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ON DUAL-BIMODULES

(Dedicated to Prof. G. Azumaya for his seventieth birthday)

By

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A ring R with identity in which $I=r_Rl_R(I)$ for every right ideal I and $J=l_Rr_R(J)$ for every left ideal J of R is called *a dual ring*. This ring has been investigated by many authors. As is well-known, an Artinian dual ring is a QF-ring and, recently, Hajarnavis and Norton [4] have studied dual rings and pointed out that certain properties well-known for QF-rings are also seen to hold without the Artinian assumption.

In this paper, we shall introduce the notion of dual-bimodules and try to give a module-theoretic characterization of dual rings. Let R and S be rings with identity and ${}_RQ_S$ an (R, S)-bimodule. We shall call Q *a left dual-bimodule* if

(1) $l_Rr_Q(A)=A$ for every left ideal A of R, and

(2) $r_Ql_R(Q')=Q'$ for every S-submodule Q' of Q.

A right dual-bimodule is similarly defined and we shall call Q *a dual-bimodule* if it is a left dual-bimodule and is a right dual-bimodule as well. A left dual-bimodule need not be a right dual-bimodule in general (see Example 4.2).

Trivially a dual ring is a dual-bimodule. A bimodule which defines a Morita duality is a dual-bimodule [1, Exercise 24.7]. Furthermore, a dual-bimodule is a quasi-Frobenius bimodule in the sense of Azumaya [2] (cf. also [5, Theorem 4]).

In Section 1, we shall study basic properties of left dual-bimodules and show that, among other things, an (R, S)-bimodule Q such that the mapping

$$\lambda: R \longrightarrow \mathrm{End}\,(Q_S)$$

given by $a\to a_L$, the left multiplication by a, is surjective is a left dual-bimodule if and only if every factor module of ${}_RR$ and Q_S is Q-torsionless (Theorem 1.4), for a left dual-bimodule ${}_RQ_S$ the ring R is semilocal (Theorem 1.10) and that for every R-module ${}_RQ\neq 0$, ${}_RQ_S$ is a left dual-bimodule with $S=\mathrm{End}\,({}_RQ)$ if and

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only if R is simple Artinian (Theorem 1.16). Finally, in closing this section, we shall show that the notion of left dual-bimodules is closed under Morita equivalence (Theorem 1.20).

We shall treat in Section 2 dual-bimodules. It is shown that for an (R, S)-bimodule Q with both ${}_RQ$ and Q_S finitely generated, Q is a dual-bimodule if and only if ${}_RR$ and S_S are Q-reflexive and every factor module of ${}_RR$, S_S, ${}_RQ$ and Q_S is Q-torsionless (Theorem 2.8). Furthermore, if Q_S is finitely generated and $\mathrm{rad}\,({}_RQ) \leqq \mathrm{rad}\,(Q_S)$, then the ring R is semiperfect (Theorem 2.10).

In Section 3, we shall consider a duality defined by a left dual-bimodule ${}_RQ_S$. It is shown that, in case ${}_RQ$ is finitely generated, a duality defined by Q exists if and only if Q_S is quasi-injective and λ is surjective (Theorem 3.3) and that the duality is one between the full subcategory of R-mod of finitely generated Q-reflexive R-modules and the full subcategory of mod-S of finitely cogenerated Q-reflexive S-modules (Proposition 3.4).

Finally we shall provide, in Section 4, some examples of left dual-bimodules to illustrate the results given in this paper.

Throughout this paper, R and S will denote rings with identity. If ${}_RM$ is a left R-module and M' is a submodule of M, then we shall write $M' \leqq_R M$, in particular, $A \leqq_R R$ will mean A is a left ideal of R. For $M' \leqq_R M$, $M' \leqq_{e(s)R} M$ will mean M' is essential (small) in M. We shall use similar notations for right S-modules. For an (R, S)-bimodule Q, we write $(\ )^* = \mathrm{Hom}\,(-, Q)$ to denote the Q-dual functor.

For notations, definitions and familiar results concerning the ring theory we shall mainly follow [1].

1. Left Dual-Bimodules.

We shall begin with the following

LEMMA 1.1 [1, Exercise 24.3]. *Let Q be a left dual-bimodule. Then for each indexed set $(A_\lambda)_\Lambda$ of left ideals of R and each indexed set $(Q'_\lambda)_\Delta$ of submodules of Q_S*

$$r_Q(\textstyle\bigcap_\Lambda A_\lambda) = \sum_\Lambda r_Q(A_\lambda) \quad \text{and} \quad l_R(\bigcap_\Lambda Q'_\lambda) = \sum_\Lambda l_R(Q'_\lambda)\,.$$

The preceding lemma implies that if Q is a left dual-bimodule, then the mapping $A \to r_Q(A)$ is a lattice anti-isomorphism with inverse $Q' \to l_R(Q')$ between the submodule lattices of ${}_RR$ and Q_S. In particular, we have $l_R(Q) = 0$, i.e. ${}_RQ$ is faithful.

LEMMA 1.2. *Let Q be an (R, S)-bimodule. Then for $A \leqq_R R$ the following conditions are equivalent*:

(1) $l_R r_Q(A)=A$.

(2) *R/A is a Q-torsionless R-module.*

PROOF. This follows from the fact that $l_R r_Q(A)/A=\mathrm{Rej}_{R/A}(Q)$ for every $A \leqq_R R$ [1, Lemma 24.4], where $\mathrm{Rej}_{R/A}(Q)=\cap\{\mathrm{Ker}\, h \mid h \in \mathrm{Hom}_R(R/A, Q)\}$ [1, p. 109].

Let Q be a left dual-bimodule. Then by (1.2) ${}_R R$ is Q-torsionless. Hence, not only cyclic R-modules, but also left ideals of R are Q-torsionless.

Note that if a bimodule Q defines a Morita duality, then every left ideal of R is Q-reflexive [1, p. 278]. However, there is a dual-bimodule Q which has no Q-reflexive left ideal of R (see Example 4.1). Hence a dual-bimodule need not define a Morita duality, in general.

Recall that λ: $R \rightarrow \mathrm{End}(Q_S)$ is the mapping given by $a \rightarrow a_L$, the left multiplication by a. If Q is a left dual-bimodule, then ${}_R Q$ is faithful and hence λ is injective.

LEMMA 1.3. *Let Q be an (R, S)-bimodule. Then for $Q' \leqq Q_S$ the following conditions are equivalent*:

(1) $r_Q l_R(Q')=Q'$.

(2) *$Q' \cong^{\phi} (R/l_R(Q'))^*$, where $\phi: Q' \rightarrow (R/l_R(Q'))^*$ denotes the monomorphism given by $\phi(u)(a+l_R(Q'))=au$ for $u \in Q'$, $a \in R$.*

Furthermore, (1) *implies*

(3) *Q/Q' is Q-torsionless,*

and if λ is surjective, then (3) *implies* (1).

PROOF. Since $Q' \leqq r_Q l_R(Q')$ and the composite map of the canonical isomorphism $(R/l_R(Q'))^* \cong r_Q l_R(Q')$ with ϕ is the identity map of Q', the equality holds if and only if ϕ is onto. This means that (1) and (2) are equivalent.

(1)$\Rightarrow$(3) follows from the fact that $\mathrm{Rej}_{Q/Q'}(Q) \leqq r_Q l_R(Q')/Q'$. If λ is surjective, these are the same and (3) implies (1).

Clearly Q_S is Q-torsionless. Hence, for a left dual-bimodule Q by (1.3) not only submodules of Q_S, but also factor modules of Q_S are Q-torsionless.

Combining these two lemmas, we have

THEOREM 1.4. *Let Q be an (R, S)-bimodule. If λ is surjective, then the* *following conditions are equivalent*:

(1) *Q is a left dual-bimodule.*

(2) *Every factor module of ${}_RR$ and Q_S is Q-torsionless.*

As we shall show in (2.7), if Q is a dual-bimodule and Q_S is finitely generated, then λ is surjective. However, in case λ is not surjective, though every factor module of ${}_RR$ and Q_S is Q-torsionless, we can not conclude that Q is a left dual-bimodule, in general (see Example 4.4).

