and $p \in \mathcal{B}$ imply $2k - p \in \mathcal{A}$, namely

$$\sum_{n \in \mathcal{A}} \int_{-1/2}^{1/2} e((n+p-2k)\beta) d\beta = 1.$$

Hence the main contribution of Ω_0 is equal to

$$(3.5) \quad \begin{aligned} &\Xi \Gamma \sum_{q \leq Q_1} \frac{\mu(q)}{\varphi(q)} \sum_{\substack{d|q \\ (d, 2k)=1}} d \mu\left(\frac{q}{d}\right) \sum_{\substack{p \in \mathcal{B} \\ p \equiv 2k(d)}} \log p + O\left(\mathcal{L}^{-1} \sum_{q \leq Q_1} \frac{|\mu(q)|}{\varphi(q)} \sum_{d|q} d \mathcal{L}^2\right) \\ &= \Xi \Gamma \sum_{q \leq Q_1} \frac{\mu(q)}{\varphi(q)} \sum_{\substack{d|q \\ (d, 2k)=1}} d \mu\left(\frac{q}{d}\right) \frac{Y}{\varphi(d)} \\ &\quad + O\left(\mathcal{L}^{-1} \sum_{q \leq Q_1} \frac{|\mu(q)|}{\varphi(q)} \sum_{\substack{d|q \\ (d, 2k)=1}} d \left| \sum_{\substack{p \in \mathcal{B} \\ p \equiv 2k(d)}} \log p - \frac{Y}{\varphi(d)} \right| \right) + O(Q_1 \mathcal{L}^2) \\ &= Y \Xi \Gamma \sum_{q \leq Q_1} \frac{\mu^2(q)}{\varphi^2(q)} c_q(2k) \\ &\quad + O\left(\sum_{\substack{d \leq Q_1 \\ (d, 2k)=1}} \left| \sum_{\substack{x-3Y < p \leq x-2Y \\ p \equiv 2k(d)}} \log p - \frac{Y}{\varphi(d)} \right| + Y^{3/4}\right). \end{aligned}$$

By Lemmas 2 and 3, the above O -term is

$$(3.6) \quad \ll Y \mathcal{L}^{-E},$$

for any $E > 0$. Put

$$(3.7) \quad \mathfrak{N}_0 = \mathfrak{N}_0(2k) = Y \Xi \Gamma \sum_{q > Q_1} \frac{\mu^2(q)}{\varphi^2(q)} c_q(2k).$$

Then, by [15, Kap. VI p. 201], we see

$$(3.8) \quad \sum_{2k \in \mathcal{K}} \mathfrak{N}_0(2k)^2 \ll y Y^2 Q_1^{-1}.$$

In conjunction with (3.4)-(3.7) we obtain

$$(3.9) \quad \Omega_0 = \mathfrak{S}(2k) Y \Xi \Gamma - \mathfrak{N}_0 + O(Y \mathcal{L}^{-3}),$$

where

$$\mathfrak{S}(2k) = 2 \prod_{p > 2} \left(1 - \frac{1}{(p-1)^2}\right) \prod_{\substack{p|k \\ p > 2}} \left(\frac{p-1}{p-2}\right).$$

Next we consider \mathfrak{N}_1 . Let $I_{q', a'}, I_{q, a} \subset \mathcal{M}$, $a'/q' \neq a/q$. If $\alpha' \in I_{q', a'}$ and $\alpha \in I_{q, a}$ then we find

$$\begin{aligned} \|\alpha' - \alpha\| &\geq \left\| \frac{a'}{q'} - \frac{a}{q} \right\| - \left\| \alpha' - \frac{a'}{q'} \right\| - \left\| \alpha - \frac{a}{q} \right\| \\ &\geq \frac{1}{q'q} - \frac{1}{q'Q} - \frac{1}{qQ} \\ &\geq (2Q_1)^{-2}. \end{aligned}$$

We therefore have

$$\begin{aligned} &\sum_{2k \in \mathcal{K}} \mathfrak{R}_1(2k)^2 \\ &\ll \int_M \int_M |V(\alpha')W(\alpha')| |V(\alpha)W(\alpha)| \min\left(y, \frac{1}{\|\alpha' - \alpha\|}\right) d\alpha' d\alpha \\ &\ll Q_1^2 \left(\int_M |V(\alpha)W(\alpha)| d\alpha \right)^2 + y \sum_{q \leq Q_1} \sum_{a=1}^q * \left(\int_{I_{q,a}} |V(\alpha)W(\alpha)| d\alpha \right)^2 \\ &\ll Q_1^2 \int_M |V(\alpha)|^2 d\alpha \int_M |W(\alpha)|^2 d\alpha \\ &\quad + y \sup_{\substack{(\alpha, q)=1 \\ q \leq Q_1}} \int_{I_{q,a}} |V(\alpha)W(\alpha)| d\alpha \cdot \int_M |V(\alpha)W(\alpha)| d\alpha \\ &\ll Q_1^2 Y^2 \mathcal{L} + y Y^{3/2} \mathcal{L} \sup_{\substack{(\alpha, q)=1 \\ q \leq Q_1}} \left(\int_{I_{q,a}} |V(\alpha)|^2 d\alpha \right)^{1/2}, \end{aligned}$$

by Cauchy's inequality. Here we appeal to the following lemma. We shall prove Lemma 9 in the next section.

LEMMA 9. *Let $\varepsilon, B, E > 0$ be given. If $Y^{1/4+\varepsilon} \leq \Delta \leq Y^{1/2}$ and $4Y/z \leq Y^{8/15}$ then*

$$\Delta^2 \int_{-1/2\Delta}^{1/2\Delta} \left| V\left(\frac{a}{q} + \beta\right) \right|^2 d\beta \ll \Delta^2 Y \mathcal{L}^{-E}$$

uniformly for $(a, q) = 1$ and $q \leq (\log Y)^B$. Here the implied constant depends only on ε, B and E .

Now, we take $2\Delta = Q$ and

$$(3.10) \quad z = Y^{7/15+\delta}$$

with a sufficiently small $\delta > 0$. Then this choice of z satisfies the assumption of Lemma 9, also (3.1). We thus obtain

$$(3.11) \quad \sum_{2k \in \mathcal{K}} \mathfrak{R}_1(2k)^2 \ll y Y^2 \mathcal{L}^{-E}$$

for any $E > 0$.

We turn to \mathfrak{R}_2 . For $2\Delta < y$, we have

$$\begin{aligned}
 \sum_{2k \in \mathcal{K}} \mathfrak{R}_2(2k)^2 &\ll \int_{\mathfrak{m}} \int_{\mathfrak{m}} |T(\alpha')W(\alpha')| |T(\alpha)W(\alpha)| \min\left(y, \frac{1}{\|\alpha' - \alpha\|}\right) d\alpha' d\alpha \\
 &\ll \Delta \left(\int_{\mathfrak{m}} |T(\alpha)W(\alpha)| d\alpha \right)^2 \\
 &\quad + y \int_{\mathfrak{m}} |T(\alpha)W(\alpha)| \left(\int_{\substack{\mathfrak{m} \\ \|\alpha' - \alpha\| \leq 1/2\Delta}} |T(\alpha')W(\alpha')| d\alpha' \right) d\alpha \\
 &\ll \Delta \int_{\mathfrak{m}} |T(\alpha)|^2 d\alpha \int_{\mathfrak{m}} |W(\alpha)|^2 d\alpha \\
 &\quad + y \left(\sup_{\substack{\alpha \in \mathfrak{m} \\ \|\alpha' - \alpha\| \leq 1/2\Delta}} \int_{\mathfrak{m}} |T(\alpha')|^2 d\alpha' \right)^{1/2} \left(\int_{\mathfrak{m}} |T(\alpha)|^2 d\alpha \right)^{1/2} \int_{\mathfrak{m}} |W(\alpha)|^2 d\alpha \\
 (3.12) \quad &\ll \Delta Y^2 \mathcal{L}^4 + y Y^{3/2} \mathcal{L}^3 \sup_{\alpha \in \mathfrak{m}} \left(\int_{-1/2\Delta}^{1/2\Delta} |T(\alpha + \beta)|^2 d\beta \right)^{1/2},
 \end{aligned}$$

since $T(\alpha)$ has the period 1.

We now use the following lemma. We postpone the proof of Lemma 10 until the final section.

LEMMA 10. *Let $1 < \Delta < Y^{1/2}$. Suppose that $|\alpha - a/q| \leq 1/q^2$ with $(a, q) = 1$. Let $E, \varepsilon > 0$ be given. If $4Y/z \leq Y^{8/15-\varepsilon}$ then*

$$\Delta^2 \int_{-1/2\Delta}^{1/2\Delta} |T(\alpha + \beta)|^2 d\beta \ll \Delta Y \mathcal{L}^F \{Y^{1/4} + \Delta q^{-1/2} + (q\Delta)^{1/2}\} + \Delta^2 Y \mathcal{L}^{-E}$$

where the implied constant depends only on E and ε .

