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HARMONIC FOLIATIONS ON THE SPHERE

By

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Introduction.

Let M be a compact orientable manifold and let $\mathcal{F}$ be a harmonic foliation on M with respect to a bundle-like metric. Kamber and Tondeur [4] proved a fundamental formula for a special variation of $\mathcal{F}$, and making use of it they proved that the index of a harmonic foliation $\mathcal{F}$ on the sphere S^n $(n>2)$ for which the standard metric is bundle-like is not smaller than $q+1$, where q is the codimension of $\mathcal{F}$. On the other hand, Nakagawa and Takagi [6] proved that any harmonic foliation on a compact space form $M^n(c)$, $c\geqq 0$, for which the normal plane field is minimal is totally geodesic. Here a complete Riemannian manifold of constant curvature is called a *space form* and an n-dimensional space form of constant curvature c is denoted by $M^n(c)$. However a formula in [6] contains an error, and hence the above result is yet open.

The purpose of this paper is to study a harmonic foliation on the sphere. We use the method of Nakagawa and Takagi [6] to calculate the divergence of a vector field and obtain a formula of Simons' type. Then, after Chern, do Carmo and Kobayashi [2] it is proved that a harmonic foliation $\mathcal{F}$ of codimension q on an n-dimensional unit sphere satisfying $S\leqq(n-q)/(2-1/q)$ for which the normal plane field is minimal, is totally geodesic or $n=4$, $q=2$, where S denotes the square of the norm of the second fundamental form of each leaf. Moreover, was also prove that if $S\leqq(n-q)/(2-1/q)$ or $K\geqq(q-1)/(2q-1)$ for a harmonic foliation $\mathcal{F}$ of codimension q on the unit sphere with respect to a bundle-like metric, here K denotes the sectional curvature of leaves, then $\mathcal{F}$ is totally geodesic. Thus they have been completely classified by the theorem due to Escobales [3].

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1. Preliminaries.

Let (M, g) be an n-dimensional Riemannian manifold and $\mathcal{F}$ a foliation of codimension q on M. We may choose a suitable Riemannian metric on the tangent bundle $T(M)$ of M and decompose $T(M)$ as the direct product $\mathcal{F}\oplus\mathcal{F}^{\perp}$, where $\mathcal{F}^{\perp}$ is called a *normal plane field*. For any vector field X on M we decompose it as

$$X=X'+X'',$$

where X' (resp. X'') is tangent (resp. normal) to $\mathcal{F}$.

We define two tensors A and h of type (1, 2) on M by

$$A(X, Y)=-(\nabla_{Y''}X'')', \qquad h(X, Y)=(\nabla_{Y'}X')'' \tag{1.1}$$

for any vector fields X and Y on M, where ∇ denotes the Riemannian connection with respect to g. The restriction of h to each leaf of $\mathcal{F}$ is identified with the second fundamental from of the leaf.

After Reinhart [7] we define the second fundamental from B of the normal field $\mathcal{F}^{\perp}$ by

$$B(X, Y)=\frac{1}{2}\{A(X, Y)+A(Y, X)\} \tag{1.2}$$

for any vector fields X and Y on M.

The following convention on the range of indices will be used throughout this paper:

$$A, B, C, \cdots=1, \cdots, n;$$

$$i, j, k, \cdots=1, \cdots, p;$$

$$\alpha, \beta, \gamma, \cdots=p+1, \cdots, p+q=n,$$

where $p=n-q$ denotes the dimension of $\mathcal{F}$. The summation $\sum$ is taken over all repeated indices, unless otherwise stated. We take a local orthonormal frame field $\{e_A\}$ in $(M, g, \mathcal{F})$ such that $e_1, \cdots, e_p$ are tangent to $\mathcal{F}$ and hence $e_{p+1}, \cdots, e_n$ are orthogonal to $\mathcal{F}$. The dual coframe field is denoted by $\{\omega_A\}$.

The structure equations of M are given as follows:

$$\begin{cases} d\omega_A+\sum\omega_{AB}\wedge\omega_B=0, \\ \omega_{AB}+\omega_{BA}=0, \end{cases} \tag{1.3}$$

$$\begin{cases} d\omega_{AB}+\sum\omega_{AC}\wedge\omega_{CB}=\Omega_{AB}, \\ \Omega_{AB}=-\frac{1}{2}\sum R_{ABCD}\omega_C\wedge\omega_D, \end{cases} \tag{1.4}$$

where ω_{AB} is the connection from with respect to ω_A, Ω_{AB} denotes the curvature form of M and R_{ABCD} are its components, which are the Riemannian curvature tensor with respect to g.