The followmg lemma is often useful.

LEMMA 1.5. *Let Q be a left dual-bimodule, $A \leqq {}_RR$ and $Q' \leqq Q_S$. Then we have*

(1) *$A \leqq_{e(s)} {}_RR$ if and only if $r_Q(A) \leqq_{s(e)} Q_S$.*

(2) *$Q' \leqq_{e(s)} Q_S$ if and only if $l_R(Q') \leqq_{s(e)} {}_RR$.*

PROOF. (1) Suppose that $A \leqq_s {}_RR$ and $r_Q(A) \cap Q' = 0$ for some $Q' \leqq Q_S$. Then by (1.1) $A + l_R(Q') = R$ and hence $l_R(Q') = R$. Thus we have $Q' = 0$, from which we see that $r_Q(A) \leqq_e Q_S$.

Conversely, suppose that $r_Q(A) \leqq_e Q_S$ and $A + A' = R$ for some $A' \leqq {}_RR$. Then $r_Q(A) \cap r_Q(A') = 0$ and hence $r_Q(A') = 0$. Thus we have $A' = R$, which shows that $A \leqq_s {}_RR$.

(2) follows from (1) at once.

From this lemma, we can see that the socle corresponds to the radical to each other under the lattice anti-isomorphism between the submodule lattices of ${}_RR$ and Q_S. Indeed, we have

PROPOSITION 1.6. *Let Q be a left dual-bimodule. Then*

(1) $Z({}_RQ) = \mathrm{rad}(Q_S) = r_Q(\mathrm{soc}({}_RR))$ *where $Z({}_RQ)$ denotes the singular submodule of ${}_RQ$.*

(2) $\mathrm{rad}(R) = l_R(\mathrm{soc}(Q_S))$.

PROOF. (1) If $u \in Z({}_RQ)$, then by (1.5) $uS \leqq_s Q_S$ and hence $u \in \mathrm{rad}(Q_S)$. Conversely, if $u \in \mathrm{rad}(Q_S)$, then u is contained in some small submodule Q' of Q_S. Hence, uS is also small in Q_S. Again by (1.5) $l_R(u) \leqq_e {}_RR$ and u is in $Z({}_RQ)$.

Furthermore, $\mathrm{rad}(Q_S) = \cap\{Q' \leqq Q_S \mid Q'$ is maximal in $Q_S\} = \cap\{r_Q(A) \mid A$ is minimal in ${}_RR\} = r_Q(\mathrm{soc}({}_RR))$.

Likewise (2) follows from (1.1).

PROPOSITION 1.7. *Let Q be a left dual-bimodule. Then Q_S has finite Goldie dimension.*

PROOF. Let $0 \neq u \in Q$. If there is no nonzero submodule of Q_S not containing u, then Q_S is indeed uniform. Otherwise there exists a submodule Q_u of Q_S maximal with respect to not containing u by Zorn's lemma. Then Q/Q_u is uniform.

Now clearly $\bigcap_{0 \neq u \in Q} Q_u = 0$. Therefore $R = \sum_{0 \neq u \in Q} l_R(Q_u)$ and hence there exist $u_1, \cdots, u_n$ in Q such that $l_R(Q_{u_1}) + \cdots + l_R(Q_{u_n}) = R$. We therefore have $\bigcap_{i=1}^n Q_{u_i} = 0$. Thus Q is embedded into $Q/Q_{u_1} \oplus \cdots \oplus Q/Q_{u_n}$, from which we see that Q_S has finite Goldie dimension.

From this proof we see at once

PROPOSITION 1.8. *Let Q be a left dual-bimodule. Then*

(1) *Q_S is finitely cogenerated.*

(2) $\mathrm{soc}(Q_S)$ *is finitely generated and is the smallest essential submodule of Q_S* [1, *Proposition* 10.7].

(3) *There are only finitely many non-isomorphic simple submodules of Q_S.*

The preceding proposition is based on the fact that ${}_RR$ is finitely generated. Hence, we have

PROPOSITION 1.9. *Let Q be a left dual-bimodule. Then Q_S is finitely generated if and only if ${}_RR$ is finitely cogenerated.*

If this is the case, $\mathrm{soc}({}_RR)$ *is finitely generated and is the smallest essential left ideal of R.*

PROOF. The proof of the "only if" part is similar to that of (1.8). To prove the "if" part, suppose that ${}_RR$ is finitely cogenerated. Since $Q_S = \sum_{u \in Q} uS$, it follows that $0 = \bigcap_{u \in Q} l_R(uS)$. By assumption there exist $u_1, \cdots, u_n$ in Q such that $0 = \bigcap_{i=1}^n l_R(u_i S)$ and hence we have $Q = \sum_{i=1}^n u_i S$. This shows that Q is finitely generated.

THEOREM 1.10. *Let Q be a left dual-bimodule. Then R is semilocal, i.e.* $R/\mathrm{rad}(R)$ *is semisimple.*

PROOF. Let $\mathrm{soc}(Q_S) = \bigoplus_{i=1}^n Q_i$, where each Q_i is a simple submodule of Q_S. Then $\mathrm{rad}(R) = \bigcap_{i=1}^n l_R(Q_i)$. Since each $l_R(Q_i)$ is a maximal left ideal of R and $0 \to R/\mathrm{rad}(R) \to \bigoplus_{i=1}^n R/l_R(Q_i)$ is exact, $R/\mathrm{rad}(R)$ is semisimple.

In particular, we have by [1, Proposition 15.17]

PROPOSITION 1.11. *For a left dual-bimodule Q, we have*

$$\mathrm{soc}({}_RQ)=r_Q(\mathrm{rad}(R))=\mathrm{soc}(Q_S)\,.$$

Henceforth we shall denote $\mathrm{soc}({}_RQ)=\mathrm{soc}(Q_S)$ simply by $\mathrm{soc}(Q)$.

Using [1, Corollary 15.18], for any R-module ${}_RM$,

$$\mathrm{rad}({}_RM)=\mathrm{rad}(R)\cdot M$$

and $M/\mathrm{rad}({}_RM)$ is semisimple, i.e. ${}_RM$ is semisimple if and only if $\mathrm{rad}({}_RM)=0$.

As an application of (1.6) and (1.11), we have

PROPOSITION 1.12. *Let Q be a left dual-bimodule. Then the following conditions are equivalent*:

(1) *R is semisimple.*

(2) *Q_S is semisimple.*

(3) *${}_RQ$ is semisimple.*

(4) $Z({}_RQ)=0$.

(5) $\mathrm{rad}(Q_S)=0$.

LEMMA 1.13. *Let Q be a left dual-bimodule. Then every R-homomorphism from a left ideal of R to Q with finitely generated image is given by a right multiplication of an element of Q.*

PROOF. Cf. [4, Proposition 5.2].

The preceding lemma implies that, for every finitely generated left ideal A of R, every diagram of the form

$$\begin{tikzcd} A \arrow[d] \arrow[r, phantom, "\leqq"] & R \\ Q & \end{tikzcd}$$

is completed by an R-homomorphism $R\to Q$.

