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DUALITIES AND DIMENSIONS OF ENDOMORPHISM RINGS

By

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Introduction and Terminology

A duality between two categories $\mathcal{C}$ and $\mathcal{D}$ is an equivalence of categories between the dual category $\mathcal{C}^{\mathrm{op}}$ of $\mathcal{C}$ and $\mathcal{D}$. The idea of studying dualities between full subcategories of module categories stems from Morita [6], who tried to dualize his theorem for equivalences of categories. He showed that, with a few conditions, every duality between full subcategories of R-Mod and Mod-S was given by the functors $\mathrm{Hom}_R(-, Q)$ and $\mathrm{Hom}_S(-, Q)$ for a certain bimodule ${}_RQ_S$. However, a classical result from Osofsky (see [1; lemma 24.7]) showed that there is no duality between $R-$Mod and Mod$-S$, for rings R and S.

This problem could be partially avoided by studying dualities between a full subcategory $\mathcal{C}$ of the category $R-$TMod of topological left modules and a full subcategory $\mathcal{D}$ of Mod$-S$. These dualities, which extend the classical Pontryagin and Lefschetz dualities, have been studied by some authors, for instance ([4], [7], [9], [11], [14], [15]). Similarly to the algebraic case, Zelmanowitz and Jansen [15, Theorem 1.3] have proved that, under a few conditions, any such duality is a restriction of the natural duality between the full subcategories $\mathrm{Ref}({}_RQ)$ and $\mathrm{Ref}(Q_S)$ of Q-reflexive modules of $R-$TMod and Mod$-S$, given by the functors $\mathrm{CHom}_R(-, Q)$ and $\mathrm{Hom}_S(-, Q)$, for a certain bimodule ${}_RQ_S$, such that ${}_RQ$ is a topological left module and any $s\in S$ defines a continuous R-endomorphism (see [14]). On the other hand, in [4], [9] and [14] the authors have given some particular cases where $\mathrm{Ref}(Q_S)=\mathrm{Cogen}(Q_S)$. This is a very interesting case, because, when Q_S is faithful, $\mathrm{Cogen}(Q_S)$ contains all the free right S-modules and is closed under submodules. Therefore the duality allows to transport properties of ${}_RQ$ to properties of S (see [9]).

The main purpose of this paper is to give necessary and sufficient conditions for having such nice dualities and apply them to the study of certain dimensions of the ring of continuous endomorphisms of a topological module. In Theorem 1.4 we characterize when $\mathrm{Ref}(Q_S)=\mathrm{Cogen}(Q_S)$ and in Theorem 1.7 we do it for $\mathrm{Ref}(Q_S)=\mathrm{Mod}-S$. In the last part of the paper we apply the foregoing technics

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to the study of the global and weak dimension of the ring $S=\mathrm{CEnd}_R(Q)$, for a certain topological module ${}_RQ$.

We refer the reader to [1] and [10] for all the ring-theoretical notions used in the text.

Throughout, R will denote an associative ring with unit. Unless specific mention we will consider R endowed with the discrete topology and will denote by $R-\mathrm{TMod}$ the category of topological left R-modules and continuous R-homomorphisms. We shall write ${}_RM$ (resp. M_R) when we want to emphasize that M is a left (resp. right) R-module. If $M\in R-\mathrm{TMod}$, we will call topological submodules of ${}_RM$ (resp. topological quotients of ${}_RM$) to the submodules of ${}_RM$ (resp. quotient modules of ${}_RM$) endowed with the relative (resp. quotient) topology. If $\{M_i \mid i\in I\}$ is a family of objects of $R-\mathrm{TMod}$, the direct product $\prod_{i\in I} M_i$, endowed with its product topology, is called the topological product of the M_i's. A topological submodule L of ${}_RM$ is said to be a topological direct summand of M if there exists another topological submodule N of ${}_RM$ such that M is the topological product of L and N. If $M, N\in R-\mathrm{TMod}$, then $\mathrm{CHom}_R(M, N)$ will denote the group of continuous homomorphisms from M to N. In particular, $\mathrm{CEnd}_R(M)$ will denote the ring of continuous endomorphisms of ${}_RM$. For convenience homomorphisms will act opposite to scalars. Accordingly the action of a composition of homomorphisms is evaluated from the closest to the element to the furthest (i.e. $[g\cdot f](x)=g(f(x))$ for right modules and $(x)[f\cdot g]=((x)f)g$ for left modules).