The Riemannian connection ∇ on M is given by

(1.5) $$\nabla_{e_A} e_B = \sum \omega_{CB}(e_A) e_C .$$

It follows from (1.1) and (1.5) that

(1.6) $$\begin{cases} h(e_i, e_j) = \sum \omega_{\alpha i}(e_j) e_\alpha , \\ A(e_\alpha, e_\beta) = \sum \omega_{\alpha j}(e_\beta) e_j . \end{cases}$$

Thus the only components h^A_{BC} (resp. A^B_{CD}) of h (resp. A) which may not vanish are

(1.7) $$h^\alpha_{ij} = \omega_{\alpha i}(e_j), \qquad (resp.\ A^i_{\alpha\beta} = \omega_{\alpha i}(e_\beta)) .$$

Moreover the connection form $\omega_{\alpha i}$ are given by

(1.8) $$\omega_{\alpha i} = \sum h^\alpha_{ij} \omega_j + \sum A^i_{\alpha\beta} \omega_\beta .$$

The foliation $\mathcal{F}$ is said to be *harmonic* or *minimal* if $\sum h^\alpha_{jj} = 0$. The foliation $\mathcal{F}$ is said to be *totally geodesic* if $h^\alpha_{ij} = 0$. The normal plane field $\mathcal{F}^\perp$ is said to be *minimal* if $\mathrm{Tr}\, B = \sum A^i_{\alpha\alpha} e_i = 0$. The normal plane field $\mathcal{F}^\perp$ is said to be *totally geodesic* if $B=0$. The Riemannian metric tensor g is bundle-like (see Molino [5]) if and only if

(1.9) $$A^i_{\alpha\beta} = -A^i_{\beta\alpha} .$$

This is equivalent to that $B=0$. Since the distribution $\omega_\alpha = 0$ is integrably by definition, it yields

(1.10) $$h^\alpha_{ij} = h^\alpha_{ji} .$$

Now, for a tensor field $T = (T^{A_1 \cdots A_r}_{B_1 \cdots B_s})$ on M, we define the covariant derivative $T^{A_1 \cdots A_r}_{B_1 \cdots B_s C}$ by

(1.11) $$\sum T^{A_1 \cdots A_r}_{B_1 \cdots B_s C} \omega_C = dT^{A_1 \cdots A_r}_{B_1 \cdots B_s} - \sum T^{A_1 \cdots A_{a-1} C A_{a+1} \cdots A_r}_{B_1 \cdots B_s} \omega_{C A_a}$$
$$- \sum T^{A_1 \cdots A_r}_{B_1 \cdots B_{b-1} C B_{b+1} \cdots B_s} \omega_{C B_b} .$$

Then, from the definition of (h^A_{BCD}), (A^A_{BCD}) and (1.8), it follows that we have

(1.12) $$h^l_{ijk} = -\sum h^\alpha_{ij} h^\alpha_{lk} ,$$

(1.13) $$h^l_{ij\alpha} = -\sum h^\beta_{ij} A^l_{\beta\alpha} ,$$

(1.14) $$h^\alpha_{i\beta j} = h^\alpha_{\beta ij} = \sum h^\alpha_{ik} h^\beta_{kj} ,$$

(1.15) $$h^{\alpha}_{i\beta\gamma}=A^{\alpha}_{\beta i\gamma}=\sum h^{\alpha}_{ik}A^{k}_{\beta\gamma}\,,$$

(1.16) $$h^{A}_{\beta\gamma C}=h^{i}_{\alpha CD}=A^{i}_{C\beta D}=0\,,$$

(1.17) $$A^{i}_{j\alpha\beta}=-\sum A^{i}_{\gamma\alpha}A^{j}_{\gamma\beta}\,,$$

(1.18) $$A^{i}_{\alpha j\beta}=-\sum A^{i}_{\alpha\gamma}A^{j}_{\gamma\beta}\,,$$