Hence, by [6, Proposition 2.8] we have

COROLLARY 1.14. *Let Q be a left dual-bimodule. If either ${}_RQ$ or ${}_RR$ is Noetherian, then ${}_RQ$ must be an injective cogenerator.*

In general, for a finitely generated left ideal A of R, the mapping $Q/r_Q(A)\to A^*$ given by $u+r_Q(A)\to u_R|_A$ is an S-monomorphism. The lemma also shows that this mapping is surjective and hence

$$Q/r_Q(A)\cong A^* .$$

Let Q be a left dual-bimodule and Q' an (R, S)-submodule. If ${}_RQ'_S$ is also a left dual-bimodule, then ${}_RQ'$ must be faithful. Hence, $Q'=r_Ql_R(Q')=r_Q(0)=Q$. Thus, there is no proper (R, S)-submodule which is also a left dual-bimodule. However, we have

PROPOSITION 1.15. *Let Q be a left dual-bimodule, Q' an (R, S)-submodule and $\bar{R}=R/l_R(Q')$. Then ${}_{\bar{R}}Q'_S$ is a left dual-bimodule.*

PROOF. Since Q' can be regarded as an $\bar{R}$-module by defining $a+l_R(Q')\cdot u'=au'$ for $a\in R$ and $u'\in Q'$, we have $r_{Q'}(A/l_R(Q'))=r_{Q'}(A)$ for $A/l_R(Q')\leqq_{\bar{R}}\bar{R}$ and $l_{\bar{R}}(Q'')=l_R(Q'')/l_R(Q')$ for $Q''\leqq Q'_S$. Therefore, we have $l_{\bar{R}}r_{Q'}(A/l_R(Q'))=l_{\bar{R}}r_{Q'}(A)=l_Rr_{Q'}(A)/l_R(Q')=(l_Rr_Q(A)+l_R(Q'))/l_R(Q')=l_Rr_Q(A)/l_R(Q')=A/l_R(Q')$ and $r_{Q'}l_{\bar{R}}(Q'')=r_{Q'}(l_R(Q'')/l_R(Q'))=r_{Q'}l_R(Q'')=r_Ql_R(Q'')\cap Q'=Q''\cap Q'=Q''$.

In particular, for a left dual-bimodule Q, $\mathrm{soc}(Q)$ is an (R, S)-submodule and hence ${}_{\bar{R}}\mathrm{soc}(Q)_S$ is a left dual-bimodule satisfying the equivalent condition of (1.12), where $\bar{R}=R/\mathrm{rad}(R)$.

The following theorem characterizes simple Artinian rings by means of the notion of left dual-bimodules.

THEOREM 1.16. *For a ring R the following conditions are equivalent:*

(1) *R is simple Artinian.*

(2) *For every R-module ${}_RQ\neq 0$, ${}_RQ_S$ is a left dual-bimodule with $S=\mathrm{End}({}_RQ)$.*

(3) *For every finitely generated R-module ${}_RQ\neq 0$, ${}_RQ_S$ is a left dual-bimodule with $S=\mathrm{End}({}_RQ)$.*

(4) *For every simple R-module ${}_RQ$, ${}_RQ_S$ is a left dual-bimodule with $S=\mathrm{End}({}_RQ)$.*

(5) *There exists a simple R-module ${}_RQ$ such that ${}_RQ_S$ is a left dual-bimodule with $S=\mathrm{End}({}_RQ)$.*

If this is the case, $R\cong^{\lambda}\mathrm{End}(Q_S)$ for every R-module ${}_RQ\neq 0$ with $S=\mathrm{End}({}_RQ)$. Furthermore, in case ${}_RQ$ is finitely generated, S is also simple Artinian.

PROOF. (1)$\Rightarrow$(2). Let R be a simple Artinian ring, ${}_RQ\neq 0$ and $S=\mathrm{End}({}_RQ)$. Then by [1, Exercise 13.10] ${}_RQ$ is a cogenerator. Hence every (cyclic) R-module is Q-torsionless.

Furthermore, ${}_RQ$ is balanced by [1, Excrcise 18.32] which means that λ is surjective. However, $\mathrm{Ker}\,\lambda$ must be zero, since R is a simple ring and $Q\neq 0$. Thns, we have $R\cong^{\lambda}\mathrm{End}(Q_S)$.

Since ${}_RR$ is semisimple, we can write R as $R=m_1\oplus\cdots\oplus m_n$ with each m_i a minimal left ideal of R. Using this decomposition, $Q_S\cong\mathrm{Hom}_R(R, Q)\cong \mathrm{Hom}_R(m_1, Q)\oplus\cdots\oplus\mathrm{Hom}_R(m_n, Q)$, where each $\mathrm{Hom}_R(m_i, Q)$ is either simple or zero by [1, Exercise 16.18]. It follows that Q_S is semisimple.

Now let $Q'\leqq Q_S$. Then Q/Q' is isomorphic to a submodule of Q_S and hence is Q-torsionless. Thus, by (1.4) ${}_RQ_S$ is a left dual-bimodule.

(2)⇒(3)⇒(4)⇒(5) are clear.

(5)⇒(1). Let ${}_RQ$ be a simple R-module such that ${}_RQ_S$ is a left dual-bimodule with $S=\mathrm{End}({}_RQ)$. Then for any ideal A of R $r_Q(A)$ is an (R, S)-submodule of Q and is either Q or 0. Hence A must be either 0 or R and R is a simple ring. Since $\mathrm{rad}(R)=0$, ${}_RR$ is semisimple by (1.10). Thus R is simple Artinian by [1, Proposition 13.5].

Note that in the preceding theorem each condition of (1) to (5) is also equivalent to each one of the following

(3′) For every finitely generated R-module ${}_RQ\neq 0$, ${}_RQ_S$ is a dual-bimodule with $S=\mathrm{End}({}_RQ)$.

(4′) For every simple R-module ${}_RQ$, ${}_RQ_S$ is a dual-bimodule with $S=\mathrm{End}({}_RQ)$.

(5′) There exists a simple R-module ${}_RQ$ such that ${}_RQ_S$ is a dual-bimodule with $S=\mathrm{End}({}_RQ)$.

To see this, assume that R is simple Artinian and ${}_RQ\neq 0$ is a finitely generated R-module with $S=\mathrm{End}({}_RQ)$. As was shown in the proof of (1)⇒(2) of the preceding theorem, ${}_RQ_S$ is a left dual-bimodule and $R\cong^{\lambda}\mathrm{End}(Q_S)$. Hence, to prove (3′) it is sufficient to show that S is simple Artinian. Since ${}_RQ$ is semisimple by (1.12) and is finitely generated, we can write ${}_RQ$ as $Q=Q_1\oplus\cdots\oplus Q_n$ with each ${}_RQ_i$ simple and $Q_i\cong Q_j$ for all i and j [1, Exercise 13.1]. Therefore, S is isomorphic to the ring of all $n\times n$ matrices over the division ring $\mathrm{End}({}_RQ_i)$ and thus it is simple Artinian. This shows that (1)⇒(3′) and (3′)⇒(4′)⇒(5′)⇒(5) are evident.

As we shall show in Example 4.5, the condition (2′) corresponding to the condition (2) of (1.16) does not hold in general.

The following proposition follows from (1.14) and the proof of (1)⇒(2) of (1.16).

PROPOSITION 1.17. *Let R be a semisimple ring and ${}_RQ$ an R-module with $S=\mathrm{End}({}_RQ)$. Then ${}_RQ_S$ is a left dual-bimodule if and only if ${}_RQ$ is a cogenerator.*

Let Q be a left dual-bimodule. Then Q_S is finitely cogenerated and hence by [1, Exercise 10.15] Q_S has a finite indecomposable decomposition $Q_S=Q_1\oplus\cdots\oplus Q_n$ with each Q_i indecomposable. Each Q_i can be written as $Q_i=r_Q(A_i)$ for some $A_i\leqq_R R$ and R/A_i is indecomposable. For, if R/A_i is decomposable and $R/A_i=A'/A_i\oplus A''/A_i$ for $A_i\leqq A'$, $A''\leqq_R R$, then we have $Q_i=r_Q(A')\oplus r_Q(A'')$, a contradiction. Since $r_Q(A_i+\bigcap_{j\neq i}A_j)=Q_i\cap\sum_{j\neq i}Q_j=0$, $(A_i)_{1\leq i\leq n}$ is coindependent and hence the R-homomorphism $f: R\to\bigoplus_{i=1}^n R/A_i$ defined by $f(a)=(a+A_i)$ for $a\in R$ is surjective by [1, Exercise 6.18]. Furthermore, $\operatorname{Ker} f=\bigcap_{i=1}^n A_i=l_R(Q)=0$. Thus, we have $R\cong^f\bigoplus_{i=1}^n R/A_i$ and R has a finite indecomposable decomposition.

