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EXCEPTIONAL MINIMAL SURFACES WITH THE RICCI CONDITION

By

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0. Introduction.

Let $X^{N}(c)$ denote the N-dimensional simply connected space form of constant curvature c, and let M be a minimal surface in $X^{N}(c)$ with Gaussian curvature K $(\leqq c)$ with respect to the induced metric ds^{2}. When $N=3$, M satisfies the Ricci condition with respect to c, that is, the metric $d\hat{s}^{2}=\sqrt{c-K}\,ds^{2}$ is flat at points where $K<c$. Conversely, every 2-dimensional Riemannian manifold with Gaussian curvature less than c which satisfies the Ricci condition with respect to c, can be realized locally as a minimal surface in $X^{3}(c)$ (see [2]). Then it is an interesting problem to classify those minimal surfaces in $X^{N}(c)$ which satisfy the Ricci condition with respect to c, that is, to classify those minimal surfaces in $X^{N}(c)$ which are locally isometric to minimal surfaces in $X^{3}(c)$. In the case where $c=0$, Lawson [3] solved this problem completely. In [4] Naka (=Miyaoka) obtained some results in the case where $c>0$.

In [1] Johnson studied a class of minimal surfaces in $X^{N}(c)$, called exceptional minimal surfaces. In this paper, we discuss exceptional minimal surfaces in $X^{N}(c)$ which satisfiy the Ricci condition with respect to c. Our results are as follows:

THEOREM 1. *Let M be an exceptional minimal surface lying fully in $X^{N}(c)$ where $c>0$. We denote by K the Gaussian curvature of M with respect to the induced metric ds^{2}. Suppose that the metric $d\hat{s}^{2}=\sqrt{c-K}\,ds^{2}$ is flat at points where $K<c$. Then either* (i) *$N=4m+1$ and M is flat, or* (ii) *$N=4m+3$.*

THEOREM 2. *Let M be an exceptional minimal surface lying fully in $X^{N}(c)$ where $c<0$. We denote by K the Gaussian curvature of M with respect to the induced metric ds^{2}. Suppose that the metric $d\hat{s}^{2}=\sqrt{c-K}\,ds^{2}$ is flat at points where $K<c$. Then $N=3$.*

REMARK. We note that every flat minimal surface in $X^{N}(c)$, where $c>0$,

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automatically satisfies the Ricci condition with respect to c. In Section 3, we show that there are flat exceptional minimal surfaces lying fully in $X^{2n+1}(c)$, where $c>0$. We also show that there are non-flat exceptional minimal surfaces lying fully in $X^{4m+3}(c)$ which satisfy the Ricci condition with respect to c, where $c>0$.

In Section 1, we follow [1] and recall the definition of exceptional minimal surfaces. In Section 2, we give lemmas for exceptional minimal surfaces in $X^N(c)$ which satisfy the Ricci condition with respect to c. In Sections 3 we prove Theorem 1, and in Section 4 we prove Theorem 2.

1. Exceptional minimal surfaces.

Suppose M is a minimal surface in $X^N(c)$. Assume that M lies fully in $X^N(c)$, namely, does not lie in a totally geodesic submanifold of $X^N(c)$. Let the integer n be given by $N=2n+1$ or $2n+2$, and let indices have the following ranges:

$$1\leqq i,\ j\leqq 2, \qquad 3\leqq \alpha\leqq N, \qquad 1\leqq A,\ B\leqq N.$$

Let $\tilde{e}_A$ be a local orthonormal frame field on $X^N(c)$, and let $\tilde{\theta}_A$ be the coframe dual to $\tilde{e}_A$. Then $d\tilde{\theta}_A=\sum_B\tilde{\omega}_{AB}\wedge\tilde{\theta}_B$, where $\tilde{\omega}_{AB}$ are the connection forms on $X^N(c)$.

Suppose that e_i is a local orthonormal frame field on M and that the frame $\tilde{e}_A$ is chosen so that on M, $e_i=\tilde{e}_i$ and $\tilde{e}_\alpha$ are normal to M. When forms and vectors on $X^N(c)$ are restricted to M, let them be denoted by the same symbol without tilde: $\theta_A=\tilde{\theta}_A|_M$, $\omega_{AB}=\tilde{\omega}_{AB}|_M$ and $e_A=\tilde{e}_A|_M$. Then $\omega_{\alpha i}=\sum_j h_{\alpha ij}\theta_j$, where $h_{\alpha ij}$ are the coefficients of the second fundamental form of M.