(1.19) $$A^{i}_{j\alpha k}=-\sum A^{i}_{\beta\alpha}h^{\beta}_{jk}\,,$$

(1.20) $$A^{i}_{\alpha jk}=-\sum A^{i}_{\alpha\beta}h^{\beta}_{jk}\,,$$

(2.21) $$A^{\gamma}_{\alpha\beta j}=\sum A^{l}_{\alpha\beta}h^{\gamma}_{lj}\,,$$

(1.22) $$A^{\gamma}_{\alpha\beta\delta}=\sum A^{l}_{\alpha\beta}A^{l}_{\gamma\delta}\,,$$

(1.23) $$A^{C}_{ijD}=A^{\alpha}_{iCD}=A^{\alpha}_{CjD}=0\,.$$

Moreover, by the exterior derivatives of (1.8) and by means of (1.14), (1.15) and (1.18), we have

(1.24) $$h^{\alpha}_{ijk}-h^{\alpha}_{ikj}=R_{\alpha ijk}\,,$$

(1·25) $$h^{\alpha}_{ij\beta}-h^{\alpha}_{i\beta j}+A^{i}_{\alpha j\beta}-A^{i}_{\alpha\beta j}=R_{\alpha ij\beta}\,,$$

(1.26) $$h^{\alpha}_{i\beta\gamma}-h^{\alpha}_{i\gamma\beta}+A^{i}_{\alpha\beta\gamma}-A^{i}_{\alpha\gamma\beta}=R_{\alpha i\beta\gamma}\,.$$

Next, the Ricci formulas for the second covariant derivatives of h are given by

(1.27) $$h^{A}_{BCDE}-h^{A}_{BCDE}=\sum(h^{F}_{BC}R_{AFDE}+h^{A}_{FC}R_{BFDE}+h^{A}_{BF}R_{CFDE})\,.$$

2. The divergence of a vector field.

Let (M, g) be a locally symmetric Riemannian manifold and $\mathcal{F}$ be a harmonic foliation on M. We consider a global vector field $v=\sum v_A e_A$ on M defined by

$$v_k=\sum h^{\alpha}_{ij}h^{\alpha}_{ijk}, \qquad v_\alpha=0\,.$$

We calculate the divergence δv of v as follows: First, noting $\sum h^{\beta}_{kk}=0$, we have

(2.1) $$\sum v_{kk}=\sum h^{\alpha}_{ijk}h^{\alpha}_{ijk}+\sum h^{\alpha}_{ij}h^{\alpha}_{ijkk}+\sum h^{\alpha}_{ij}h^{\beta}_{ij}h^{\alpha}_{lk}h^{\beta}_{lk}$$
$$+\sum h^{\alpha}_{ij}h^{\alpha}_{jl}h^{\beta}_{lk}h^{\beta}_{ki}+\sum h^{\alpha}_{ij}h^{\alpha}_{il}h^{\beta}_{lk}h^{\beta}_{jk}\,,$$

(2.2) $$\sum v_{\alpha\alpha}=\sum v_k A^{k}_{\alpha\alpha}\,.$$

To calculate h^{α}_{ijkk}, we take tқe exterior derivative of (1.24):

$$d(h^{\alpha}_{ijk}-h^{\alpha}_{ikj})=dR_{\alpha ijk}\,.$$

Then, noting $R_{\alpha ijkl}=0$, it yields

(2.3) $$h^{\alpha}_{ijkl}-h^{\alpha}_{ikjl}=\sum(h^{m}_{ijk}-h^{m}_{ikj})h^{\alpha}_{ml}-\sum\{(h^{\alpha}_{\beta jk}-h^{\alpha}_{\beta kj})h^{\beta}_{il}+(h^{\alpha}_{i\beta k}-h^{\alpha}_{ik\beta})h^{\beta}_{jl}+(h^{\alpha}_{ij\beta}-h^{\alpha}_{i\beta j})h^{\beta}_{kl}\}-\sum R_{mijk}h^{\alpha}_{ml}+\sum(R_{\alpha\beta jk}h^{\beta}_{il}+R_{\alpha i\beta k}h^{\beta}_{jl}+R_{\alpha ij\beta}h^{\beta}_{kl}).$$

Remark. In [6] this formula is wrongly derived.