PROPOSITION 1.18. *Let Q be a left dual-bimodule. Then both Q_S and ${}_RR$ have finite indecomposable decompositions. In particular, Q_S is indecomposable if and only if ${}_RR$ is indecomposable.*

Finally, in closing this section, we shall show that the notion of left dual-bimodules is closed under Morita equivalence.

To see this, let ${}_RQ_S$ be a left dual-bimodule and T a ring equivalent to S via an equivalence H: mod-$S\to$mod-T. There exists a (T, S)-bimodule P such that ${}_TP$ and P_S are progenerators and H is given by $H\cong\operatorname{Hom}_S(P,\text{-})$ [1, Theorem 22.1]. We assume that for simplicity $H=\operatorname{Hom}_S(P,\text{-})$. Using [1, Proposition 21.7], each submodule of $H(Q)_T$ is of the form $\operatorname{Im} H(\nu)$ for some $Q'\leqq Q_S$ and the inclusion map $\nu: Q'\to Q$.

LEMMA 1.19. *With the same notation as above, we have*

(1) $l_R(\operatorname{Im} H(\nu))=l_R(Q')$.

(2) $r_{H(Q)}l_R(\operatorname{Im} H(\nu))=\operatorname{Im} H(\nu)$.

For a left ideal A of R and the inclusion map $\mu: r_Q(A)\to Q$,

(3) $r_{H(Q)}(A)=\operatorname{Im} H(\mu)$.

(4) $l_Rr_{H(Q)}(A)=A$.

PROOF. (1) Suppose that $a\in l_R(Q')$. Then for any $f\in H(Q')$ and any $p\in P$ $(a\cdot\nu f)(p)=a\cdot f(p)\in aQ'=0$ and hence $l_R(Q')\leqq l_R(\operatorname{Im} H(\nu))$. Conversely, suppose that $a\in l_R(\operatorname{Im} H(\nu))$. Since P_S is a generator, there exists a set Λ such that $P^{(\Lambda)}\to^{\alpha}Q'\to 0$ is exact. For the injection map $\nu_\lambda: P\to P^{(\Lambda)}$, $\lambda\in\Lambda$, $\alpha\nu_\lambda$ is in $H(Q')$ and hence by assumption $(a\cdot\alpha\nu_\lambda)(p)=0$ for each $p\in P$ and each $\lambda\in\Lambda$. Let $u'\in Q'$ and let $x\in P^{(\Lambda)}$ such that $\alpha(x)=u'$. Then x can be written as $x=\nu_{\lambda_1}(p_1)+\cdots+\nu_{\lambda_k}(p_k)$ for some $\lambda_1,\cdots,\lambda_k\in\Lambda$ and $p_1,\cdots,p_k\in P$. Then $au'=a\cdot\alpha(x)=(a\cdot\alpha\nu_{\lambda_1})(p_1)+\cdots+(a\cdot\alpha\nu_{\lambda_k})(p_k)=0$. Hence, $a\in l_R(Q')$ and thus $l_R(\operatorname{Im} H(\nu))$

$\leq l_R(Q')$.

(2) Let $f \in r_{H(Q)} l_R(\mathrm{Im}\, H(\nu)) = r_{H(Q)} l_R(Q')$. Then for each $a \in l_R(Q')$ and each $p \in P$ $a \cdot f(p) = (a \cdot f)(p) = 0$. Hence $f(p) \in r_Q l_R(Q') = Q'$, since ${}_RQ_S$ is a left dual-bimodule. It follows that $f(p) = \nu(f(p)) = (\nu f)(p)$ and thus $f = \nu f \in \mathrm{Im}\, H(\nu)$. Hence we have $r_{H(Q)} l_R(\mathrm{Im}\, H(\nu)) \leq \mathrm{Im}\, H(\nu)$ and thus (2) follows.

(3) Let $f \in r_{H(Q)}(A)$. Then for each $a \in A$ and each $p \in P$ $a \cdot f(p) = (af)(p) = 0$. It follows that $f(p) \in r_Q(A)$ and hence $f = \mu f \in \mathrm{Im}\, H(\mu)$. Conversely, let $\mu f \in \mathrm{Im}\, H(\mu)$, where $f \in H(r_Q(A))$. Then for each $a \in A$ and each $p \in P$ $(a \cdot \mu f)(p)$ $a \cdot f(p) = 0$. Hence, $a \cdot \mu f = 0$ and $\mu f \in r_{H(Q)}(A)$. Thus, we have $r_{H(Q)}(A) = \mathrm{Im}\, H(\mu)$.

(4) Using (1), $l_R(\mathrm{Im}\, H(\mu)) = l_R r_Q(A)$ and hence by (3) $l_R r_{H(Q)}(A) = l_R r_Q(A) = A$, since ${}_RQ_S$ is a left dual-bimodule.

THEOREM 1.20. *Let Q be a left dual-bimodule and let T be a ring equivalent to S via an equivalence H: mod-$S \to$ mod-T. Then ${}_RH(Q)_T$ is also a left dual-bimodule.*

As is well-known, S and the ring $(S)_n$ of all $n \times n$ matrices over S are equivalent via $H = - \otimes_S S^n$: mod-$S \to$ mod-$(S)_n$. Hence, we have

COROLLARY 1.21. *Let Q be a left dual-bimodule. Then for each $n > 0$, ${}_RQ^n_{(S)_n}$ is also a left dual-bimodule.*

In particular, if R is a dual ring, then for each $n > 0$, ${}_RR^n_{(R)_n}$ is a left dual-bimodule.

2. Dual-Bimodules.

If Q is a left dual-bimodule, then there are only finitely many non-isomorphic simple submodules of Q_S. However, in case Q is a dual-bimodule, by (1.10) there are only finitely many non-isomorphic simple right S-modules and each of which is isomorphic to a submodule of Q_S [6, Proposition 2.8]. Furthermore, we have

THEOREM 2.1. *Let Q be a dual-bimodule. Then*

(1) *The Q-dual of every simple left R-module as well as that of every simple right S-module is simple.*

(2) *Every simple left R-module as well as every simple right S-module is Q-reflexive.*

(3) *There is a bijection between the irredundant sets of representatives of the simple left R-modules and the simple right S-modules.*

PROOF. Suppose first that Q is a left dual-bimodule and ${}_RM$ is a simple R-module. Then by [6, Proposition 2.8] M is isomorphic to a simple submodule Ru of ${}_RQ$ for some $u\in Q$. Therefore, $M^*\cong(R/l_R(u))^*\cong r_Ql_R(u)=uS$ and hence M^*_S is simple, since $l_R(u)$ is a maximal left ideal. Thus, for each $u\in Q$, ${}_RRu$ is simple if and only if uS_S is simple and further we have $uS\cong(Ru)^*$ via $us\to s_R|_{Ru}$.

However, if in addition Q is a right dual-bimodule, then every simple right S-module is of the form uS for some $u\in Q$. Since $(uS)^*\cong(S/r_S(u))^*\cong l_Qr_S(u)=Ru$, $Ru\cong(uS)^*$ via $au\to a_L|_{uS}$ and thus for each $u\in Q$ the mapping $Ru\to uS$ can be seen as a bijection between irredundant sets of representatives of the simple left R-modules and the simple right S-modules.

Finally it is easy to see that isomorphisms mentioned above yield the condition (2).