We take $\Delta = yQ_1^{-1} = QQ_1$. The assumption of Lemma 10 is satisfied, because of (3.10). For any $\alpha \in \mathfrak{m}$ there exist a and q such that $|\alpha - a/q| \leq 1/q^2$, $(a, q) = 1$ and $Q_1 < q \leq Q$. Thus, Lemma 10 yields

$$\begin{aligned}
 \sup_{\alpha \in \mathfrak{m}} \Delta^2 \int_{-1/2\Delta}^{1/2\Delta} |T(\alpha + \beta)|^2 d\beta &\ll \sup_{Q_1 < q \leq Q} \Delta Y \mathcal{L}^F \{Y^{1/4} + \Delta q^{-1/2} + (q\Delta)^{1/2}\} + \Delta^2 Y \mathcal{L}^{-E} \\
 &\ll \Delta^2 Y \mathcal{L}^F Q_1^{-1/2}.
 \end{aligned}$$

Combining this with (3.12) we obtain

$$(3.13) \quad \sum_{2k \in \mathcal{K}} \mathfrak{R}_2(2k)^2 \ll y Y^2 \mathcal{L}^F Q_1^{-1/4}.$$

For any given $E > 0$, we take $D = 4(E + F)$. Then (3.8), (3.11) and (3.13) become

$$(3.14) \quad \sum_{i=0}^2 \sum_{2k \in \mathcal{K}} \mathfrak{R}_i(2k)^2 \ll y Y^2 \mathcal{L}^{-E}.$$

Finally we calculate $\mathcal{E}F$. Because of (3.10),

$$\begin{aligned}
 (\log Y) \mathcal{E}I &= \frac{(1/2) \log(8Y/z^2) \cdot (\log Y)}{\log(Y/z) \cdot \log z} \\
 &\leq \frac{(1/2) \cdot (1/15)}{(8/15) \cdot (7/15)} + \delta_1 \\
 (3.15) \quad &< \frac{1}{7} + \delta_1,
 \end{aligned}$$

for sufficiently large Y and small δ_1 . In conjunction with (3.3), (3.9), (3.14) and (3.15) we therefore have

PROPOSITION 1.

$$\Omega \leq \left(\frac{1}{7} + \delta_1\right) \mathfrak{S}(2k) \frac{Y}{\log Y} - \sum_{i=0}^2 \mathfrak{R}_i + O(Y(\log Y)^{-3}).$$

Here, \mathfrak{R}_i 's satisfy (3.14).

In the next stage of the proof we use Sieve method. We remember

$$\mathcal{S} = \sum_{\substack{2k=p+n \\ p \in \mathcal{B} \\ (n, \mathcal{P}(2k, z))=1}} \log p = \sum_{\substack{p \in \mathcal{B} \\ (2k-p, \mathcal{P}(2k, z))=1}} \log p.$$

For d with $(d, 2k)=1$, put

$$\begin{aligned}
 r(\mathcal{B}, d, 2k) &= \sum_{\substack{p \in \mathcal{B} \\ p=2k(d)}} \log p - \frac{Y}{\varphi(d)} \\
 &= \sum_{\substack{2k=p+n \\ p \in \mathcal{B} \\ d|n}} \log p - \frac{Y}{\varphi(d)}.
 \end{aligned}$$

Then Lemma 5 yields, on taking $D=Y^{1-\delta} > z^2$,

$$\begin{aligned}
 \mathcal{S} &\geq Y \prod_{\substack{p < z \\ p \nmid 2k}} \left(1 - \frac{1}{\varphi(p)}\right) \left\{ f\left(\frac{\log D}{\log z}\right) + O((\log \mathcal{L})^{-1}) \right\} \\
 &\quad + \sum_{\substack{(d, 2k)=1 \\ d < D}} \lambda_d r(\mathcal{B}, d, 2k) \\
 (3.16) \quad &= \mathcal{I} + \Sigma, \text{ say.}
 \end{aligned}$$

We first consider Σ .

$$\begin{aligned}
 \Sigma &= \sum_{\substack{(d, 2k)=1 \\ d < D}} \lambda_d \sum_{\substack{p \in \mathcal{B} \\ p=2k(d)}} \log p - Y \sum_{\substack{(d, 2k)=1 \\ d < D}} \frac{\lambda_d}{\varphi(d)} \\
 &= \sum_{d < D} \lambda_d \sum_{\substack{2k=p+n \\ p \in \mathcal{B} \\ d|n}} \log p - Y \sum_{d < D} \frac{\lambda_d}{d} \sum_{q|d} \frac{\mu(q)}{\varphi(q)} c_q(2k) \\
 &\quad + O\left(\sum_{\substack{(d, 2k) > 1 \\ d < D}} |\lambda_d| \sum_{\substack{p \in \mathcal{B} \\ p=2k(d)}} \log p \right)
 \end{aligned}$$

$$\begin{aligned}
&= \sum_{\substack{2k=p+n \\ p \in \mathcal{B}}} (\log p) \left(\sum_{\substack{d|n \\ d < D}} \lambda_d \right) - Y \sum_{q \leq Q_1} \frac{\mu(q)}{\varphi(q)} \left(\sum_{\substack{d < n \\ q|d}} \frac{\lambda_d}{d} \right) c_q(2k) \\
&\quad + O\left(D \mathcal{L}^2 + Y \mathcal{L} \sum_{q_1 < q < D} \frac{|\mu(q)|}{\varphi(q)q} |c_q(2k)| \right) \\
(3.17) \quad &= Z - Z_1 + \mathfrak{R}_3, \text{ say.}
\end{aligned}$$

By [15, Kap. VI], as before, we see

$$(3.18) \quad \sum_{2k \in \mathcal{B}} \mathfrak{R}_3(2k)^2 \ll y Y^2 \mathcal{L}^4 Q_1^{-1}.$$

We next appeal to Circle method. Put

$$S(\alpha) = \sum_{n \in \mathcal{A}} \left(\sum_{\substack{d|n \\ d < D}} \lambda_d \right) e(\alpha n).$$

For $\alpha = a/q + \beta \in I_{q,a}$, we write

$$U(\alpha) = \sum_{n \in \mathcal{A}} \left(\left(\sum_{\substack{d|n \\ d < D}} \lambda_d \right) e\left(\frac{a}{q}n\right) - \left(\sum_{\substack{d < D \\ q|d}} \frac{\lambda_d}{d} \right) \right) e(\beta n).$$

We then have

$$\begin{aligned}
Z &= \int_{1/Q}^{1+1/Q} S(\alpha) W(\alpha) e(-2k\alpha) d\alpha \\
&= \int_{\mathcal{M}} (S(\alpha) - U(\alpha)) W(\alpha) e(-2k\alpha) d\alpha \\
&\quad + \int_{\mathcal{M}} U(\alpha) W(\alpha) e(-2k\alpha) d\alpha \\
&\quad + \int_{\mathcal{m}} S(\alpha) W(\alpha) e(-2k\alpha) d\alpha \\
(3.19) \quad &= Z_0 + \mathfrak{R}_4 + \mathfrak{R}_5, \text{ say.}
\end{aligned}$$

By the argument similar to that for Ω , we infer that

$$\begin{aligned}
(3.20) \quad Z_0 &= Z_1 + O(Y \mathcal{L}^{-3}), \\
\sum_{2k \in \mathcal{K}} \mathfrak{R}_4(2k)^2 &\ll Q_1^2 Y^2 \mathcal{L}^4 + y Y^{3/2} \mathcal{L}^3 \sup_{\substack{(\alpha, q)=1 \\ q \leq Q_1}} \left(\int_{I_{q,a}} |U(\alpha)|^2 d\alpha \right)^{1/2}, \\
\sum_{2k \in \mathcal{K}} \mathfrak{R}_5(2k)^2 &\ll \Delta Y^2 \mathcal{L}^4 + y Y^{3/2} \mathcal{L}^3 \sup_{\alpha \in \mathfrak{m}} \left(\int_{-1/2d}^{1/2d} |S(\alpha + \beta)|^2 d\beta \right)^{1/2}.
\end{aligned}$$

We now use the following lemmas, which will be proven later.

LEMMA 11. *In Lemma 9, replace the sum V and the condition $4Y/z \leq Y^{8/15}$ by U and $D \leq Y^{1-\epsilon}$, respectively. Then, the resulting assertion holds true.*

LEMMA 12. *In Lemma 10, replace the sum T and the condition $4Y/z \leq Y^{8/15-\epsilon}$*

by S and $D \leq Y^{1-\varepsilon}$, respectively. Then, the resulting assertion holds true.

On taking the parameters as before, we find

$$(3.21) \quad \sum_{i=3}^5 \sum_{2k \in \mathcal{K}} \mathfrak{R}_i(2k)^2 \ll yY^2 \mathcal{L}^{-E},$$

for any $E > 0$. In conjunction with (3.17)-(3.20) we obtain

$$(3.22) \quad \Sigma = \sum_{i=3}^5 \mathfrak{R}_i + O(Y \mathcal{L}^{-3}).$$

We turn now to Π . By Mertens' theorem,

$$\begin{aligned} \prod_{\substack{p < z \\ p \nmid 2k}} \left(1 - \frac{1}{\varphi(p)}\right) &= \prod_{2 < p < z} \left(\frac{p-1}{p} \frac{p(p-2)}{(p-1)^2}\right) \cdot \prod_{\substack{2 < p < z \\ p \mid k}} \left(\frac{p-1}{p-2}\right) \\ &= \prod_{p < z} \left(1 - \frac{1}{p}\right) \cdot 2 \prod_{2 < p < z} \left(1 - \frac{1}{(p-1)^2}\right) \prod_{\substack{2 < p < z \\ p \mid k}} \left(\frac{p-1}{p-2}\right) \\ &= \prod_{p < z} \left(1 - \frac{1}{p}\right) \cdot 2 \prod_{2 < p < z} \left(1 - \frac{1}{(p-1)^2}\right) \prod_{\substack{2 < p < z \\ p \mid k}} \left(\frac{p-1}{p-2}\right) \\ &= \frac{e^{-\gamma}}{\log z} \mathfrak{S}(2k) (1 + O((\log z)^{-1})). \end{aligned}$$

Thus,

$$(3.23) \quad \begin{aligned} \Pi &= \mathfrak{S}(2k) Y \frac{e^{-\gamma}}{\log z} \frac{2e^{\gamma} \log(\log D / \log z - 1)}{\log D / \log z} (1 + O((\log \log Y)^{-1})) \\ &= \mathfrak{S}(2k) Y \cdot K, \text{ say.} \end{aligned}$$

Since $\log x > 1 - 1/x$ for $x > 1$, we see

$$\begin{aligned} (\log Y)K &= \frac{2 \log(\log D / \log z - 1)}{\log D / \log Y} (1 + O((\log \log Y)^{-1})) \\ &\geq 2 \log\left(\frac{15}{7} - 1\right) - \delta_2 \\ &> \frac{1}{4} - \delta_2, \end{aligned}$$

for sufficiently large Y and small δ_2 . Hence, (3.23) becomes

$$\Pi > \left(\frac{1}{4} - \delta_2\right) \frac{\mathfrak{S}(2k)Y}{\log Y}.$$

Combining this with (3.16) and (3.22) we therefore obtain

PROPOSITION 2.

$$S > \left(\frac{1}{4} - \delta_2\right) \mathfrak{S}(2k) \frac{Y}{\log Y} + \sum_{i=3}^5 \mathfrak{R}_i + O(Y(\log Y)^{-3}).$$

Here \mathfrak{R}_i 's satisfy (3.21).