By a topological homomorphism (resp. monomorphism, epimorphism) we shall mean a continuous homomorphism (resp. monomorphism, epimorphism) open over the image.

Let us fix a bimodule ${}_RQ_S$ such that ${}_RQ$ is a topological module for which any $s\in S$ defines a continuous R-endomorphism of Q in the obvious way. There are canonical contravariant functors $-^*=\mathrm{CHom}_R(-, Q): R-\mathrm{TMod}\to\mathrm{Mod}-S$ and $-^*=\mathrm{Hom}_S(-, Q): \mathrm{Mod}-S\to R-\mathrm{TMod}$ (endowing $\mathrm{Hom}_S(X, Q)$ with the relative topology of the topological product Q^X via the natural inclusion). We will use the same notation $-^{**}$ to denote each composition of these two functors. There are canonical natural transformations $\sigma: 1_{R-\mathrm{TMod}}\to -^{**}$ and $\sigma: 1_{\mathrm{Mod}-S}\to -^{**}$ satisfying the equalities $\sigma_M^*\cdot\sigma_{M^*}=1_{M^*}$ and $\sigma_{X^*}\cdot\sigma_X^*=1_{X^*}$ for each $M\in R-\mathrm{TMod}$ and $X\in\mathrm{Mod}-S$ (see [14]). It is also useful to notice that if $M'\to M\xrightarrow{p}M''\to 0$ is an exact sequence of continuous R-homomorphism and p is open, then its dual sequence $0\to M''^*\to M^*\to M'^*$ is an exact sequence and if $X'\to X\xrightarrow{p}X''\to 0$ is an exact sequence in $\mathrm{Mod}-S$, then its dual $0\to X''^*\xrightarrow{p^*}X^*\to X'^*$ is an exact sequence and p^* is a topological monomorphism (see [14; 1.2 and 1.3]).

A topological left R-module M(resp. right S-module X) will be called Q-reflexive in case that σ_M(resp. σ_X) is a topological (resp. algebraic) isomorphism. We will denote by $\mathrm{Ref}({}_RQ)$(resp. $\mathrm{Ref}(Q_S)$) the full subcategory of $R-\mathrm{TMod}$ (resp. $\mathrm{Mod}-S$) whose objects are the Q-reflexive topological left R-modules (resp. right S-modules). It is well-known [14; 2.1] that the $-^*$ functors define a duality between $\mathrm{Ref}({}_RQ)$ and $\mathrm{Ref}(Q_S)$. An $M\in R-\mathrm{TMod}$ is said to be (finitely) Q-copresented if there exists an exact sequence $0\to M\xrightarrow{f}Q^I\to Q^J$ of continuous homomorphisms, where I and J are suitable (finite) sets and f is a topological monomorphism. We will denote by $\mathrm{Copres}({}_RQ)$ the class of Q-copresented topological left R-modules. Also $\mathrm{CEm}({}_RQ)$ will denote the class of topological left R-modules which are topologically isomorphic to closed topological submodules of topological products of copies of ${}_RQ$.

If every continuous homomorphism from a topological submodule of ${}_RQ$ (resp. ${}_RQ^I$ for any set I) to Q extends to a continuous endomorphism of ${}_RQ$ (resp. a continuous homomorphism $Q^I\to Q$) we will say that ${}_RQ$ is quasi-injective (resp. Π-quasi-injective).