Now, interchanging i, j and k, l in (2.3), we have also

(2.4) $$h^{\alpha}_{klij}-h^{\alpha}_{kilj}=\sum(h^{m}_{kli}-h^{m}_{kil})h^{\alpha}_{mj}-\sum\{(h^{\alpha}_{\beta li}-h^{\alpha}_{\beta il})h^{\beta}_{kj}+(h^{\alpha}_{k\beta i}-h^{\alpha}_{ki\beta})h^{\beta}_{lj}+(h^{\alpha}_{kl\beta}-h^{\alpha}_{k\beta l})h^{\beta}_{ij}\}-\sum R_{mkli}h^{\alpha}_{mj}+\sum(R_{\alpha\beta li}h^{\beta}_{kj}+R_{\alpha k\beta i}h^{\beta}_{lj}+R_{\alpha kl\beta}h^{\beta}_{ij}),$$

from which together with (2.3) it follows that we get

(2.5) $$h^{\alpha}_{ijkl}-h^{\alpha}_{klij}=(\textit{the right hand side of } (2.3))-(\textit{the right hand side of}(2.4))+(h^{\alpha}_{ikjl}-h^{\alpha}_{kilj}).$$

Noticing that

(2.6) $$h^{\alpha}_{kilj}=h^{\alpha}_{iklj},$$

and, by means of the Ricci formula (1.27) for h^{α}_{ij}, we can derive the following equation from (2.5):

(2.7) $$h^{\alpha}_{ijkl}-h^{\alpha}_{klij}=(\textit{the right hand side of } (2.3))-(\textit{the right hand side of } (2.4))+\sum(R_{\alpha\beta jl}h^{\beta}_{ik}+R_{imjl}h^{\alpha}_{mk}+R_{kmjl}h^{\alpha}_{mi}).$$

Putting $l=k$ in (2.7) and noting $\sum h^{\beta}_{kk}=0$, we have

(2.8) $$h^{\alpha}_{ijkk}-h^{\alpha}_{kkij}=\sum(h^{m}_{ijk}-h^{m}_{ikj})h^{\alpha}_{km}-\sum(h^{\alpha}_{\beta jk}-h^{\alpha}_{\beta kj})h^{\beta}_{ik}+\sum h^{m}_{kik}h^{\alpha}_{mj}+2\sum(h^{\alpha}_{\beta ki}-h^{\alpha}_{\beta ik})h^{\beta}_{kj}-\sum h^{\alpha}_{kk\beta}h^{\beta}_{ij}+2\sum R_{imjk}h^{\alpha}_{mk}-2\sum R_{\alpha\beta ki}h^{\beta}_{kj}+2\sum R_{\alpha\beta jk}h^{\beta}_{ik}+\sum R_{mkki}h^{\alpha}_{mj}-\sum R_{\alpha kk\beta}h^{\beta}_{ij}+\sum R_{kmjk}h^{\alpha}_{im}.$$

It can be easily seen in [6, Lemma 2.2] that

(2.9) $$\sum h^{\alpha}_{kkij}=-2\sum h^{\alpha}_{kl}h^{\beta}_{li}h^{\beta}_{kj}.$$

Hence we have

$$\begin{aligned}(2.10)\qquad \sum h_{ij}^{\alpha}h_{ijkk}^{\alpha}= & -2\sum h_{ij}^{\alpha}h_{kl}^{\alpha}h_{li}^{\beta}h_{kj}^{\beta}-\sum h_{ij}^{\alpha}h_{kl}^{\alpha}(h_{ij}^{\beta}h_{kl}^{\beta}-h_{ik}^{\beta}h_{jl}^{\beta}) \\ & -\sum h_{ij}^{\alpha}h_{ik}^{\beta}(h_{jl}^{\alpha}h_{lk}^{\beta}-h_{kl}^{\alpha}h_{lj}^{\beta})-\sum h_{ij}^{\alpha}h_{jl}^{\alpha}h_{jk}^{\beta}h_{kl}^{\beta} \\ & +2\sum h_{ij}^{\alpha}h_{jk}^{\beta}(h_{kl}^{\alpha}h_{li}^{\beta}-h_{il}^{\alpha}h_{lk}^{\beta})-h_{ij}^{\alpha}h_{ij}^{\beta}h_{kl}^{\alpha}h_{kl}^{\beta} \\ & +2\sum R_{imjk}h_{ij}^{\alpha}h_{mk}^{\alpha}+4\sum R_{\alpha\beta jk}h_{ij}^{\alpha}h_{ik}^{\beta} \\ & +2\sum R_{mkki}h_{ij}^{\alpha}h_{mj}^{\alpha}-\sum R_{\alpha kk\beta}h_{ij}^{\alpha}h_{ij}^{\beta}.\end{aligned}$$