In conjunction with (3.2), Propositions 1 and 2, we finally have

$$\begin{aligned} \sum_{\substack{2k=p+p' \\ p \in \mathcal{B}}} \log p &> S(2k) - \Omega(2k) \\ &> \left(\frac{1}{4} - \frac{1}{7} - (\delta_1 + \delta_2)\right) \mathfrak{S}(2k) \frac{Y}{\log Y} + \sum_{i=0}^5 \mathfrak{R}_i(2k) + O(Y(\log Y)^{-3}) \\ (3.22) \quad &> \frac{\mathfrak{S}(2k)Y}{10 \log Y} + R(2k), \end{aligned}$$

for sufficiently large Y . Here, $R(2k) = \sum_{i=0}^5 \mathfrak{R}_i(2k)$.

We shall derive Theorem from (3.22), (3.14) and (3.19). We first note $\mathfrak{S}(2k) > c_6$, see [15, Kap. VI]. Let \mathcal{E} denote the exceptional set in Goldbach's problem. For all $2k \in \mathcal{E}$, the left hand side of (3.22) is zero. If $|R(2k)| \leq \mathfrak{S}(2k)Y/11 \log Y$ for some $2k \in \mathcal{K} = (x, x + y]$, then the right hand side of (3.22) is positive, whence $2k \notin \mathcal{E} \cap \mathcal{K}$. Consequently, it follows from (3.14) and (3.19) that, for any $A > 0$,

$$\begin{aligned} \#\mathcal{E} \cap \mathcal{K} \left(\frac{Y}{\log Y}\right)^2 &\ll \sum_{2k \in \mathcal{E} \cap \mathcal{K}} |R(2k)|^2 \\ &\leq \sum_{2k \in \mathcal{K}} |R(2k)|^2 \\ &\ll yY^2 \mathcal{L}^{-(A+2)} \end{aligned}$$

or

$$E(x + y) - E(x) = \#\mathcal{E} \cap \mathcal{K} \ll y \mathcal{L}^{-A},$$

as required.

This completes our proof of Theorem, apart from the verification of Lemmas 9, 10, 11 and 12.

4. Major arc.

Let $\varepsilon', B > 0$ be given. Throughout this section we assume

$$(4.1) \quad x^{1/4+\varepsilon'} \leq \Delta \leq x^{1/2}, \quad (a, q) = 1, \quad q \leq (\log x)^B.$$

We use the convention

$$|f|_i = \int_x^{2x} |f(t, \Delta, q, a)|^i dt, \quad i = 1, 2.$$

We call $f(t, \Delta, q, a)$ "admissible" if for any $E > 0$

$$|f|_2 \ll \Delta^2 x \mathcal{L}^{-E}$$

where the implied constant depends only on ε' , B and E . An admissible function is abbreviated to "A.R." in a formula. Under the assumption (4.1) we shall show Lemmas 9[†] and 11[†] below. By Lemma 7, Lemmas 9 and 11 follow from Lemmas 9[†] and 11[†], respectively.

LEMMA 9[†]. Define

$$v = v(t, \Delta, q, a; M, N) = \sum_{\substack{t < mn \leq t + \Delta \\ M < n \leq N}} \Lambda(m) \Lambda(n) e\left(\frac{a}{q} mn\right).$$

If $x^{1/2} \leq M < N \leq x^{8/15}$ then

$$v = \frac{\mu(q)}{\varphi(q)} \Delta \log\left(\frac{N}{M}\right) + A. R..$$

PROOF. The terms with $(mn, q) > 1$ contribute to v at most

$$\begin{aligned} \sum_{\substack{(mn, q) > 1 \\ t < mn \leq t + \Delta \\ x^\varepsilon \leq m, n \leq x^{1-\varepsilon}}} \Lambda(m) \Lambda(n) &\ll \sum_{\substack{x^\varepsilon \leq m \leq x^{1-\varepsilon} \\ (m, q) > 1}} \Lambda(m) \left(\frac{\Delta}{m} + 1\right) \mathcal{L} \\ &\ll (\Delta x^{-\varepsilon} + 1) \mathcal{L}^3, \end{aligned}$$

which is admissible trivially. We then have

$$v = \frac{1}{\varphi(q)} \sum_{\chi(q)} \chi(a) \tau(\bar{\chi}) \sum_{\substack{t < mn \leq t + \Delta \\ n \in \mathcal{J}}} \chi(mn) \Lambda(m) \Lambda(n) + A. R..$$

Here $\mathcal{J} = (M, N]$. By the explicit formula [15, Kap. VII Satz 4.4.],

$$\begin{aligned} \sum_{t/n < m \leq (t+\Delta)/n} \chi(m) \Lambda(m) &= E_0 \frac{\Delta}{n} \\ &\quad - \sum_{\substack{\rho(\chi) \\ |\operatorname{Im}(\rho)| \leq x^2}} \frac{(t+\Delta)^\rho - t^\rho}{n^\rho \rho} \\ &\quad + O\left(\frac{x/n}{x^2} (\log qx)^2 + (\log qx) \sum_{j=0,1} \min\left(1, \frac{x/n}{x^2 \|(t+j\Delta)/n\|}\right)\right) \end{aligned}$$

where $E_0 = 1$ when χ is the principal character and $E_0 = 0$ otherwise. Thus,

$$\begin{aligned} v &= \frac{\mu(q)}{\varphi(q)} \Delta \sum_{\substack{n \in \mathcal{J} \\ (n, q) = 1}} \frac{\Lambda(n)}{n} \\ &\quad - \frac{1}{\varphi(q)} \sum_{\chi(q)} \chi(a) \tau(\bar{\chi}) \sum_{\substack{\rho(\chi) \\ |\operatorname{Im}(\rho)| \leq x^2}} \frac{(t+\Delta)^\rho - t^\rho}{\rho} \sum_{n \in \mathcal{J}} \frac{\chi(n) \Lambda(n)}{n^\rho} \\ &\quad + O\left(\sqrt{q} \left(\mathcal{L}^3 x^{-1} + \mathcal{L}^2 \sum_{j=0,1} \sum_{n \in \mathcal{J}} \min\left(1, \frac{1/x}{n \|(t+j\Delta)/n\|}\right)\right)\right) + A. R. \end{aligned}$$

$$\begin{aligned}
 &= \frac{\mu(q)}{\varphi(q)} \Delta \log\left(\frac{N}{M}\right) \\
 &\quad - \frac{1}{\varphi(q)} \sum_{\chi(q)} \chi(a) \tau(\bar{\chi}) \left(\sum_{\substack{\rho(\chi) \\ |\operatorname{Im}(\rho)| \leq x}} + \sum_{\substack{\rho(\chi) \\ x < |\operatorname{Im}(\rho)| \leq x^2}} \right) \theta(\rho) N(\rho, \chi) \\
 &\quad + O\left(\sqrt{q} \mathcal{L}^2 \sum_{j=0,1} \sum_{n \in \mathcal{J}} \min\left(1, \frac{1}{nx \|(t+j\Delta)/n\|}\right)\right) + A. R.
 \end{aligned}$$

(4.2) $= v_0 - (v_1 + v_2) + O(v_3) + A. R.$, say,

where

$$\begin{aligned}
 N(s, \chi) &= \sum_{n \in \mathcal{J}} \chi(n) A(n) n^{-s}, \\
 \theta(s) &= \theta(s, t, \Delta) = \frac{(t+\Delta)^s - t^s}{s}.
 \end{aligned}$$

First we consider v_3 .

$$\begin{aligned}
 |v_3|_1 &\ll \sqrt{q} \mathcal{L}^2 \sum_{n \in \mathcal{J}} \int_x^{3x} \min\left(1, \frac{1}{nx \|t/n\|}\right) dt \\
 &\ll \sqrt{q} \mathcal{L}^2 \sum_{n \in \mathcal{J}} x^{-1} \int_{x/n}^{3x/n} \min\left(nx, \frac{1}{\|u\|}\right) du \\
 &\ll \sqrt{q} \mathcal{L}^4.
 \end{aligned}$$

Since $v_3 \ll \sqrt{q} \mathcal{L}^2 N$ trivially, we have

(4.3) $|v_3|_2 \ll qN \mathcal{L}^6$.

Hence v_3 is admissible.

We proceed to v_2 . We note that the number of ρ with $T < |\operatorname{Im}(\rho)| \leq T+1$ is $O(\log T)$. By partial summation and Lemma 4, we then have

$$\begin{aligned}
 |v_2|_2 &\ll \mathcal{L} \sum_{\chi(q)} \int_x^{2x} \left| \sum_{\substack{\rho(\chi) \\ x < |\operatorname{Im}(\rho)| \leq x^2}} \frac{t^\rho}{\rho} N(\rho, \chi) \right|^2 dt \\
 &\ll \mathcal{L} \sum_{\chi(q)} \sum_{\substack{\rho(\chi) \\ x < |\operatorname{Im}(\rho)| \leq x^2}} \frac{x^{1+2\operatorname{Re}(\rho)}}{|\rho|^2} |N(\rho, \chi)|^2 \sum_{\substack{\rho'(\chi) \\ |\operatorname{Im}(\rho')| \leq x^2}} \frac{1}{1+|\rho'+\bar{\rho}|} \\
 &\ll \mathcal{L}^5 \sup_{\substack{x < U \leq x^2 \\ 0 \leq \sigma \leq 1}} \frac{x^{1+2\sigma}}{U^2} (qU+N) M^{1-2\sigma} \mathcal{L}^{12} \\
 &\ll \mathcal{L}^{17} \sup_{\sigma} \left(qM \left(\frac{x}{N}\right)^{2\sigma} + N \frac{x}{M} \right)^{2\sigma-1} \\
 (4.4) \quad &\ll qx^{3/2} \mathcal{L}^{17}.
 \end{aligned}$$

Because of (4.1), v_2 is also admissible.