More precisely, we have

PROPOSITION 2.2. *For a dual-bimodule Q, let $\bar{e}_1, \cdots, \bar{e}_m$ and $\bar{f}_1, \cdots, \bar{f}_m$ be basic sets of idempotents of the semisimple ring $\bar{R}=R/\mathrm{rad}(R)$ and $\bar{S}=S/\mathrm{rad}(S)$, respectively. Then*

$$e_1\cdot\mathrm{soc}(Q),\ e_2\cdot\mathrm{soc}(Q),\ \cdots,\ e_m\cdot\mathrm{soc}(Q)$$

and

$$\mathrm{soc}(Q)\cdot f_1,\ \mathrm{soc}(Q)\cdot f_2,\ \cdots,\ \mathrm{soc}(Q)\cdot f_m$$

exhaust non-isomorphic simple right S-modules and that of simple left R-modules, respectively.

PROOF. For each i, $l_R(e_i\cdot\mathrm{soc}(Q))=\{a\in R\mid ae_i\in\mathrm{rad}(R)\}$ and hence the mapping $R\to\bar{R}\bar{e}_i$, given by $a\to\bar{a}\bar{e}_i$, is an R-epimorphism with kernel $l_R(e_i\cdot\mathrm{soc}(Q))$. Therefore, $e_i\cdot\mathrm{soc}(Q)$ is a simple submodule of Q_S. Furthermore, $e_i\cdot\mathrm{soc}(Q)\cong(R/l_R(e_i\cdot\mathrm{soc}(Q)))^*\cong(\bar{R}\bar{e}_i)^*$. Thus, the proposition follows from (2.1).

THEOREM 2.3. *Let Q be a dual-bimodule. Then every finitely generated submodule of Q_S as well as that of ${}_RQ$ is Q-reflexive.*

To see this, we need a lemma which is shown by a similar way as in [4, Proposition 5.2].

LEMMA 2.4. *Let Q be a dual-bimodule and $Q'\leq Q_S$. Then every S-homomorphism from Q' to Q with finitely generated image is given by a left multiplication of an element of R.*

It follows from this lemma that if Q_S is Noetherian, then Q_S is quasi-injective.

PROOF OF (2.3). For every finitely generated submodule Q' of Q_S, the R-momomorphism $R/l_R(Q')\to Q'^*$ given by $a+l_R(Q')\to a_L|_{Q'}$ yields by (2.4)

$$R/l_R(Q')\cong Q'^*$$

Therefore, using the natural isomorphism $Q'=r_Q l_R(Q')\cong(R/l_R(Q'))^*$, we see that Q' is Q-reflexive.

The preceding theorem is not true without the assumption that Q' is finitely generated (see Example 4.1).

Since $\mathrm{soc}(Q_S)$ is finitely generated, the above isomorphism $R/l_R(Q')\cong Q'^*$ yields, in particular,

$$R/\mathrm{rad}(R)\cong \mathrm{End}(\mathrm{soc}(Q)_S)$$

as rings.

From (2.3) and [1, Proposition 20.14] we have

COROLLARY 2.5. *Let Q be a dual-bimodule. Then for every finitely generated submodule Q' of Q_S, $R/l_R(Q')$ is Q-reflexive.*

The proof of [4, Theorem 5.3] carries over almost word for word to the case of dual-bimodules.

PROPOSITION 2.6. *Let Q be a dual-bimodule. Then for each $n>0$ every factor module of Q_S^n has finite Goldie dimension.*

In particular, in case where Q_S is a generator, every finitely generated right S-module has finite Goldie dimension.

PROOF. First we shall prove by induction on n that every semisimple submodule of any factor module of Q_S^n is finitely generated.

Let $n=1$ and $K\leqq Q'\leqq Q_S$. Suppose that Q'/K is semisimple, which is not finitely generated. Then by (2.2) Q'/K contains a countably infinite direct sum $\bigoplus_{i\geq 1}(u_iS+K)/K$, where each $(u_iS+K)/K$ is simple and is isomorphic to the same simple S-module $e\cdot\mathrm{soc}(Q)$. Let $f_i: u_iS+K\to e\cdot\mathrm{soc}(Q)$ be the composite of the canonical map $\pi_i: u_iS+K\to(u_iS+K)/K$ with the isomorphism. Then by (2.4) $f_i=a_{iL}$ for some $a_i\in l_R(K)$ and hence we have $a_iu_iS=e\cdot\mathrm{soc}(Q)$.

We now define for any subset Λ of N, where N denotes the set of positive integers, an S-homomorphism

$$h_\Lambda : \sum_{i\leq 1}(u_iS+K) \longrightarrow e\cdot\mathrm{soc}(Q)$$

to be $h_\Lambda(u_i)=a_iu_i$ whenever $i\in\Lambda$, $h_\Lambda(u_i)=0$ whenever $i\notin\Lambda$, $h_\Lambda(K)=0$ and extending this definition by linearity. By (2.4) $h_\Lambda=b_{\Lambda L}$ for some $b_\Lambda\in l_R(K)$ and hence for each Λ and $i\in N$ we have $eb_\Lambda u_i=b_\Lambda u_i$, since the image of h_Λ is $e\cdot\mathrm{soc}(Q)$.

Using [4, Lemma 5.1] there is an uncountable independent collection C of subsets of N. We shall show that

$$\sum_{\Lambda\in C}(Reb_\Lambda+l_R(Q'))/l_R(Q')$$

is a direct sum. To this end, let $\Lambda_1, \cdots, \Lambda_n$ be distinct elements of C and let $c_1eb_{\Lambda_1}+\cdots+c_neb_{\Lambda_n}\in l_R(Q')$ where $c_1, \cdots, c_n\in R$. For each j, $1\leqq j\leqq n$, take an $t_j\in\Lambda_j\cap(\Lambda_1^{-1}\cap\cdots\cap\Lambda_{j-1}^{-1}\cap\Lambda_{j+1}^{-1}\cap\cdots\cap\Lambda_n^{-1})$, where Λ_i^{-1} means the set $N\setminus\Lambda_i$. Since $u_{t_j}\in Q'$, $c_1eb_{\Lambda_1}u_{t_j}+\cdots+c_neb_{\Lambda_n}u_{t_j}=0$. But if $k\neq j$, then $t_j\notin\Lambda_k$. Hence $b_{\Lambda_k}u_{t_j}=0$ and therefore we have $c_jeb_{\Lambda_j}u_{t_j}=0$ for $1\leqq j\leqq n$. We now show that $c_je\in\mathrm{rad}(R)$ for $1\leqq j\leqq n$. Suppose that $c_je\notin\mathrm{rad}(R)$ for some j. Then $Rc_je+\mathrm{rad}(R)\neq\mathrm{rad}(R)$ and hence $\bar{R}\bar{e}=(Re+\mathrm{rad}(R))/\mathrm{rad}(R)\geqq(Rc_je+\mathrm{rad}(R))/\mathrm{rad}(R)\neq 0$. Since $\bar{R}\bar{e}$ is simple, $Re+\mathrm{rad}(R)=Rc_je+\mathrm{rad}(R)$ and therefore we have $eb_{\Lambda_j}u_{t_j}\in(Re+\mathrm{rad}(R))b_{\Lambda_j}u_{t_j}=(Rc_je+\mathrm{rad}(R))b_{\Lambda_j}u_{t_j}=Rc_jeb_{\Lambda_j}u_{t_j}+\mathrm{rad}(R)b_{\Lambda_j}u_{t_j}=0$, since $c_jeb_{\Lambda_j}u_{t_j}=0$ and $b_{\Lambda_j}u_{t_j}\in e\cdot\mathrm{soc}(Q)\leqq\mathrm{soc}(Q)=r_Q(\mathrm{rad}(R))$. However, $eb_{\Lambda_j}u_{t_j}=b_{\Lambda_j}u_{t_j}=a_{t_j}u_{t_j}\neq 0$, a contradiction.