Now, let M be a space of constant curvature $c(\geqq 0)$. For each index α, we denote by H_α the symmetric matrix (h_{ij}^{α}) and set

$$(2.11)\qquad S_{\alpha\beta}=\sum h_{ij}^{\alpha}h_{ij}^{\beta}.$$

Since the matrix $S_{\alpha\beta}$ of order q is also symmetric and it is diagonalizable, a local field of orthonormal frames $\{e_\alpha\}$ can be chosen in such a way that $S_{\alpha\beta}=S_\alpha\delta_{\alpha\beta}$, where the eigenvalues S_α's are real-valued functions on M. We denote by S the squence of the length of the second fundamental form h:

$$(2.12)\qquad S=\sum h_{ij}^{\alpha}h_{ij}^{\alpha}=\sum S_\alpha.$$

From (2.1) and (2.10) we have

$$\begin{aligned}(2.13)\qquad \sum v_{kk}= & \sum h_{ijk}^{\alpha}h_{ijk}^{\alpha} \\ & +2\sum Tr(H^\alpha H^\beta H^\alpha H^\beta-H^\alpha H^\alpha H^\beta H^\beta) \\ & -\sum S_\alpha^2+pcS.\end{aligned}$$

Thus the divergence δv becomes

$$\begin{aligned}(2.14)\qquad \delta v= & \sum v_{\alpha\alpha}+\sum h_{ijk}^{\alpha}h_{ijk}^{\alpha} \\ & +\sum Tr[(H^\alpha H^\beta-H^\beta H^\alpha)(H^\alpha H^\beta-H^\beta H^\alpha)] \\ & -\sum S_\alpha^2+pcS.\end{aligned}$$

3. The main result.

In the present section we follow Chern, do-Carmo akd Kobayashi [2] closely. For an $n\times n$ matrix A with components (a_{ij}) we denote by $N(A)$ the trace of the matrix A^tA, i.e., we put $N(A)=\sum(a_{ij})^2$. First of all, we need the following

LEMMA [2]. *Let A and B be symmetric $q\times q$ watrices. Then*

$$N(AB-BA)\leqq 2N(A)N(B)$$

and the equality holds for nonzero matrices A and B if and only if A and B can be transformed simultaneously by an othogonal matrix into scalar multiples of $\tilde{A}$

and $\tilde{B}$ *respectively, where*

$$\tilde{A}=\begin{pmatrix}0 & 1 & \\ 1 & 0 & \\ & & 0\end{pmatrix} \qquad \tilde{B}=\begin{pmatrix}1 & 0 & \\ 0 & -1 & \\ & & 0\end{pmatrix}$$

Moreover, if A_1, A_2 *and* A_3 *are* $(n\times n)$*--symmetric matrices and if*

$$N(A_\alpha A_\beta - A_\beta A_\alpha)=2N(A_\alpha)N(A_\beta), \qquad 1\leqq\alpha,\ \beta\leqq 3\,,$$

then at least one of the matrices A_α *must be zero.*

THEOREM 1. *Let* $(S^n(c), g)$ *be an* $n=(p+q)$*-dimensional sphere of constant curvature* c *and let* $\mathcal{F}$ *be a harmonic foliation of codimension* q *on* $S^n(c)$. *If the normal plane field* $\mathcal{F}^\perp$ *is minimal, then we have*

$$\int_{S^n(c)} S\left\{\left(2-\frac{1}{q}\right)S-pc\right\}*1\geqq 0\,,$$

where $*1$ *denotes the volume element of* $S^n(c)$.

PROOF. Since the normal plane field $\mathcal{F}^\perp$ is minimal, we get $\sum v_{\alpha\alpha}=0$ by (2.2), which implies that (2.14) becomes

(3.1) $$\delta v=\sum h_{ijk}^\alpha h_{ijk}^\alpha-\sum N(H^\alpha H^\beta-H^\beta H^\alpha)-\sum S_\alpha^2+pcS\,.$$