We turn now to v_1 . We write $\theta(\rho) = \int_1^{1+\Delta/t} u^{\rho-1} du \cdot t^\rho$ for $|\text{Im}(\rho)| \leq x/\Delta$. On splitting the remaining range of $|\text{Im}(\rho)|$ into intervals $((x/\Delta)2^j, (x/\Delta)2^{j+1}]$, we use Cauchy's inequality. Thus,

$$\begin{aligned} |\sum_{\rho} \theta(\rho) N(\rho, \chi)|_2 &\ll \left(\frac{\Delta}{x}\right)^2 \sup_{1 \leq u \leq 1+\Delta/x} \int_x^{2x} \left| \sum_{\substack{\rho(\chi) \\ |\text{Im}(\rho)| \leq x/\Delta}} u^{\rho-1} t^\rho N(\rho, \chi) \right|^2 dt \\ &\quad + \mathcal{L}^2 \sup_{x/\Delta \leq U \leq x/2} \int_x^{3x} \left| \sum_{\substack{\rho(\chi) \\ U < |\text{Im}(\rho)| \leq 2U}} \frac{t^\rho}{\rho} N(\rho, \chi) \right|^2 dt \\ &\ll \mathcal{L}^4 \sup_{x/\Delta \leq U \leq x} \sum_{\substack{\rho(\chi) \\ |\text{Im}(\rho)| \leq U}} \frac{x^{1+2\text{Re}(\rho)}}{U^2} |N(\rho, \chi)|^2. \end{aligned}$$

Hence, by partial summation,

$$\begin{aligned} |v_1|_2 &\leq \frac{q}{\varphi(q)} \sum_{\chi(q)} \left| \sum_{\rho(\chi)} \theta N \right|_2 \\ &\ll \mathcal{L}^6 \sup_{\substack{x/\Delta \leq U \leq x \\ 0 \leq \sigma \leq 1}} \frac{x^{1+2\sigma}}{U^2} \sum_{\chi(q)} \sum_{\substack{\rho(\chi) \\ \text{Re}(\rho) \geq \sigma \\ |\text{Im}(\rho)| \leq U}} |N(\rho, \chi)|^2 \\ (4.5) \quad &= \mathcal{L}^6 \sup_{\substack{x/\Delta \leq U \leq x \\ 0 \leq \sigma \leq 1}} x^{1+2\sigma} U^{-2} K_\sigma(U), \text{ say.} \end{aligned}$$

In case of $0 \leq \sigma \leq 1/2$ Lemma 4 yields

$$K_\sigma(U) \ll \mathcal{L}^F (qU + N) N^{1-2\sigma},$$

since $1-2\sigma, 2-2\sigma \geq 0$. Moreover the supremum over U is attained at $U = x/\Delta$. This contributes to $|v_1|_2$

$$\begin{aligned} &\ll \mathcal{L}^F \sup_{0 \leq \sigma \leq 1/2} \Delta^2 x^{2\sigma-1} \left(q \frac{x}{\Delta} + N \right) N^{1-2\sigma} \\ &\ll \mathcal{L}^F \Delta^2 x \sup_{0 \leq \sigma \leq 1/2} \left(\frac{q}{\Delta} \left(\frac{x}{N} \right)^{2\sigma-1} + \left(\frac{x}{N} \right)^{2\sigma-2} \right) \\ &\ll \Delta^2 x \mathcal{L}^F \left(\frac{q}{\Delta} + \frac{N}{x} \right) \\ (4.6) \quad &\ll \Delta^2 x^{3/4}. \end{aligned}$$

In another case $1/2 \leq \sigma \leq 1$, Lemma 2 yields that the supremum may be taken over $1/2 \leq \sigma \leq 1 - \eta(x)$ only, where $\eta(x) = c_4(\log x)^{-4/5}$. Furthermore, by Lemmas 3 and 4, we have

$$\begin{aligned} K_\sigma(U) &\leq \left(\sum_{\chi(q)} \sum_{\substack{\rho(\chi) \\ \text{Re}(\rho) \geq \sigma \\ |\text{Im}(\rho)| \leq U}} 1 \right)^{1/2} \left(\sum_{\chi(q)} \sum_{\substack{\rho(\chi) \\ \text{Re}(\rho) \geq \sigma \\ |\text{Im}(\rho)| \leq U}} |N(\rho, \chi)|^4 \right)^{1/2} \\ &\ll \mathcal{L}^F ((qU)^*(1-\sigma))^{1/2} \left(\sup_{M \leq V \leq N} (qU + V^2) V^{2(1-2\sigma)} \right)^{1/2}. \end{aligned}$$

Here $k > 12/5$. We see the supremum over U is attained at $U = x/\Delta$ again. Then, since $x/\Delta \leq M^2$ and $1 - \sigma \geq 0$, we have that the contribution to $|v_1|_2$ is at most

$$(4.7) \quad \Delta^2 x q \mathcal{L}^F \sup_{1/2 \leq \sigma \leq 1 - \eta(x)} \left(\left(\frac{x}{\Delta} \right)^{k/4} \frac{N}{x} \right)^{2(1-\sigma)}.$$

We now note that

$$\left(1 - \frac{1}{4} \right) \frac{12}{5} \frac{1}{4} + \left(\frac{8}{15} - 1 \right) = -\frac{1}{60}.$$

Hence, there exists a constant $\xi > 0$ such that

$$\left(\frac{x}{\Delta} \right)^{k/4} \frac{N}{x} \leq x^{-\xi}.$$

Thus (4.7) becomes

$$(4.8) \quad \begin{aligned} &\ll \Delta^2 x q \mathcal{L}^F \sup_{1/2 \leq \sigma \leq 1 - \eta(x)} x^{-2\xi(1-\sigma)} \\ &\ll \Delta^2 x q \mathcal{L}^F \exp(-c_6(\log x)^{-1/5}) \\ &\ll \Delta^2 x \mathcal{L}^{-E}, \end{aligned}$$

for any $E > 0$.

In conjunction with (4.5), (4.6) and (4.8), v_1 is admissible. Combining this with (4.2), (4.3) and (4.4), we conclude that $v - v_0$ is also admissible, as required.

LEMMA 11'. Define, for arbitrary sequence (λ_d) with $|\lambda_d| \leq 1$,

$$u = u(t, \Delta, q, a; D) = \sum_{t < n \leq t + \Delta} \left(\sum_{\substack{d|n \\ d < D}} \lambda_d \right) e\left(\frac{a}{q} n\right).$$

For any $\varepsilon > 0$, if $D \leq x^{1-\varepsilon}$ then

$$u = \left(\sum_{\substack{d < D \\ q|d}} \frac{\lambda_d}{d} \right) \Delta + A. R..$$

PROOF. Put

$$f(s) = \sum_{n=1}^{\infty} \left(\sum_{\substack{d|n \\ d < D}} \lambda_d \right) e\left(\frac{a}{q} n\right) n^{-s},$$

which is absolutely convergent for $\text{Re}(s) > 1$, since $|f(s)| \leq \zeta^2(\text{Re}(s))$. On writing $n = dm$, $m = (d, q)l$ and $q = (d, q)r$, we eliminate n and m . Then the conditions on l and r are

$$r|q, \quad (l, r) = 1.$$

Thus, we have

$$\begin{aligned}
 f(s) &= \sum_{d < D} \sum_{r|q} \sum_{b=1}^r * \sum_{\substack{l > 0 \\ l \equiv b(r)}} \lambda_d e\left(\frac{a}{q} \frac{dql}{r}\right) \left(\frac{dql}{r}\right)^{-s} \\
 &= \frac{1}{q^s} \sum_{r|q} \sum_{b=1}^r * \left(\sum_{d < D} \lambda_d e\left(\frac{abd}{r}\right) d^{-s} \right) r^s \left(\sum_{\substack{l > 0 \\ l \equiv b(r)}} l^{-s} \right) \\
 (4.9) \quad &= \frac{1}{q^s} \sum_{r|q} \sum_{b=1}^r * D\left(s, \frac{b}{r}\right) \zeta\left(s, \frac{b}{r}\right), \text{ say.}
 \end{aligned}$$

Hence $f(s)$ is regular, except for a simple pole at $s=1$. By Perron's formula [15, Anhang Satz 3.1.],

$$u = \frac{1}{2\pi i} \int_{c-i x}^{c+i x} f(s) \theta(s) ds + O((\log x)^3),$$

where $c=1+(\log x)^{-1}$. Now,

$$D\left(s, \frac{b}{r}\right) = \sum_{d \leq x/d} + \sum_{x/d < d < D} = D_1 + D_2, \text{ say.}$$

Let the corresponding expressions be f_1 and f_2 , respectively. Put $\eta = \eta(x) = c_2(\log x)^{-4/5}$. On moving the line of integration, we have

$$\begin{aligned}
 u &= \text{Res}_{s=1} f(s) \theta(s) \\
 &+ \frac{1}{2\pi i} \int_{1/2-i x}^{1/2+i x} f_1(s) \theta(s) ds + O\left(\int_{1/2 \pm i x}^{c \pm i x} |f_1(s) \theta(s)| |ds|\right) \\
 &+ \frac{1}{2\pi i} \int_{1-\eta-i x}^{1-\eta+i x} f_2(s) \theta(s) ds + O\left(\int_{1-\eta \pm i x}^{c \pm i x} |f_2(s) \theta(s)| |ds|\right) + A. R. \\
 (4.10) \quad &= u_0 + u_1 + u_2 + O(u_3) + A. R., \text{ say.}
 \end{aligned}$$

First,

$$(4.11) \quad u_0 = \text{Res}_{s=1} = \frac{\Delta}{q} \sum_{r|q} \sum_{b=1}^r * D\left(1, \frac{b}{r}\right) = \Delta \left(\sum_{\substack{d < D \\ q|d}} \frac{\lambda_d}{d} \right).$$

Since $\theta(s) \ll x^{\text{Re}(s)-1}$ for $|\text{Im}(s)| = x$, we have

$$\begin{aligned}
 u_3 &\ll \sup_{\substack{1/2 \leq \sigma \leq c \\ j=1,2}} |f_j(\sigma \pm i x)| x^{\sigma-1} \\
 &\ll \sup_{1/2 \leq \sigma \leq c} \sum_{r|q} \sum_{b=1}^r * D^{1-\sigma} \mathcal{L} \left| \zeta\left(\sigma \pm i x, \frac{b}{r}\right) \right| x^{\sigma-1} \\
 &\ll \sup_{1/2 \leq \sigma \leq c} q^2 \mathcal{L}^2(Dx^{-2/3})^{1-\sigma} \\
 &\ll q^2 x^{1/6},
 \end{aligned}$$

by Lemma 1. Hence, because of (4.1), u_3 is admissible.