Since Q'/K is semisimple, $Q'/K\cdot\mathrm{rad}(S)\leqq\mathrm{rad}(Q'/K)=0$ and so we have $Q'\cdot\mathrm{rad}(S)\leqq K$, which means that $l_R(K)\cdot Q'\cdot\mathrm{rad}(S)=0$ and $l_R(K)\cdot Q'\leqq l_Q(\mathrm{rad}(S))=r_Q(\mathrm{rad}(R))$ by (1.11). Therefore, $\mathrm{rad}(R)\cdot l_R(K)\cdot Q'=0$ and we have $\mathrm{rad}(R)\cdot l_R(K)\leqq l_R(Q')$. Since $c_je\in\mathrm{rad}(R)$ and $b_{\Lambda_j}\in l_R(K)$, $c_jeb_{\Lambda_j}\in l_R(Q')$. This shows that $\sum_{\Lambda\in C}(Reb_\Lambda+l_R(Q'))/l_R(Q')$ is a direct sum.

As we have shown above, $\mathrm{rad}(R)\cdot l_R(K)\leqq l_R(Q')$, from which we see that $l_R(K)/l_R(Q')$ is an $\bar{R}$-module and is a semisimple R-module. On the other hand, $b_\Lambda\in l_R(K)$ implies that $(Reb_\Lambda+l_R(Q'))/l_R(Q')$ is a submodule of $l_R(K)/l_R(Q')$. Hence it is semisimple and so $\bigoplus_{\Lambda\in C}(Reb_\Lambda+l_R(Q'))/l_R(Q')$ is also semisimple. Thus, we see that $\dim\,(l_R(K)/l_R(Q'))\geqq|C|>|N|$ (see [4, p. 259] for the definition). A symmetrical argument now gives $\dim(Q'/K)>|N|$. But this holds whenever Q'/K is a non finitely generated semisimple S-module and in particular when $Q'/K=\bigoplus_{i\geq 1}(u_iS+K)/K$. However, clearly in this case $\dim(Q'/K)=|N|$, a contradiction. Thus, we have established that every semisimple submodule of any factor module of Q_S is finitely generated.

Now suppose that, for $k\leqq n-1$, every semisimple submodule of any factor module of Q_S^k is finitely generated. Let $k=n$ and $K\leqq Q_S^n$. Then $(Q^{n-1}+K)/K\leqq$

Q^n/K and we have $\mathrm{soc}(Q^n/K)\cong\mathrm{soc}((Q^{n-1}+K)/K)\oplus\mathrm{soc}(Q^n/K)/\mathrm{soc}((Q^{n-1}+K)/K)$, Since $(Q^{n-1}+K)/K\cong Q^{n-1}/(Q^{n-1}\cap K)$, it follows that by induction hypothesis $\mathrm{soc}((Q^{n-1}+K)/K)$ is finitely generated. On the other hand, $\mathrm{soc}(Q^n/K)/\mathrm{soc}((Q^{n-1}+K)/K)\cong\mathrm{soc}(Q^n/K)/(\mathrm{soc}(Q^n/K)\cap(Q^{n-1}+K)/K)\cong(\mathrm{soc}(Q^n/K)+(Q^{n-1}+K)/K)/((Q^{n-1}+K)/K)\leqq Q^n/(Q^{n-1}+K)$. Let K_n denote the submodule of all the n-th coordinates of elements of K. Then $Q^n/(Q^{n-1}+K)$ is isomorphic to Q/K_n via $\overline{(v_1, \cdots, v_n)}\to\bar{v}_n$ and hence $\mathrm{soc}(Q^n/K)/\mathrm{soc}((Q^{n-1}+K)/K)$ can be seen as a semisimple submodule of Q/K_n. Hence, it is finitely generated. Therefore, we see that $\mathrm{soc}(Q^n/K)$ is also finitely generated.

Finally, for any $K\leqq Q_S^n$, we shall show that Q^n/K has finite Goldie dimension. Let $0\neq Q_\alpha/K\leqq Q^n/K$ for $\alpha\in A$ and suppose that $(Q_\alpha/K)_{\alpha\in A}$ is independent. For each $\alpha\in A$, take $0\neq\bar{x}_\alpha=x_\alpha+K\in Q_\alpha/K$. Then $\bar{x}_\alpha\cdot\mathrm{rad}(S)\neq\bar{x}_\alpha S$ by Nakayama's lemma and hence $\bar{x}_\alpha S/\bar{x}_\alpha\cdot\mathrm{rad}(S)$ is a nonzero semisimple S-module. Using [1, Exercise 6.3] we have $\oplus_A\bar{x}_\alpha S/\oplus_A\bar{x}_\alpha\cdot\mathrm{rad}(S)\cong\oplus_A(\bar{x}_\alpha S/\bar{x}_\alpha\cdot\mathrm{rad}(S))$ and both $\oplus_A\bar{x}_\alpha S$ and $\oplus_A\bar{x}_\alpha\cdot\mathrm{rad}(S)$ are submodules of Q^n/K. Hence we can see that $\oplus_A(\bar{x}_\alpha S/\bar{x}_\alpha\cdot\mathrm{rad}(S))$ is a semisimple submodule of Q^n/K' for some $K'\leqq Q_S^n$ and is finitely generated. It follows that A is a finite set, which completes the proof of the proposition.

THEOREM 2.7. *Let Q be a dual-bimodule. Then R is a dense subring of* $\mathrm{End}(Q_S)$.

In particular, if Q_S is finitely generated, then we have

$$R\cong^\lambda \mathrm{End}(Q_S).$$

PROOF. Let $f\in\mathrm{End}(Q_S)$, $u_1, \cdots, u_n$ finitely many elements of Q and $Q'=u_1S+\cdots+u_nS$. Then the mapping $f|_{Q'}$ belongs to Q'^* and hence by (2.4) there exists an $a\in R$ such that $f|_{Q'}=a_L|_{Q'}$. Thus, $f(u_i)=au_i$, $1\leqq i\leqq n$, and R is dense in $\mathrm{End}(Q_S)$.

If Q_S is not finitely generated, the theorem is not always true in general (see Example 4.1). We note that the last part of the theorem also follows from (2.5).

By [1, Theorem 24.1], an (R, S)-bimodule Q defines a Morita duality if and only if every factor module of ${}_RR$, S_S, ${}_RQ$ and Q_S is Q-reflexive. However, for a dual-bimodule by (1.4) and (2.7) we have

THEOREM 2.8. *Let Q be an (R, S)-bimodule such that both ${}_RQ$ and Q_S are finitely generated. Then the following conditions are equivalent*:

(1) *Q is a dual-bimodule.*

(2) *${}_RR$ and S_S are Q-reflexive and every factor module of ${}_RR$, S_S, ${}_RQ$ and Q_S is Q-torsionless.*

LEMMA 2.9. *Let Q be a dual-bimodule with λ surjective. Assume that* $\mathrm{rad}({}_RQ)\leqq_s Q_S$. *Then every idempotent of R can be lifted modulo* $\mathrm{rad}(R)$.

PROOF. Cf. [4, Theorem 3.8].

Thus, we have

THEOREM 2.10. *Let Q be a dual-bimodule with Q_S finitely generated and* $\mathrm{rad}({}_RQ)\leqq \mathrm{rad}(Q_S)$. *Then R is semiperfect.*

As we shall show in Example 4.1, there is a dual-bimodule Q for which R is semiperfect, but Q_S is not finitely generated.

3. Dualities.

For a left dual-bimodule Q, it is shown in (1.2) every cyclic R-module is Q-torsionless. The following theorem gives a criterion for every cyclic R-module being Q-reflexive. First, we need a lemma.