Let T_xM and $T_xX^N(c)$ denote the tangent space of M and $X^N(c)$, respectively, at a point x. Curves on M through x have their first derivatives at x in T_xM, but higher order derivatives will have components normal to M. The space spanned by the derivatives of order up to r is called the r-th osculating space of M at x, denoted $T_x^{(r)}M$.

The r-th normal space of M at x, denoted $Nor_x^{(r)}M$, is the orthogonal complement of $T_x^{(r)}M$ in $T_x^{(r+1)}M$. At generic points of M, the dimension of $Nor_x^{(r)}M$ is 2 when $1\leqq r\leqq n-1$, and the dimension of $Nor_x^{(n)}M$ is 1 or 2, depending on whether N is odd or even. Those normal spaces that have dimension 2 is called the normal planes of M. Let β_N denote the number of normal planes possessed by M at generic points: $\beta_N=n-1$ if $N=2n+1$, and $\beta_N=n$ if $N=2n+2$.

Choose the normal vectors e_α so that $Nor_x^{(r)}M$ is spanned by $\{e_{2r+1}, e_{2r+2}\}$, where $1\leqq r\leqq\beta_N$. When $N=2n+1$, $Nor_x^{(n)}M$ is spanned by $\{e_{2n+1}\}$. Set $\varphi=\theta_1+\sqrt{-1}\theta_2$.

PROPOSITION ([1]). *There are* H_α *such that* $H_\alpha=h_{\alpha11}+\sqrt{-1}h_{\alpha12}$ *for* $\alpha=3$ *and* 4, *for each* r *such that* $2\leqq r\leqq\beta_N$

$$H_{2r-1}\omega_{\alpha,2r-1}+H_{2r}\omega_{\alpha,2r}=H_\alpha\bar{\varphi}$$

where $\alpha=2r+1$ *and* $2r+2$, *and when* $N=2n+1$

$$H_{2n-1}\omega_{2n+1,2n-1}+H_{2n}\omega_{2n+1,2n}=H_{2n+1}\bar{\varphi}\ .$$

The r-th normal plane, $Nor_x^{(r)}M$, of M is called exceptional if $H_{2r+2}=\pm\sqrt{-1}H_{2r+1}$. The minimal surface M is called exceptional if all of its normal planes are exceptional. Note that when $N=2n+1$, $Nor_x^{(n)}M$ is a line, not a plane, and the notion of exceptionality does not apply. So, every minimal surface in $X^3(c)$ is exceptional.

2. Lemmas.

Let M be an exceptional minimal surface lying fully in $X^N(c)$. We denote by K and Δ the Gaussian curvature and the Laplacian of M, respectively, with respect to the induced metric ds^2. Set

$$A_0^c=1/2\ ,\qquad A_1^c=c-K\ ,$$

(1)
$$A_{p+1}^c=\begin{cases} A_p^c[\Delta\log(A_p^c)+A_p^c/A_{p-1}^c-2(p+1)K], & \text{if } A_p^c>0,\\ 0, & \text{otherwise.}\end{cases}$$

Set $M_1=\{x\in M;\ K<c\}$ and $M_2=\{x\in M;\ K=c\}$. Suppose that the metric $d\hat{s}^2=\sqrt{c-K}\,ds^2$ is flat on M_1. Then by the lemma in Section 3 of [1] for $n=1$,

(2)
$$\Delta\log(c-K)=4K$$

on M_1.

LEMMA 1. *When* $c>0$,

$$A_{4k}^c=2^{4k-1}c^{2k}(c-K)^{2k},\qquad A_{4k+1}^c=2^{4k}c^{2k}(c-K)^{2k+1}\ ,$$
$$A_{4k+2}^c=2^{4k+1}c^{2k}(c-K)^{2k+2},\qquad A_{4k+3}^c=2^{4k+2}c^{2k+1}(c-K)^{2k+2}\ .$$