We proceed to u_1 . Lemma 1 yields that

$$\begin{aligned}
 |u_1|_2 &\ll \mathcal{L}^2 \sup_{x/D \leq U \leq x} \left(\frac{x}{U}\right)^2 \sum_{r|q} \sum_{b=1}^r * \int_{1/2-iU}^{1/2+iU} \left|D_1\left(s, \frac{b}{r}\right) \zeta\left(s, \frac{b}{r}\right)\right|^2 |ds| \\
 &\ll \mathcal{L}^2 \sup_{x/D \leq U \leq x} \left(\frac{x}{U}\right)^2 \sum_{r|q} \sum_{b=1}^r * \left(\sup_{\substack{\operatorname{Re}(s)=1/2 \\ |\operatorname{Im}(s)| \leq U}} \left|\zeta\left(s, \frac{b}{r}\right)\right|^2\right) \int_{1/2-iU}^{1/2+iU} \left|D_1\left(s, \frac{b}{r}\right)\right|^2 |ds| \\
 &\ll \mathcal{L}^5 \sup_{x/D \leq U \leq x} \left(\frac{x}{U}\right)^2 q U^{1/3} \left(U + \frac{x}{\Delta}\right) \\
 (4.13) \quad &\ll \mathcal{L}^5 \Delta^2 q \left(\frac{x}{\Delta}\right)^{4/3}.
 \end{aligned}$$

Since $\Delta \geq x^{1/4+\varepsilon'}$, u_1 is admissible.

We turn now to u_2 . As before, by Lemma 1, we have

$$\begin{aligned}
 |u_2|_2 &\ll \mathcal{L}^2 \sup_{x/D \leq U \leq x} x^{1+2(1-\eta)} U^{-2} \sum_{r|q} \sum_{b=1}^r * \int_{1-\eta-iU}^{1-\eta+iU} \left|D_2\left(s, \frac{b}{r}\right) \zeta\left(s, \frac{b}{r}\right)\right|^2 |ds| \\
 &\ll q \mathcal{L}^4 \exp(2c_1(\log \log x)^3) x^{3-2\eta} \sup_{\substack{x/D \leq U \leq x \\ x/D \leq N \leq D}} U^{-2} (U+N) N^{1-2(1-\eta)} \\
 &\ll q \mathcal{L}^4 \exp(2c_1(\log \log x)^3) \Delta^2 x \left(\frac{D}{x}\right)^{2\eta}.
 \end{aligned}$$

Since $\eta = c_2(\log x)^{-4/5}$, $D \leq x^{1-\varepsilon}$ and $q \leq (\log x)^B$, we finally obtain

$$(4.14) \quad |u_2|_2 \ll \Delta^2 x \exp(-\varepsilon c_2(\log x)^{-1/6}).$$

Thus, u_2 is also admissible.

In conjunction with (4.10)–(4.14), we have

$$u = u_0 + A.R.,$$

as required.

5. Auxiliary results.

In this section we provide for the proof of Lemmas 16 and 17 in the next section. For real numbers a, b and c , $a \leq b$, we set

$$\begin{aligned}
 \Phi(a, b; c) &= \phi(b-c) - \phi(a-c) \\
 &= [b-c] - [a-c] - (b-a).
 \end{aligned}$$

Let α, β, γ and δ be arbitrary sequences with modulus ≤ 1 . Moreover, let 1 denote the arithmetical function $1(n) \equiv 1$ or $\equiv \log n$. We consider the following linear forms involving Φ .

$$\begin{aligned}
 F_1 &= F_1(K, D, M, N; x, s, t) \\
 &= \sum_{0 < k \leq K} \sum_{d \sim D} \sum_{\substack{m \sim M \\ (d, mn)=1 \\ mn \leq t}} \sum_{n \sim N} \alpha_d \beta_m 1(n) \delta_k \Phi\left(\frac{2x}{dmn}, \frac{\max(x, ds, mns)}{dmn}; k \frac{\overline{mn}}{d}\right).
 \end{aligned}$$

$$F_2 = F_2(K, D, M, N; x, s, t) = \sum_{0 < k \leq K} \sum_{d \sim D} \sum_{\substack{m \sim M \\ (d, mn)=1 \\ mn \leq t}} \sum_{n \sim N} \alpha_d \beta_m \gamma_n \delta_k \Phi\left(\frac{2x}{dmn}, \frac{\max(x, ds, mns)}{dmn}; k \frac{\bar{d}}{mn}\right).$$

Here, $1 \ll K, D, M, N \ll x, MN \leq t$ and $s \leq 2x/D, 2x/MN$.

Our aim is to prove $F \ll Kx^{1-\epsilon}, \epsilon > 0$. Since $F \ll KDMN$ trivially, we assume

$$(5.1) \quad DMN > x^{1-2\epsilon}.$$

LEMMA 13. For any $\epsilon > 0$, if $MN, DM \leq x^{2/3-4\epsilon}$ then

$$F_1 \ll Kx^{1-\epsilon}.$$

PROOF. We appeal to the Erdős-Turán inequality [8, Lemme 2], which runs as follows. For arbitrary real numbers x_n and $H > 2$,

$$\left| \sum_{n \sim N} \psi(x_n) \right| \ll \frac{N}{H} + \sum_{0 < h \leq H} \frac{1}{h} \left| \sum_{n \sim N} e(hx_n) \right|.$$

Now, by partial summation, we have

$$\begin{aligned} \sum_{\substack{n \sim N \\ (n, d)=1 \\ mn \leq t}} 1(n) \Phi &\ll \mathcal{L} \sup_{\substack{y \leq 2x/DM \\ \mathcal{J}, \bar{\mathcal{J}} \subset [N, 2N]}} \left(\left| \sum_{\substack{n \in \mathcal{J} \\ (n, d)=1}} \psi\left(\frac{y}{n} - k \frac{\overline{mn}}{d}\right) \right| + \left| \sum_{\substack{n \in \bar{\mathcal{J}} \\ (n, d)=1}} \psi\left(\frac{s}{d} - k \frac{\overline{mn}}{d}\right) \right| \right) \\ &\ll \mathcal{L} \sup_{\substack{y \leq 2x/DM \\ \mathcal{J} \subset [N, 2N]}} \left(\frac{N}{H} + \sum_{0 < h \leq H} \frac{1}{h} \left| \sum_{\substack{n \in \mathcal{J} \\ (n, d)=1}} e\left(\frac{hy}{n}\right) e\left(-hk \frac{\overline{mn}}{d}\right) \right| \right). \end{aligned}$$

It follows from the argument of [11, p. 37] that

$$\begin{aligned} F_1 &\ll \mathcal{L} \frac{KDMN}{H} + \mathcal{L}^2 \sum_{0 < h \leq H} \frac{1}{h} x^\epsilon \left(1 + \frac{Hx}{DMN}\right) \tau(h) KM(D^{3/2} + D^{1/2}N) \\ &\ll \mathcal{L} \frac{KDMN}{H} + \mathcal{L}^4 x^\epsilon \left(1 + \frac{Hx}{DMN}\right) Kx^{1-6\epsilon}. \end{aligned}$$

On taking $H = DMN/x^{1-3\epsilon} > 2$, because of (5.1), we obtain Lemma 13.

LEMMA 14. For any $\epsilon > 0$, if $D, MN \leq x^{8/15-4\epsilon}$ and $N \geq x^{1/15}$ then

$$F_2 \ll Kx^{1-\epsilon}.$$

PROOF. We follow the argument of [11] with a minor modification. We use a well known Fourier expansion. For $H > 2$,

$$(5.2) \quad \begin{aligned} \Phi(a, b; c) &= \sum_{0 < |h| \leq H} \frac{e(hb) - e(ha)}{2\pi ih} e(-hc) \\ &\quad + O(\min(1, 1/H\|b-c\|) + \min(1, 1/H\|a-c\|)). \end{aligned}$$

Also,

$$(5.3) \quad \min(1, 1/H\|x\|) = \sum_{h \in \mathbb{Z}} A_h e(hx)$$

where

$$|A_h| \ll \min\left(\frac{\log H}{H}, \frac{H}{h^2}\right).$$

Now, Φ in F_2 is expressed as a sum of the main part of Fourier series, say Φ' , and the tail part, say Φ'' . On putting the resulting linear forms to F_2' and F_2'' , we may write

$$F_2 = F_2' + F_2''.$$

First we consider F_2'' . (5.3) yields that

$$\begin{aligned} F_2'' &\ll \sum_h |A_h| \left(\left| \sum_{0 < k \leq K} \sum_{\substack{d \sim D \\ (d, mn)=1}} \sum_{m \sim M} \sum_{n \sim N} e\left(\frac{2hx}{dmn}\right) e\left(-hk \frac{\bar{d}}{mn}\right) \right| \right. \\ &\quad \left. + \left| \sum_{0 < k \leq K} \sum_{\substack{d \sim D \\ (d, mn)=1}} \sum_{m \sim M} \sum_{n \sim N} e\left(\frac{h \max(x, ds, mns)}{dmn}\right) e\left(-hk \frac{\bar{d}}{mn}\right) \right| \right) \\ &= \sum_h |A_h| S(h), \text{ say.} \end{aligned}$$