LEMMA 3.1 (cf. [3, Proposition 1.1]). *Let ${}_RQ_S$ be an (R, S)-bimodule and N_S an S-module such that Q_S is N-injective and N_S is Q-reflexive. Then for $K\leqq N_S$, N/K is Q-torsionless if and only if K is Q-reflexive.*

PROOF. Let Q_S be N-injective and $K\leqq N_S$. Then we have a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \longrightarrow & N & \longrightarrow & N/K & \longrightarrow & 0 \\ & & \sigma_K\downarrow & & \downarrow\sigma_N & & \downarrow\sigma_{N/K} & & \\ 0 & \longrightarrow & K^{**} & \longrightarrow & N^{**} & \longrightarrow & (N/K)^{**} & & \end{array}$$

where σ_* means the evaluation map. Assume that N_S is Q-reflexive. Then by [1, Lemma 3.14] we see that $\sigma_{N/K}$ is monic if and only if σ_K is epic and this is so if and only if σ_K is an isomorphism, since K is Q-torsionless as a submodule of N_S.

THEOREM 3.2. *Let Q be a left dual-bimodule. Then the following conditions are equivalent:*

(1) *Q_S is quasi-injective and λ is surjective.*

(2) *Every cyclic R-module is Q-reflexive.*

(3) *Every finitely generated Q-torsionless R-module is Q-reflexive.*

Moreover, if each one of these conditions holds, then R is semiperfect and every submodule of Q_S is finitely cogenerated Q-reflexive.

PROOF. (1)⇒(3). Let ${}_RM$ be a finitely generated Q-torsionless R-module. Then $R^n \to M \to 0$ is exact for some $n>0$. Since $(R^n)^* \cong Q^n$ and Q is Q^n-injective, we have a commutative diagram with exact rows

$$\begin{array}{ccccc} R^n & \longrightarrow & M & \longrightarrow & 0 \\ \sigma_{R^n}\downarrow & & \downarrow\sigma_M & & \\ (R^n)^{**} & \longrightarrow & M^{**} & \longrightarrow & 0 \end{array}$$

Since λ is surjective, σ_{R^n} is an epimorphism and hence σ_M is also an epimorphism. Thus, M is Q-reflexive.

(3)⇒(2). This is evident by (1.2).

(2)⇒(1). For any $A \leqq_R R$ the mapping $\lambda_A: R \to r_Q(A)^*$ given by $a \to a_L | r_{Q(A)}$ is an R-homomorphism. With the canonical S-isomorphism $h: (R/A)^* \to r_Q(A)$, λ_A yields a commutative diagram

$$\begin{array}{ccc} R & \xrightarrow{\ \pi\ } & R/A \\ \lambda_A\downarrow & & \downarrow\sigma_{R/A} \\ r_Q(A)^* & \xrightarrow{\ h^*\ } & (R/A)^{**} \end{array}$$

Since $\sigma_{R/A}$ is an epimorphism, so is λ_A. Therefore, for every S-homomorphism $f: r_Q(A) \to Q$, there exists an $a \in R$ such that $f = a_L | r_{Q(A)}$. Thus, Q_S is quasi-injective. In particular, if we take $A=0$, then we see that λ is surjective.

As was pointed out in [3, p. 120], if Q_S is quasi-injective, then $\mathrm{End}(Q_S)$ is semiperfect if and only if Q_S has finite Goldie dimension. Hence, the last part of the theorem follows from (1.3), (1.7) and (3.1).

As is seen from (2.4) and (2.7), if Q is a dual-bimodule and Q_S is Noetherian, then Q satisfies the equivalent condition of the preceding theorem.

It is also to be noted that the equivalence in the preceding theorem is closely related to the assumption that Q is a left dual-bimodule and without this assumption we can not prove (3)⇒(1). See Example 4.6.

We shall give another criterion for every cyclic R-module being Q-reflexive. To do this, for an (R, S)-bimodule ${}_RQ_S$, consider the full subcategory $\underline{M}$ of R-mod of finitely generated Q-torsionless R-modules and the full subcategory $\underline{N}$

of mod-S whose objects are all the S-modules N such that there exists an exact sequence of the form $0 \to N \to Q^n \to Q^I$ for some $n > 0$ and a set I.

THEOREM 3.3. *For an* (R, S)*-bimodule* Q*, consider the following conditions:*

(1) Q_S *is quasi-injective and* λ *is surjective.*

(2) *The pair* (H', H'') *of functors*

$$H' = \mathrm{Hom}_R(-, Q) : \underline{M} \longrightarrow \underline{N} \quad \text{and} \quad H'' = \mathrm{Hom}_S(-, Q) : \underline{N} \longrightarrow \underline{M}$$

defines a duality between $\underline{M}$ *and* $\underline{N}$.

Then (1) *implies* (2). *If* Q *is a left dual-bimodule with* ${}_RQ$ *finitely generated, then* (2) *implies* (1) *and in this case* (1) *and* (2) *are equivalent.*

PROOF. (1)$\Rightarrow$(2) (cf. [3, Proposition 1.3]). First, we note that from the proof of (1)$\Rightarrow$(3) of (3.2) each ${}_RM \in \underline{M}$ is Q-reflexive.

Next we show that $M^* \in \underline{N}$ for every ${}_RM \in \underline{M}$. Since M is finitely generated, $R^n \to M \to 0$ is exact for some $n > 0$. Hence $0 \to M^* \to^{\alpha} Q^n$ is exact. We may show that $Q^n/\alpha(M^*)$ is Q-torsionless. Since λ is surjective, σ_R is an epimorphism and hence R^* is Q-reflexive. Therefore, Q is Q-reflexive and so is Q^n. Applying (3.1) to Q_S and Q_S^n, we see that $Q^n/\alpha(M^*)$ is Q-torsionless, since M^* is Q-reflexive.

Now we show that $N_S \in \underline{N}$ implies $N^* \in \underline{M}$. Let $0 \to N \to Q^n \to Q^I$ be exact for some $n > 0$ and I. Since Q_S is Q^n-injective, $(Q^n)^* \to N^* \to 0$ is exact. Furthermore, λ is surjective and hence $R^n \to (Q^*)^n \to 0$ is exact. Thus, $R^n \to N^* \to 0$ must be exact, from which we see that N^* is finitely generated. By [1, Proposition 20.14] N^* is Q-torsionless.

Finally we see that N is Q-reflexive for $N_S \in \underline{N}$, applying (3.1) again.

(2)$\Rightarrow$(1). This follows from a similar way as in the proof of [1, Theorem 23.5]. Note that, by the assumption that ${}_RQ$ is finitely generated, we may use [1, Exercise 20.5].

As is shown above, the quasi-injectivity of Q_S implies a duality between $\underline{M}$ and $\underline{N}$. The converse, however, is not the case without the assumption that Q is a left dual-bimodule. See Example 4.6.

Now let ${}_RQ_S$ be an (R, S)-bimodule and let $\underline{M}$ and $\underline{N}$ be as above. Assume that Q_S is quasi-injective and λ is surjective. Then as is remarked in the proof of (3.3), $\underline{M}$ is the full subcategory of finitely generated Q-reflexive R-modules. On the other hand, if we assume further that Q_S is finitely cogenerated, then $\underline{N}$ becomes the full subcategory of mod-S of finitely cogenerated Q-reflexive S-

modules.

PROPOSITION 3.4. *Let Q be an (R, S)-bimodule such that Q_S is quasi-injective and λ is surjective. Assume further that Q_S is finitely cogenerated. Then*

$$\underline{M}=\{{}_RM \mid M \text{ is finitely generated and } Q\text{-reflexive}\},$$

and

$$\underline{N}=\{N_S \mid N \text{ is finitely cogenerated and } Q\text{-reflexive}\}.$$

PROOF. It is clear that each $N_S \in \underline{N}$ is finitely cogenerated and Q-reflexive. Conversely, suppose that N_S is finitely cogenerated Q-reflexive. Then there exists an $n>0$ for which $0 \to N_S \to^{\alpha} Q^n$ is exact. By (3.1) $Q^n/\alpha(N)$ is Q-torsionless and thus $0 \to N_S \to^{\alpha} Q^n \to Q^I$ is exact for some set I.