LEMMA 2. *When* $c\leqq0$,

$$A_2^c=2(c-K)^2,\qquad A_3^c=4c(c-K)^2,\qquad A_p^c=0\ \ for\ p\geqq4\ .$$

PEOOF OF LEMMA 1. By (1) and (2),

$$\begin{aligned}A_2^c&=A_1^c[\Delta\log(A_1^c)+A_1^c/A_0^c-4K]\\&=(c-K)[\Delta\log(c-K)+2(c-K)-4K]\\&=2(c-K)^2\end{aligned}$$

on M_1, and $A_2^c=0$ on M_2. So $A_2^c=2(c-K)^2$ on M. By (1) and (2)

$$\begin{aligned}A_3^c&=A_2^c[\Delta\log(A_2^c)+A_2^c/A_1^c-6K]\\&=2(c-K)^2[2\Delta\log(c-K)+2(c-K)-6K]\\&=4c(c-K)^2\end{aligned}$$

on M_1, and $A_3^c=0$ on M_2. So $A_3^c=4c(c-K)^2$ on M. Thus Lemma 1 is true foı $k=0$.

Assume that Lemma 1 is true for some k. Then, by (1), (2) and the assumption,

$$\begin{aligned}A_{4k+4}^c&=A_{4k+3}^c[\Delta\log(A_{4k+3}^c)+A_{4k+3}^c/A_{4k+2}^c-2(4k+4)K]\\&=2^{4k+2}c^{2k+1}(c-K)^{2k+2}[(2k+2)\Delta\log(c-K)+2c-2(4k+4)K]\\&=2^{4k+3}c^{2k+2}(c-K)^{2k+2}\end{aligned}$$

on M_1, and $A_{4k+4}^c=0$ on M_2. So $A_{4k+4}^c=2^{4k+3}c^{2k+2}(c-K)^{2k+2}$ on M. Using (1), (2) and the assumption we have

$$\begin{aligned}A_{4k+5}^c&=A_{4k+4}^c[\Delta\log(A_{4k+4}^c)+A_{4k+4}^c/A_{4k+3}^c-2(4k+5)K]\\&=2^{4k+3}c^{2k+2}(c-K)^{2k+2}[(2k+2)\Delta\log(c-K)+2c-2(4k+5)K]\\&=2^{4k+4}c^{2k+2}(c-K)^{2k+3}\end{aligned}$$

on M_1, and $A_{4k+5}^c=0$ on M_2. So $A_{4k+5}^c=2^{4k+4}c^{2k+2}(c-K)^{2k+3}$ on M. By (1) and (2),

$$\begin{aligned}A_{4k+6}^c&=A_{4k+5}^c[\Delta\log(A_{4k+5}^c)+A_{4k+5}^c/A_{4k+4}^c-2(4k+6)K]\\&=2^{4k+4}c^{2k+2}(c-K)^{2k+3}[(2k+3)\Delta\log(c-K)+2(c-K)-2(4k+6)K]\\&=2^{4k+5}c^{2k+2}(c-K)^{2k+4}\end{aligned}$$

on M_1, and $A_{4k+6}^c=0$ on M_2. So $A_{4k+6}^c=2^{4k+5}c^{2k+2}(c-K)^{2k+4}$ on M. By (1) and (2),

$$\begin{aligned}A_{4k+7}^c&=A_{4k+6}^c[\Delta\log(A_{4k+6}^c)+A_{4k}^c{}^{+}{}_6/A_{4k+5}^c-2(4k+7)K]\\&=2^{4k+5}c^{2k+2}(c-K)^{2k+4}[(2k+4)\Delta\log(c-K)+2(c-K)-2(4k+7)K]\\&=2^{4k+6}c^{2k+3}(c-K)^{2k+4}\end{aligned}$$

on M_1, and $A_{4k+7}^c=0$ on M_2. So $A_{4k+7}^c=2^{4k+6}c^{2k+3}(c-K)^{2k+4}$ on M. Therefore, by induction, Lemma 1 is proved. q.e.d.

PROOF OF LEMMA 2. By the same argument as in the proof of Lemma 1, we have $A_2^c=2(c-K)^2$ and $A_3^c=4c(c-K)^2$. As $c\leqq 0$, $A_3^c=4c(c-K)^2\leqq 0$. Hence by (1) we have $A_p^c=0$ for $p\geqq 4$. q.e.d.