Summing by parts, we see

$$S(h) \ll \sum_{0 < k \leq K} \sum_{m \sim M} \sum_{n \sim N} \left(1 + \frac{hx}{DMN}\right) \sup_{\mathcal{J} \subset [D, 2D]} \left| \sum_{\substack{d \in \mathcal{J} \\ (d, mn)=1}} e\left(hk \frac{\bar{d}}{mn}\right) \right|.$$

By the same argument as that in [11, (17)(18)] with $q=1$, we then obtain

$$F_2'' \ll Kx^{1-2\epsilon} + \mathcal{L}^3 x^{4\epsilon} K \{(MN)^{3/2} + D(MN)^{1/2}\}.$$

Here we have taken $H = DMN x^{3\epsilon-1} > 2$, because of (5.1). Since $D, MN < x^{2/3-4\epsilon}$, we have

$$F_2'' \ll Kx^{1-\epsilon}.$$

We turn now to F_2' . By (5.2),

$$\begin{aligned} F_2' &= \sum_{0 < k \leq K} \sum_{\substack{d \sim D \\ (d, mn)=1}} \sum_{m \sim M} \sum_{n \sim N} \alpha_d \beta_m \gamma_n \delta_k \\ &\quad \times \sum_{0 < |h| \leq H} \int_{\max(x, ds, mns)/g}^{2xg/dmn} e\left(\frac{h}{g}y\right) \frac{dy}{g} \cdot e\left(-hk \frac{\bar{d}}{mn}\right), \end{aligned}$$

where g is arbitrary. After some elementary rearrangement, we reach the inequality

$$(5.4) \quad F_2' \ll \frac{x}{DMN} \sup_B \sum_{\substack{d \sim D \\ (d, m)=1}} \sum_{\substack{m \sim M \\ mn \leq t}} \left| \sum_{0 < h \leq H} \sum_{\substack{0 < k \leq K \\ (d, n)=1 \\ mn \leq t}} \sum_{n \sim N} B(h, k, n) e\left(hk \frac{\bar{d}}{mn}\right) \right|,$$

where the supremum is over all arithmetical functions B with $|B| \leq 1$. To deal with the condition $mn \leq t$, we use a lemma in Fourier analysis [1, Lemma 2.2.], which runs as follows. Let a_n be arbitrary finite sequence. If $X \leq Y$ then

$$\left| \sum_{n \leq X} a_n \right| \leq \int_{-\infty}^{\infty} f(\theta, Y) \left| \sum_n a_n e(\theta n) \right| d\theta.$$

Here, the kernel $f \geq 0$ satisfies

$$\int_{-\infty}^{\infty} f(\theta, Y) d\theta \ll \log Y.$$

Now, applying this lemma with $Y = t/M$, we may remove the condition $mn \leq t$ in (5.4) with the cost of $(\log x)$ time. Thus, the method in [11, p. 38, 39] is applicable. We therefore have

$$F_2' \ll \mathcal{L} x^{3\epsilon} \{ (KDM)^{1/2} x^{1/2-\epsilon} + Kx^{5\epsilon/2} (D^{1/2} M^{5/4} N^{3/2} + DM^{3/4} N^{1/2}) \}.$$

Since $DM < x^{8/15-4\epsilon} \cdot x^{7/15-4\epsilon} = x^{1-8\epsilon}$, the above first term is

$$\begin{aligned} &\ll \mathcal{L} x^{3\epsilon} K^{1/2} x^{1-5\epsilon} \\ &\ll Kx^{1-\epsilon}. \end{aligned}$$

We may assume $M \geq N$, because of the symmetry of F_2 , so that $M \geq (MN)^{1/2}$. Hence, the second term is

$$\begin{aligned} &\ll Kx^{6\epsilon} \{ D^{1/2} (MN)^{3/2-1/8} + D(MN)^{1/2} M^{1/4} \} \\ &\ll Kx^{6\epsilon} \{ (x^{8/15-4\epsilon})^{15/8} + (x^{8/15-4\epsilon})^{3/2} (x^{7/15-4\epsilon})^{1/4} \} \\ &\ll Kx^{1-\epsilon}. \end{aligned}$$

We thus obtain Lemma 14.

6. Minor arc, preliminaries.

In this section we provide for the final section. We re-present the method of [10] with a few simplifications. Let $a(n)$, $b(n)$ and $c(n)$ be arbitrary arithmetical functions which are bounded by

$$\tau_{c_7}(n) (\log n)^{c_8}.$$

Also, $1(n) \equiv 1$ or $\equiv \log n$. Furthermore, we put

$$\begin{aligned} j \text{ I}(h; s) &= \sum_{\substack{h=dl \\ l \geq s}} a(d), \\ j \text{ II}(h; D, V) &= \sum_{\substack{h=lmn \\ m \leq D \\ n \leq V}} 1(l)a(m)1(n), \end{aligned}$$

$$j\text{III}(h; D, U) = \sum_{\substack{h=lmn \\ m, n \leq D \\ m, n \in U}} 1(l)a(m)b(n),$$

$$j\text{IV}(h; U, V, \mathcal{G}) = \sum_{\substack{h=mnk \\ V < m \leq U \\ m, k \in \mathcal{G}}} a(m)b(n)c(k).$$

Here, s, D, U, V 's are parameters and \mathcal{G} is an interval. We are devoted to estimating the mean value of trigonometrical sums

$$Ji(*) = Ji(\alpha, x, \Delta; *) = \Delta^2 \int_{-1/2\Delta}^{1/2\Delta} \left| \sum_{x < h \leq 2x} ji(h; *) e((\alpha + \beta)h) \right|^2 d\beta$$

($i = \text{I, II, III, IV}$)

under the assumption

$$(61) \quad \left| \alpha - \frac{a}{q} \right| \leq \frac{1}{q^2} \quad \text{with } (a, q) = 1 \quad \text{and } 1 < q < \Delta < x/2.$$

LEMMA 15. For any $\epsilon > 0$,

$$J\text{I}(s) \ll \Delta x \mathcal{L}^F \{ \Delta q^{-1/2} + (q\Delta)^{1/2} \} + \Delta^2 \left(\frac{x}{s} \right)^2 \mathcal{L}^F + \Delta^3 x^\epsilon.$$

PROOF. On writing $w_n = j\text{I}(n; s)$, Lemma 8 yields that

$$(6.2) \quad \begin{aligned} J\text{I} &\ll \Delta \sum_{n \sim x} |w_n|^2 + \Delta \sup_{t \leq \Delta} \left| \sum_{0 < r \leq t} e(\alpha r) \sum_{n \sim x} \bar{w}_n w_{n+r} \right| + \Delta^3 (\sup |w_n|)^2 \\ &\ll \Delta x \mathcal{L}^F + \Delta \sup |\Sigma| + \Delta^3 x^\epsilon, \end{aligned}$$

where

$$\Sigma = \sum_{0 < r \leq t} e(\alpha r) \sum_{d_1} \sum_{d_2} a(d_1)a(d_2) \# \left\{ n : \begin{array}{l} \max(x, d_1s, d_2s) < n \leq 2x \\ n \equiv 0 \pmod{d_1} \\ n \equiv r \pmod{d_2} \end{array} \right\} + O(\Delta^2 x^\epsilon).$$

The above simultaneous congruences are soluble if and only if $(d_1, d_2) | r$. We write $d_i^* = d_i / (d_1, d_2)$ and $r^* = r / (d_1, d_2)$. We then have, with the notation in section 5, that

$$\begin{aligned} \#\{n\} &= \#\left\{ l : \frac{\max(x, d_1s, d_2s)}{[d_1, d_2]} < r^* \frac{\bar{d}_1^*}{d_2^*} + l \leq \frac{2x}{[d_1, d_2]} \right\} \\ &= \frac{2x - \max(x, d_1s, d_2s)}{[d_1, d_2]} + \Phi \left(\frac{2x}{[d_1, d_2]}, \frac{\max(x, d_1s, d_2s)}{[d_1, d_2]}; r^* \frac{\bar{d}_1^*}{d_2^*} \right) \\ &= \Phi_0 + \Phi, \quad \text{say.} \end{aligned}$$

If d_1 or $d_2 > 2x/s$ then $\#\{n\} = 0$. Hence

$$(6.3) \quad \begin{aligned} \Sigma &= \sum_{0 < r \leq t} e(\alpha r) \sum_{\substack{d_1, d_2 \leq 2x/s \\ (d_1, d_2) | r}} a(d_1)a(d_2) (\Phi_0 + \Phi) + O(\Delta^2 x^\epsilon) \\ &= \Sigma_0 + \Sigma_1 + O(\Delta^2 x^\epsilon), \quad \text{say.} \end{aligned}$$

It follows from [15, Kap. IV Lemma 6.3.] that

$$\begin{aligned}
 \Sigma_0 &\ll x \sum_{\substack{d_1, d_2 \leq 2x/s \\ (d_1, d_2) \leq t}} \sum_{\substack{0 < r \leq t \\ (d_1, d_2) | r}} \frac{|a(d_1)a(d_2)|}{[d_1, d_2]} \left| \sum_{\substack{0 < r \leq t \\ (d_1, d_2) | r}} e(\alpha r) \right| \\
 &\ll x \sum_{0 < \delta \leq d} \left(\delta \sum_{\substack{d \leq 2x/s \\ \delta | d}} \frac{|a(d)|}{d} \right)^2 \delta^{-1} \min\left(\frac{\Delta}{\delta}, \frac{1}{\|\alpha\delta\|}\right) \\
 &\ll x \left(\sum_{\delta \leq d} \left(\delta \sum_{\substack{d \leq 2x/s \\ \delta | d}} \frac{|a(d)|}{d} \right)^2 \delta^{-2} \right)^{1/2} \left(\sum_{0 < \delta \leq d} \frac{\Delta}{\delta} \min\left(\frac{\Delta}{\delta}, \frac{1}{\|\alpha\delta\|}\right) \right)^{1/2} \\
 (6.4) \quad &\ll x \mathcal{L}^F \Delta^{1/2} \left(\frac{\Delta}{q} + q \right)^{1/2}.
 \end{aligned}$$

Also, since $\Phi \ll 1$,

$$\begin{aligned}
 \Sigma_1 &\ll \Delta \left(\sum_{d \leq 2x/s} |a(d)| \right)^2 \\
 (6.5) \quad &\ll \Delta \left(\frac{x}{s} \right)^2 \mathcal{L}^F.
 \end{aligned}$$

In conjunction with (6.2)-(6.5) we obtain Lemma 15.