Thus we have

$$\begin{aligned}-\delta v+\sum h_{ijk}^\alpha h_{ijk}^\alpha &=\sum N(H^\alpha H^\beta-H^\beta H^\alpha)+\sum S_\alpha^2-pcS\\ &\leqq 2\sum_{\alpha\neq\beta} N(H^\alpha)N(H^\beta)+\sum S_\alpha^2-pcS\\ &\leqq 2\sum_{\alpha\neq\beta} S_\alpha S_\beta+\sum S_\alpha^2-pcS\\ &=(\sum S_\alpha)^2+2\sum_{\alpha<\beta} S_\alpha S_\beta-pcS\\ &=q^2\sigma_1^2+q(q-1)\sigma_2-pcS\\ &=-q(q-1)(\sigma_1^2-\sigma_2)+(2q^2-q)\sigma_1^2-pcS\,,\end{aligned}$$

where $q\sigma_1=\sum S_\alpha=S$ and $q(q-1)\sigma_2=2\sum_{\alpha<\beta}S_\alpha S_\beta$. It can be easily seen that

$$q^2(q-1)(\sigma_1^2-\sigma_2)=\sum_{\alpha<\beta}(S_\alpha-S_\beta)^2\geqq 0\,,$$

and therefore we get

$$-\delta v+\sum h_{ijk}^{\alpha}h_{ijk}^{\alpha}\leqq(2q^2-q)\sigma_1^2-pcS$$
$$=S\left\{\left(2-\frac{1}{q}\right)S-pc\right\}.$$

By Green's theorem we have

$$0\leqq\int_{S^n(c)}\sum h_{ijk}^{\alpha}h_{ijk}^{\alpha}*1\leqq\int_{S^n(c)}S\left\{\left(2-\frac{1}{q}\right)S-pc\right\}*1.$$

COROLLARY. *Under the condition of Theorem* 1, *if* $\mathcal{F}$ *is not totally geodesic and if* $S\leqq pc/(2-1/q)$ *everywhere on* $S^n(c)$, *then*

$$S=\frac{pc}{2-\dfrac{1}{q}}$$

and the second fundamental form of each leaf is parallel along the leaf.

Let S^n be a unit sphere. We assume that the square length S of the second fundamental form of each leaf is equal to $p/(2-1/q)$. If the foliation $\mathcal{F}$ is harmonic on S^n, then each leaf of $\mathcal{F}$ is the minimal submanifold in M. So, the well known theorem due to Chern, do-Carmo and Kobayashi [2] implies that there are only two cases as follows:

1. $q=1$,
2. $p=q=2$.

However, by a theorem of Barbosa, Kenmotsu and Oshikiri [1] it is seen that the case 1 does not hold for our foliated Riemannian manifold. But we give here a direct simple proof of this fact. By definition we get

$$\nabla_{e_{p+1}}e_{p+1}=\sum\omega_{i\,p+1}(e_{p+1})e_j=\sum A_{p+1\,p+1}^{j}e_j,$$
$$\delta(\nabla_{e_{p+1}}e_{p+1})=\sum A_{p+1\,p+1\,j}^{j}.$$

On the other hand, from (1.14), (1.18) and (1.25) we have

$$\sum R_{p+1\,jj\,p+1}=\sum h_{jj\,p+1}^{p+1}-\sum A_{p+1\,p+1\,j}^{j}$$
$$-\sum h_{jk}^{p+1}h_{jk}^{p+1}-\sum A_{p+1\,p+1}^{j}A_{p+1\,p+1}^{j}.$$

Thus we have

$$\delta(\nabla_{e_{p+1}}e_{p+1})=-p-S-|A|<0.$$

Integrating it over M, we derived a contradiction. So we prove the following

THEOREM 2. *Let* S^n *be an* $n=(p+q)$*-dimensional unit sphere and* $\mathcal{F}$ *be a harmonic foliation of codimension* q *on* S^n *satisfying* $S=p/(2-1/q)$. *If the normal plane field* $\mathcal{F}^{\perp}$ *is minimal, then* $p=q=2$.

COROLLARY. *Let S^n be an $n=(p+q)$)-dimensional unit sphere and $\mathcal{F}$ be a harmonic foliation of codimension q on S^n. If the normal plane field $\mathcal{F}^{\perp}$ is minimal and if $S\leqq p/(2-1/q)$ holds on M, then the foliation $\mathcal{F}$ is totally geodesic or $p=q=2$.*

Remark. Yau [8] has proved the following

THEOREM. *Let M^n be a compact minimal submanifold in the unit sphere S^{p+q}. Suppose that the sectional curvature of M^n is everywhere not less than $(q-1)/(2q-1)$, then either M^n is the totally geodesic sphere, the standard immersion of the product of two spheres or the Veronese surface in $S^4(1)$.*

Following Yau's theorem we easily prove that a harmonic foliation on the sphere, for which the normal plane field $\mathcal{F}^{\perp}$ is minimal and the sectional curvature of leaves $K\geqq(q-1)/(2q-1)$, is totally geodesic or $p=q=2$.