For a dual-bimodule Q, by [6, Proposition 2.8] and [7, Lemma 4], we have

LEMMA 3.5. *For a dual-bimodule Q with λ surjective, the following conditions are equivalent:*

(1) *Q_S is injective.*

(2) *Q_S is a cogenerator,*

(3) *$E(Q_S)$ is Q-torsionless.*

Let ${}_R\mathrm{FG}$ and ${}_R\mathrm{FC}$ be the full subcategory of finitely generated and finitely cogenerated left R-modules, respectively. We shall use similar notations for right S-modules.

THEOREM 3.6. *For a dual-bimodule Q with ${}_RQ$ finitely generated, the following conidtions are equivalent:*

(1) *Q_S is injective and λ is surjective.*

(2) *(H', H'') defines a duality between $\underline{M}$ and FC_S.*

PROOF. (1)$\Rightarrow$(2) follows from [1, Excersise 10.3], (3.3), (3.4) and (3.5).

(2)$\Rightarrow$(1). As is seen from the proof of (3.3) each ${}_RM \in \underline{M}$ is Q-reflexive and hence by (3.2) λ is surjective. On the other hand, since $Q_S \in \mathrm{FC}_S$, $E(Q_S) \in \mathrm{FC}_S$. Therefore, $E(Q_S) \cong M^*$ for some $M \in \underline{M}$ and thus $E(Q_S)$ is Q-torsionless. This shows that Q_S is injective by (3.5).

Let ${}_RQ_S$ be an (R, S)-bimodule. Then by [1, Theorem 24.1] Q defines a Morita duality if and only if Q is a balanced bimodule such that ${}_RQ$ and Q_S are injective cogenerators. Hence, as a consequence of (3.6), we obtain

THEOREM 3.7. *Let Q be a dual-bimodule with ${}_RQ$ and Q_S finitely generated. Then the following conditions are equivalent:*

(1) *Q defines a Morita duality.*

(2) *Q is a balanced bimodule such that ${}_RQ$ and Q_S are injective.*

(3) *(H', H'') defines a duality between ${}_R$FG and FC$_S$ and one between ${}_R$FC and FG$_S$.*

THEOREM 3.8. *Let R and S be rings. Then the following conditions are equivalent:*

(1) *There exists a duality between ${}_R$FG and FG$_S$.*

(2) *There exists a dual-bimodule ${}_RQ_S$ such that ${}_RR$ is Artinian and ${}_RQ$ is finitely generated.*

(3) *There exists a dual-bimodule ${}_RQ_S$ such that S_S is Artinian and Q_S is finitely generated.*

Moreover, if this is the case, a left R-(right S-)module is Q-reflexive if and only if it is finitely generated if and only if it is finitely cogenerated.

PROOF. (1)⇒(2) follows from [1, Theorem 24.8].

(2)⇒(1). Assume (2). Then Q_S is Noetherian and is finitely generated. Hence, by (1.14) and its right-hand version, both ${}_RQ$ and Q_S are injective and, by (2.7) and its right-hand version, Q is a balanced bimodule. Therefore, Q defines a Morita duality by (3.7). Thus, again by [1, Theorem 24.8] there exists a duality between ${}_R$FG and FC$_S$.

Similarly we can prove the equivalence of (1) and (3). The rest of the theorem also follows from [1, Theorem 24.8].

4. Examples.

EXAMPLE 4.1. Let p be a prime number and $R=Z_{(p)}=\{a/b\in \underline{Q} \mid (a, b)=1$ and $p \nmid b\}$, where $\underline{Q}$ is the field of rational numbers. Then R is a commutative local ring with the unique maximal ideal Rp and nonzero proper ideals of R are exhausted by Rp^n, $n>0$. The quotient field of R is $\boldsymbol{Q}$.

Now let $Q=\underline{Q}/R$. Then Q is an (R, R)-bimodule and the only nonzero proper submodules of Q_R are those of the form $p^{-n}R/R$ for some $n>0$. Furthermore we have

(1) ${}_RQ_R$ is a dual-bimodule, since for each $n>0$, $r_Q(Rp^n)=p^{-n}R/R$ and $l_R(p^{-n}R/R)=Rp^n$.

(2) Q_R is an injective cogenerator. However, it is not finitely generated.

(3) ${}_RQ_R$ can not define a Morita duality. Indeed, as was pointed out in [4,

Example 6.1], λ is not surjective and hence ${}_RR$ is not Q-reflexive. However, by (2.5), each R/Rp^n is Q-reflexive. The class of Q-reflexive R-modules is closed under extensions. Hence each Rp^n can not be Q-reflexive. This shows that every factor module ($\neq R$) of ${}_RR$ is Q-reflexive, but there is no nonzero left ideal of R which is Q-reflexive.

(4) Q_R is not Q-reflexive. Indeed, if Q_R is Q-reflexive, then so is Q^*. Hence, the exactness of the sequence $0\to R\xrightarrow{\lambda} Q^*$ implies that ${}_RR$ must be Q-reflexive, a contradiction.

EXAMPLE 4.2. Using the same notations as above, let $Q'=p^{-n}R/R$ and $\bar{R}=R/Rp^n$. Then ${}_{\bar{R}}Q'_{\bar{R}}$ is a left dual-bimodule by (1.15), but not a right dual-bimodule. Indeed there is no lattice isomorphism between the submodule lattices of R_R and ${}_{\bar{R}}Q'$.

EXAMPLE 4.3. Using the same notations as above, ${}_{\bar{R}}Q'_{\bar{R}}$ can be regarded as a dual-bimodule again by (1.15). ${}_{\bar{R}}Q'_{\bar{R}}$ defines a Morita duality, since $\bar{R}$ is an Artinian ring.

EXAMPLE 4.4. Let $Q=\underline{Q}/\underline{Z}$, where $\underline{Z}$ is the ring of integers. Then Q is a $(\underline{Z}, \underline{Z})$-bimodule and every factor module of ${}_{\underline{Z}}\underline{Z}$ and $Q_{\underline{Z}}$ is Q-torsionless, since Q is a cogenerator over $\underline{Z}$. However, λ is not surjective and Q is not a left dual-bimodule by (1.10).

EXAMPLE 4.5. Let R be a simple Artinian ring and take ${}_RQ=R^N$, where N denotes the set of positive integers. Then ${}_RQ$ is not finitely cogenerated and hence by (1.16) ${}_RQ_S$ with $S=\mathrm{End}({}_RQ)$ is a left dual-bimodule but not a right dual-bimodule by (1.8).

EXAMPLE 4.6. Let R be the ring of 2×2 upper triangular matrices over a field and let $Q={}_RR_R$. Then

(1) Q is not a left dual-bimodule, since $\mathrm{soc}({}_RQ)\neq\mathrm{soc}(Q_R)$.

(2) Every finitely generated Q-torsionless left R-module is Q-reflexive, since R is left and right Artinian and hereditary and every Q-torsionless left R-module is projective.

(3) Q_R is not (quasi-)injective.

(4) $\underline{M}=\{{}_RM \mid M$ is finitely generated projective$\}$.

(5) $\underline{N}=\{N_R \mid N$ is finitely generated projective$\}$.

It is clear that each $N_R\in\underline{N}$ is finitely generated projective. Conversely, let N_R be a finitely generated projective R-module. Then $R^m\to N_R\to 0$ is split exact for some $m>0$. Hence $0 \to N_R \to^{\alpha} R^m$ is also split exact. Thus, $R^m/\alpha(N)$ is finitely generated projective and is finitely cogenerated Q-reflexive. There exists an $k>0$ such that $0 \to N_R \to^{\alpha} R^m \to R^k$ is exact.

(6) Though Q_R is not quasi-injective, the pair (H', H'') defines a duality between $\underline{M}$ and $\underline{N}$, as is well-known.

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