LEMMA 16. For any $\varepsilon > 0$, if $DV \leq x^{2/3-3\varepsilon}$ then

$$J \Pi(D, V) \ll \Delta x \mathcal{L}^F \{ \Delta q^{-1/2} + (q\Delta)^{1/2} \} + \Delta^2 x^{1-\varepsilon} + \Delta^3 x^\varepsilon.$$

PROOF. We have, by the same argument as before, that

$$\begin{aligned}
 (6.6) \quad J \Pi &\ll \Delta x \mathcal{L}^F \{ \Delta q^{-1/2} + (q\Delta)^{1/2} \} + \Delta^3 x^\varepsilon \\
 &\quad + \Delta^2 \mathcal{L}^F x^{1-2\varepsilon} + \sup_{\substack{t \leq d \\ s \leq x^{1/2+\varepsilon}}} |\Sigma_{s,t}|,
 \end{aligned}$$

where

$$\begin{aligned}
 \Sigma &= \sum_{0 < r \leq t} e(\alpha r) \sum_{\substack{d_1, d_2 \leq \min(2x/s, D) \\ (d_1, d_2) | r}} w(d_1)w(d_2) \\
 &\quad \times \Phi \left(\frac{2x}{[d_1, d_2]}, \frac{\max(x, d_1s, d_2s)}{[d_1, d_2]}; r^* \frac{\overline{d_1^*}}{d_2^*} \right)
 \end{aligned}$$

with

$$w(d) = \sum_{\substack{d = m'n' \\ m' \leq V}} a(m')l(n').$$

Write

$$\begin{aligned}
 (d_1, d_2) &= \delta, \\
 d_1 &= \delta mn, \quad \delta = ef, \quad m' = em, \quad n' = fn, \\
 d_2 &= \delta d, \\
 r^* &= k.
 \end{aligned}$$

We then find

$$\begin{aligned} \Sigma &= \sum_{0 < \delta k \leq t} \sum e(\alpha \delta k) \sum_{\substack{ef=\delta \\ e \leq V}} \sum_{m, n, d \leq \min(D/\delta, 2x/\delta s)} \sum_{\substack{m \leq V/e \\ (m, n, d)=1}} a(em) 1(fn) w(\delta d) \\ &\quad \times \Phi \left(\frac{2x/\delta}{mnd}, \frac{\max(x/\delta, mns, ds)}{mnd}; k \frac{\overline{mn}}{d} \right). \end{aligned}$$

On breaking up the ranges for m, n and d into $O((\log x)^3)$ intervals of the shape $[2^i, 2^{i+1}]$, we conclude that

$$\begin{aligned} \Sigma &\ll x^\varepsilon \sum_{\delta \leq D} \sup_{\substack{|\alpha|, |\beta|, |\gamma| \leq 1 \\ u, M, N, D' \leq D/\delta \\ M \leq V}} \left| \sum_{0 < r \leq D/\delta} \sum_{d \sim D'} \sum_{\substack{m \sim M \\ (d, mn)=1}} \sum_{n \sim N} \alpha_d \beta_m 1(n) \gamma_k \Phi \right| \\ &= x^\varepsilon \sum_{\delta \leq D} \sup_{\substack{D', M, N \\ M \leq V}} \left| F_1 \left(\frac{\Delta}{\delta}, D', M, N; \frac{x}{\delta}, s, u \right) \right|, \end{aligned}$$

where F_1 is defined in section 5. Since

$$\frac{DV}{\delta}, \frac{D}{\delta} \leq \left(\frac{x}{\delta} \right)^{2/3-8\varepsilon},$$

Lemma 13 yields

$$\begin{aligned} \Sigma &\ll x^\varepsilon \sum_{\delta \leq D} \left(\frac{\Delta}{\delta} \right) \left(\frac{x}{\delta} \right)^{1-2\varepsilon} \\ &\ll \Delta x^{1-\varepsilon}. \end{aligned}$$

Combining this with (6.6) we obtain Lemma 16.

LEMMA 17. For any $\varepsilon > 0$, if $D \leq x^{8/15-12\varepsilon}$ and $U \geq x^{1/5}$ then

$$J\text{III}(D, U) \ll \Delta x \mathcal{L}^F \{ \Delta q^{-1/2} + (q\Delta)^{1/2} \} + \Delta^2 x^{1-\varepsilon} + \Delta^3 x^\varepsilon.$$

PROOF. It suffices to show

$$\sum_{\delta \leq D} \sup_{\substack{u, D', M, N \leq D/\delta \\ M, N > \max(U/\delta, 1) \\ s \leq x^{1/2+\varepsilon}}} \left| F_2 \left(\frac{\Delta}{\delta}, D', M, N; \frac{x}{\delta}, s, u \right) \right| \ll \Delta x^{1-2\varepsilon}.$$

When $\delta > D^2 x^{2\varepsilon-1}$, we use the trivial bound

$$F_2 \ll \frac{\Delta}{\delta} \left(\frac{D}{\delta} \right)^2.$$

Otherwise, we make use of Lemma 14. Since

$$\frac{D}{\delta} \leq \left(\frac{x}{\delta} \right)^{8/15-12\varepsilon} \quad \text{and} \quad \frac{U}{\delta} \geq \left(\frac{x}{\delta} \right)^{1/15},$$

Lemma 14 yields

$$F_2 \ll \left(\frac{\Delta}{\delta} \right) x^{1-3\varepsilon}.$$

We therefore have

$$\sum_{\delta \leq \Delta} \sup |F_2| \ll \sum_{\delta \leq D^2 x^{2\epsilon-1}} \left(\frac{\Delta}{\delta}\right) x^{1-3\epsilon} + \sum_{D^2 x^{2\epsilon-1} < \delta \leq \Delta} \left(\frac{\Delta}{\delta}\right) \left(\frac{D}{\delta}\right)^2 \ll \Delta x^{1-2\epsilon},$$

as required.

LEMMA 18.

$$JIV(U, V, \mathcal{J}) \ll \Delta x \mathcal{L}^F \left(U + \frac{\Delta}{q} + \frac{\Delta}{V} + q \right).$$

PROOF. We may assume $U \leq \Delta$, for otherwise Lemma is trivial. We split the ranges for m, n and k into $O((\log x)^3)$ intervals of the form $(M, 2M]$, $(N, 2N]$ and $(K, 2K]$, so that

$$x \ll MNK \ll x \quad \text{and} \quad V \leq M \leq U.$$

Using Cauchy's inequality twice, we have

$$(6.7) \quad JIV \ll \mathcal{L}^F \sup_{\substack{V \leq M \leq U \\ x \ll MNK \ll x}} M \cdot \sum_{m \sim M} \Delta^2 \int_{-1/2\Delta}^{1/2\Delta} \left| \sum_{\substack{m \sim N \\ m \sim K}} \sum_{\substack{n \sim x \\ k \sim \mathcal{J}}} b(n)c(k)e((\alpha + \beta)mnk) \right|^2 d\beta \ll \mathcal{L}^F \sup M \cdot J, \quad \text{say.}$$

By Lemma 8,

$$\begin{aligned} J &\ll \sum_{m \sim M} \Delta \sum_{NK < l \leq 4NK} \left(\sum_{l=nk} |b(n)c(k)| \right)^2 \\ &\quad + \sum_{0 < r \leq \Delta} \sum_{\substack{r=nk-n'k' \\ n, n' \sim N \\ k, k' \sim K}} \sum |b(n)c(k)b(n')c(k')| \cdot \sum_{\substack{m \sim M \\ km, k' \sim \mathcal{J}}} \max(0, \Delta - mr)e(\alpha mr) \\ &\ll \Delta \mathcal{L}^F MNK \\ &\quad + NK \mathcal{L}^F \sum_{0 < r \leq \Delta} \sup_{\substack{\mathcal{J} \\ m \in \mathcal{J} \\ m \geq M}} \left| \sum \max(0, \Delta - mr)e(\alpha mr) \right| \\ &\ll \Delta x \mathcal{L}^F + NK \mathcal{L}^F \sum_{0 < r \leq \Delta/M} \sup_{\substack{\mathcal{J} \\ m \in \mathcal{J} \\ m \geq M \\ 0 < m r \leq \Delta}} \left| \sum (\Delta - mr)e(\alpha mr) \right| \end{aligned}$$

$$(6.8) \quad = \Delta x \mathcal{L}^F + NK \mathcal{L}^F \cdot \mathcal{I}, \quad \text{say.}$$

Summing by parts, we have

$$\begin{aligned} \mathcal{I} &\ll \sum_{0 < r \leq \Delta/M} \Delta \min\left(\frac{\Delta}{r}, \frac{1}{\|\alpha r\|}\right) \\ &\ll \Delta \left(\frac{\Delta}{q} + \frac{\Delta}{M} + q \right), \end{aligned}$$

by [15, Kap. VI Lemma 6.3.]. Combining this with (6.7) and (6.8), we finally

obtain

$$\begin{aligned} JIV &\ll \mathcal{L}^F \sup_{\substack{V \leq M \leq U \\ x \ll MNK \ll x}} M \left\{ \Delta x + NK \cdot \Delta \left(\frac{\Delta}{q} + \frac{\Delta}{M} + q \right) \right\} \\ &\ll \Delta x \mathcal{L}^F \left(U + \frac{\Delta}{q} + \frac{\Delta}{V} + q \right). \end{aligned}$$

7. Minor arc.

In this section we prove Lemmas 10[†] and 12[†] below, which cover Lemmas 10 and 12, respectively. Throughout this section we assume that

$$\left| \alpha - \frac{a}{q} \right| \leq \frac{1}{q^2} \quad \text{with } (a, q) = 1, \text{ and } 1 < \Delta < x^{2/3}.$$

LEMMA 10[†]. Put

$$\mathcal{T} = \mathcal{T}(\alpha, x, \Delta, N) = \Delta^2 \int_{-1/2\Delta}^{1/2\Delta} \left| \sum_{\substack{x < mn \leq 2x \\ \sqrt{x} < n \leq N}} \Lambda(m) \Lambda(n) e((\alpha + \beta)mn) \right|^2 d\beta.$$

For any $E, \varepsilon > 0$, if $N \leq x^{8/15 - \varepsilon}$ then

$$\mathcal{T} \ll \Delta x \mathcal{L}^F \{ x^{1/4} + \Delta q^{-1/2} + (q\Delta)^{1/2} \} + \Delta^2 x \mathcal{L}^{-E},$$

where the implied constant depends only on E and ε .