The compact condition of leaves is not necessary, because the integration is taken on the sphere.

Hereafter, we assume that the standard metric is bundle-like. Obviously, it implies that the normal plane field is minimal. If the sphere $S^4(1)$ is foliated foliated by the Veronese surfaces, then it is known in [2] that

$$(h_{ij}^3)=\begin{pmatrix} 0 & -\sqrt{\frac{1}{3}} \\ -\sqrt{\frac{1}{3}} & 0 \end{pmatrix}, \quad (h_{ij}^4)=\begin{pmatrix} \sqrt{\frac{1}{3}} & 0 \\ 0 & -\sqrt{\frac{1}{3}} \end{pmatrix}. \tag{3.2}$$

From (1.25) we have

$$\sum R_{\alpha i i \alpha}=-\sum A_{\alpha\gamma}^i A_{\gamma\alpha}^i-\sum h_{ij}^\alpha h_{ij}^\alpha ,$$

i.e.,

$$\sum A_{\alpha\beta}^i A_{\alpha\beta}^i=pq+S=\frac{16}{3}. \tag{3.3}$$

By differentiating (3.3) it yields

$$\sum A_{\alpha\beta}^i A_{\alpha\beta c}^i=0. \tag{3.4}$$

On the other hand, it follows from (1.15) and (1.26) that we get

$$\sum A_{\alpha\beta\gamma}^i-A_{\alpha\gamma\beta}^i+2\sum h_{ij}^\alpha A_{\beta\gamma}^j=0. \tag{3.5}$$

By cycling the indecies α, β and γ, it yields

$$\sum A_{\beta\gamma\alpha}^i-A_{\beta\alpha\gamma}^i+2\sum h_{ij}^\beta A_{\gamma\alpha}^j=0, \tag{3.6}$$

$$-\sum A_{\gamma\alpha\beta}^i+A_{\gamma\beta\alpha}^i-2\sum h_{ij}^\gamma A_{\alpha\beta}^j=0. \tag{3.7}$$

Taking the summation of (3.5), (3.6) and (3.7), we have

(3.8) $$\sum A^i_{\alpha\beta\gamma} = -\sum(h^\alpha_{ij}A^i_{\beta\gamma} + h^\beta_{ij}A^j_{\gamma\alpha} - h^\gamma_{ij}A^j_{\alpha\beta}).$$

By means of (3.4) and (3.8), we have

(3.9) $$\sum(h^\alpha_{ij}A^i_{\alpha\beta}A^j_{\alpha\gamma} + h^\beta_{ij}A^i_{\alpha\beta}A^j_{\gamma\alpha} - h^\gamma_{ij}A^i_{\alpha\beta}A^j_{\alpha\beta}) = 0.$$

It yields

(3.10) $$\sum_{i,j} h^\gamma_{ij}A^i_{\gamma\beta}A^j_{\gamma\beta} = 0.$$

Note that we do not take the summation with respect to α and β.

Now, taking $\gamma=3$ and then $\gamma=4$, we have

(3.11) $$A^1_{34}A^2_{34} = 0,$$

(3.12) $$(A^1_{34})^2 = (A^2_{34})^2.$$

From (3.11) and (3.12) we derive

$$A^1_{34} = A^2_{34} = 0.$$

It contradicts to (3.3). So, we can prove

THEOREM 3. *Let* $\mathcal{F}$ *be a harmonic foliation of codimension* q *on* $S^{p+q}(1)$, *for which the standard metric is bundle-like. If* $S \leqq p/(2-1/q)$ *holds on* $S^{p+q}(1)$, *then the foliation* $\mathcal{F}$ *is totally geodesic.*

THEOREM 4. *Let* $\mathcal{F}$ *be a harmonic foliation of codimension* q *on* $S^{p+q}(1)$, *for which the standard metric is bundle-like. If the sectional curvature* K *of leaves satisfy* $K \geqq (q-1)/(2q-1)$ *on* $S^{p+q}(1)$, *then the foliation* $\mathcal{F}$ *is totally geodesic.*

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