3. Proof of Theorem 1.

PROOF OF THEOREM 1. Let Δ, A_p^c and M_1 be defined as in Section 2. As M lies fully in $X^N(c)$, $K=c$ only at isolated points, and M_1 is M minus isolated points. By Lemma 1, for each $p\geqq 0$, $A_p^c>0$ on M_1. If $N=2n+2$, then $A_{n+1}^c=0$ identically by Theorem A of [1], which contradicts that $A_p^c>0$ on M_1 for each $p\geqq 0$. If $N=4m+1$, then by Theorem A of [1], the metric $(A_{2m}^c)^{1/(2m+1)}ds^2$ is flat at points where $A_{2m}^c>0$. When $m=2k$, using the lemma in Section 3 of [1], Lemma 1 and the equation (2), we have

$$\begin{aligned}0&=\Delta\log(A_{2m}^c)-2(2m+1)K\\&=\Delta\log(A_{4k}^c)-2(4k+1)K\\&=2k\Delta\log(c-K)-2(4k+1)K\\&=-2K\end{aligned}$$

on M_1. So M_1 is flat, and by continuity, M is flat. When $m=2k+1$, using the lemma in Section 3 of [1], Lemma 1 and the equation (2), we have

$$\begin{aligned}0&=\Delta\log(A_{2m}^c)-2(2m+1)K\\&=\Delta\log(A_{4k+2}^c)-2(4k+3)K\\&=(2k+2)\Delta\log(c-K)-2(4k+3)K\\&=2K\end{aligned}$$

on M_1. So M_1 is flat, and by continuity, M is flat. Therefore, either (i) $N=4m+1$ and M is flat, or (ii) $N=4m+3$. q.e.d.

By Theorem B of [1], we can see that every flat surface can be realized locally as an exceptional minimal surface lying fully in $X^{2n+1}(c)$, where $c>0$. So, there are flat exceptional minimal surfaces lying fully in $X^{2n+1}(c)$, where $c>0$.

Let M be a minimal surface in $X^3(c)$ where $c>0$. We denote by K the Gaussian curvature of M with respect to the induced metric ds^2. Let A_p^c be defined as in Section 2. Assume that $K<c$. Then M satisfies the Ricci con-

dition with respect to c. So Lemma 1 is valid, and $A_p^c>0$ for each $p\geqq 0$. Let us show that the metric $(A_{2m+1}^c)^{1/(2m+2)}ds^2$ is flat. When $m=2k$, by Lemma 1,

$$(A_{2m+1}^c)^{1/(2m+2)}=(A_{4k+1}^c)^{1/(4k+2)}=(2^{4k}c^{2k})^{1/(4k+2)}\sqrt{c-K}.$$

When $m=2k+1$, by Lemma 1,

$$(A_{2m+1}^c)^{1/(2m+2)}=(A_{4k+3}^c)^{1/(4k+4)}=(2^{4k+2}c^{2k+1})^{1/(4k+4)}\sqrt{c-K}.$$

Thus the metric $(A_{2m+1}^c)^{1/(2m+2)}ds^2$ is flat, because M satisfies the Ricci condition with respect to c. By Theorem B of [1], we find that (M, ds^2) can be realized locally as an exceptional minimal surface lying fully in $X^{4m+3}(c)$. Therefore, there are non-flat exceptional minimal surfaces lying fully in $X^{4m+3}(c)$ which satisfy the Ricci condition with respect to c, where $c>0$.

4. Proof of Theorem 2.

PROOF OF THEOREM 2. Let Δ, A_p^c and M_1 be defined as in Section 2. As M lies fully in $X^N(c)$, $K=c$ only at isolated points, and M_1 is not empty. By Lemma 2, $A_2^c>0$ and $A_3^c<0$ on M_1. If $N=4$, then $A_2^c=0$ identically by Theorem A of [1], which contradicts that $A_2^c>0$ on M_1. If $N=5$, then by Theorem A of [1], the metric $(A_2^c)^{1/3}ds^2$ is flat at points where $A_2^c>0$. Using the lemma in Section 3 of [1], Lemma 2 and the equation (2), we have

$$\begin{aligned}0&=\Delta\log(A_2^c)-6K\\&=2\Delta\log(c-K)-6K\\&=2K\end{aligned}$$

on M_1. So $K=0$ on M_1, which contradicts that $K\leqq c<0$. If $N=6$, then $A_3^c=0$ identically by Theorem A of [1], which contradicts that $A_3^c<0$ on M_1. If $N\geqq 7$, then $A_3^c\geqq 0$ by Theorem A of [1], which contradicts that $A_3^c<0$ on M_1. Therefore, $N=3$. q.e.d.

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