PROOF. We may assume $q \leq \Delta$, for otherwise Lemma is trivial. For $x < h \leq 2x$, put

$$\rho(h) = \sum_{\substack{h=mn \\ n \in \mathcal{J}}} \Lambda(m) \Lambda(n), \quad \mathcal{J} = (\sqrt{x}, N].$$

We shall decompose ρ into $O(1)$ sums of the ji -type weight, which is defined in section 6. In order to apply Lemma 6, we introduce the parameters u, v and w such that

$$\begin{aligned} u &= 2x^{1/4} \\ v &= Nx^{-1/4} \\ w &= x^\xi, \quad \varepsilon = 20\xi. \end{aligned}$$

Then, for $x < mn \leq 2x$ and $n \in \mathcal{J}$,

$$u < \frac{x}{N} < m \leq 2\sqrt{x} < u^2 \quad \text{and} \quad v < \sqrt{x} < n \leq N < uv.$$

Lemma 6 yields that, for $x < h \leq 2x$,

$$\rho(h) = \sum_{\substack{h=mn'n' \\ n'n' \in \mathcal{G} \\ m' > u \\ n' > v}} \left(\sum_{\substack{d|m \\ d \leq u}} \mu(d) \right) \Lambda(m') \left(\sum_{\substack{e|n \\ e \leq u}} \mu(e) \right) \Lambda(n').$$

We now divide the range of variables in ρ as follows.

$$\begin{aligned} d \in (0, u] &= H_2 \cup H_3, \\ e \in (0, u] &= H_2 \cup H_3, \\ m' \in (u, \infty) &= H_1 \setminus H_3, \\ n' \in (v, \infty) &= H_1 \setminus (H_3 \cup H_4). \end{aligned}$$

Here, $H_1 = (w, \infty)$, $H_2 = (0, w]$, $H_3 = (w, u]$ and $H_4 = (u, v]$. Thus, we may write

$$\rho(h) = \sum_{\substack{\delta=2,3 \\ \varepsilon=2,3 \\ \mu=1,3 \\ \nu=1,3,4}} (\pm 1) \sum_{\substack{h=mn'n' \\ n'n' \in \mathcal{G} \\ (d,e,m',n') \in H_\delta \times H_\varepsilon \times H_\mu \times H_\nu}} \left(\sum_{d|m} \mu(d) \right) \Lambda(m') \left(\sum_{e|n} \mu(e) \right) \Lambda(n').$$

Let ρ_2 and ρ_3 be the sums corresponding to

$$\begin{array}{cc} \delta=2 & \delta=2 \\ \varepsilon=2 & \varepsilon=2, 3 \\ \mu=1 & \mu=1 \\ \nu=1 & \nu=4 \end{array} \text{ and } ,$$

respectively. Moreover, let ρ_4 denote the remaining sums, so that

$$\rho(h) = \rho_2(h) - \rho_3(h) + \rho_4(h).$$

We first consider ρ_4 . We easily see that at least one of δ, ε, μ or ν must be 3. Namely, at least one of d, e, m' or n' lies in the interval $(w, u]$. We therefore have that ρ_4 is splitted into 21 sums of the $jIV(h; u, w, \mathcal{G})$ -type sum, with $\mathcal{G} = \mathcal{G}$ or $(0, 2x]$.

We proceed to ρ_2 . On writing $mm' = df$ and $nn' = eg$, we have

$$\begin{aligned} \rho_2(h) &= \sum_{\substack{h=mn'n' \\ n'n' \in \mathcal{G} \\ m, m' > w}} \sum_{\substack{d|m \\ d \leq w}} \left(\sum \mu(d) \right) \Lambda(m') \left(\sum_{\substack{e|n \\ e \leq w}} \mu(e) \right) \Lambda(n') \\ &= \sum_{h=dfe} \sum_{\substack{eg \in \mathcal{G} \\ d \leq w}} \sum \mu(d) \left(\sum_{\substack{m'|f \\ m' > w}} \Lambda(m') \right) \sum_{e \leq w} \mu(e) \left(\sum_{\substack{n'|g \\ n' > w}} \Lambda(n') \right) \\ &= \sum_{h=dfe} \sum_{\substack{d, e \leq w \\ eg \in \mathcal{G}}} \sum \mu(d) \{(\log f) - \sum_{\substack{m'|f \\ m' \leq w}} \Lambda(m')\} \mu(e) \{(\log g) - \sum_{\substack{n'|g \\ n' \leq w}} \Lambda(n')\} \end{aligned}$$

Thus, $\rho_2(h)$ is divided into 4 sums of the $jII(h; w^2N, w^4)$ -type sum.

We turn now to ρ_3 .

$$\begin{aligned} \rho_3(h) &= \sum_{\substack{h=mn'n' \\ nn' \in \mathcal{G} \\ m' > w \\ u < n' \leq v}} \sum_{\substack{d|m \\ d \leq w}} (\sum \mu(d)) \Lambda(m') (\sum_{\substack{e|n \\ e \leq u}} \mu(e)) \Lambda(n') \\ &= \sum_{\substack{h=dn'n' \\ d \leq w \\ nn' \in \mathcal{G} \\ u < n' \leq v}} \sum \mu(d) \{(\log f) - \sum_{\substack{m'|f \\ m' \leq w}} \Lambda(m')\} (\sum_{\substack{e|n \\ e \leq u}} \mu(e)) \Lambda(n'), \end{aligned}$$

since $nn' \in \mathcal{G}$ and $u < n' \leq v$ imply $n > 1$. Hence, $\rho_3(h)$ is a sum of two $j\text{III}(h; w^2N, u)$ -type sums.

We therefore conclude that

$$\mathcal{I} \ll J\text{II}(w^2N, w^4) + J\text{III}(w^2N, u) + J\text{IV}(u, w, \mathcal{G}).$$

We note that

$$\begin{aligned} w^6 N &\leq x^{8/15 - \varepsilon + 12\xi} \leq x^{2/3 - 8\xi}, \\ w^2 N &\leq x^{8/15 - \varepsilon + 4\xi} \leq x^{8/15 - 12\xi}, \\ u &= 2x^{1/4} \geq x^{1/5}. \end{aligned}$$

Thus, Lemmas 16, 17 and 18 yield

$$\begin{aligned} \mathcal{I} &\ll \Delta x \mathcal{L}^F \{ \Delta q^{-1/2} + (q\Delta)^{1/2} \} + \Delta^2 x^{1-\xi} + \Delta x \mathcal{L}^F \left(u + \frac{\Delta}{q} + \frac{\Delta}{w} + q \right) \\ &\ll \Delta x \mathcal{L}^F \{ x^{1/4} + \Delta q^{-1/2} + (q\Delta)^{1/2} \} + \Delta^2 x^{1-\xi}, \end{aligned}$$

as required.

LEMMA 12[†]. Let $(\lambda_d) = (\lambda_d(D, z))$ be the weights introduced in Lemma 5. Put

$$S = S(\alpha, x, \Delta, D, z) = \Delta^2 \int_{-1/2\Delta}^{1/2\Delta} \left| \sum_{x < n \leq 2x} (\sum_{d|n} \lambda_d) e((\alpha + \beta)n) \right|^2 d\beta.$$

For any $E, \varepsilon > 0$, if $4 < z^2 \leq D \leq x^{1-\varepsilon}$ then

$$S \ll \Delta x \mathcal{L}^F \{ x^{1/4} + \Delta q^{-1/2} + (q\Delta)^{1/2} \} + \Delta^2 x \mathcal{L}^{-E},$$

where the implied constant depends only on E and ε .

PROOF. We may assume $q \leq \Delta$. By Lemma 5, we have

$$S \ll J\text{I}(x/z_1 z) + \mathcal{L}^2 J\text{IV}(D^{1/4} z_9, z_2, (0, 2x]).$$

Here,

$$z_i = \exp((\log D)(\log \log D)^{-i/10}).$$

Lemmas 15 and 18 yield that

$$\begin{aligned} S &\ll \Delta x \mathcal{L}^F \{ \Delta q^{-1/2} + (q\Delta)^{1/2} \} + \Delta (z_1 z)^2 \mathcal{L}^F \\ &\quad + \Delta x \mathcal{L}^F \left(D^{1/4} z_9 + \frac{\Delta}{q} + \frac{\Delta}{z_2} + q \right) \end{aligned}$$

$$\ll \Delta x \mathcal{L}^F \{D^{1/4} z_0 + \Delta q^{-1/2} + (q\Delta)^{1/2}\} + \Delta^2 \mathcal{L}^F z_1^2 D + \Delta^2 x \mathcal{L}^F z_2^{-1}$$

$$\ll \Delta x \mathcal{L}^F \{x^{1/4} + \Delta q^{-1/2} + (q\Delta)^{1/2}\} + \Delta^2 x \mathcal{L}^{-E},$$

for any $E > 0$, as required.

This completes our proof of Theorem.

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