Let $M\in R-\mathrm{TMod}$, $N\subseteqq M$ and $Y\subseteqq M^*$. We will use the following notation: $N'=r_{M^*}(N)=\{f\in M^*\,|\,(N)f=0\}$ and $Y'=l_M(Y)=\{x\in M\,|\,(x)Y=0\}$. These operators define a Galois connection between the lattices of submodules of ${}_RM$ and M^*_S. The elements in the image of l_M(resp. r_{M^*}) will be called Q-closed submodules of ${}_RM$(resp. M^*_S). Analogously, if $X\in\mathrm{Mod}-S$, there is a Galois connection between the lattices of submodules of X_S and ${}_RX^*$. Let us denote $\mathrm{C}_Q({}_RM)=\mathrm{Im}(l_M)$ and $\mathrm{C}_Q(M^*_S)=\mathrm{Im}(r_{M^*})$. These are complete lattices, since they are closed under intersection and the join of a family $\{N_i\,|\,i\in I\}$ in $\mathrm{C}_Q({}_RM)$ or in $\mathrm{C}_Q(M^*_S)$ is $(\sum_{i\in I}N_i)''$.

If A is an arbitrary ring and K is a right A-module, we will denote by $\mathrm{Gen}(K_A)$(resp. $\mathrm{Cogen}(K_A)$) the class of K-generated (resp. K-cogenerated) right A-modules. In the sequel, all full subcategories of a given category will be assumed closed under isomorphic images.

1. Dualities

In order to characterize the dualities between $\mathrm{Mod}-S$ and a subcategory of $R-\mathrm{TMod}$ we give the following result whose proof is essentially the same of [15; Theorem 1.3].

1.1. PROPOSITION. *Let R be a topological ring, S an arbitrary one and*

$\mathcal{C}\subseteq R$—TMod *and* $\mathcal{D}\subseteq$Mod$-S$ *two full subcategories such that* $S\in\mathcal{D}$. *Let* $\mathcal{C}\underset{H'}{\overset{H}{\rightleftarrows}}\mathcal{D}$ *be a duality. Then, there exists an* ${}_RQ\in\mathcal{C}$ *such that* $S\cong\mathrm{CEnd}_R(Q)$, $H\cong\mathrm{CHom}_R(-, Q)$. *If, additionally,* $\mathrm{Hom}_S(\mathcal{D}, Q)\subseteq\mathcal{C}$, *then* H' *is naturally isomorphic to* $\mathrm{Hom}_S(-, Q)$, $\mathcal{C}\subseteq\mathrm{Ref}({}_RQ)$ *and* $\mathcal{D}\subseteq\mathrm{Ref}(Q_S)$.

1.2. LEMMA. *Let* ${}_RQ$ *be a topological module,* $X\in\mathrm{Ref}(Q_S)$ *and* Y *a* Q*-closed submodule of* X_S. *Then* X/Y *is* Q*-reflexive if, and only if, every* $\alpha\in Y'^*$ *extends to a continuous homomorphism* $X^*\to Q$.

PROOF. Let $j: Y'\to X^*$ the embedding map and $p: X\to X/Y$ the canonical projection. Then $p^*: (X/Y)^*\to X^*$ induces a topological isomorphism $\Phi: (X/Y)^*\to Y'$ such that $p^*=\Phi\cdot j$. Thus, the following diagram is commutative

$$\begin{array}{ccccc} X & & \xrightarrow{\quad p \quad} & & X/Y \\ {\scriptstyle\sigma_X}\downarrow\cong & & & & \downarrow{\scriptstyle\sigma_{X/Y}} \\ X^{**} & \xrightarrow{j^*} & Y'^* & \underset{\cong}{\xrightarrow{\Phi^*}} & (X/Y)^{**} \end{array}$$

and j^* is an epimorphism if, and only if, $\sigma_{X/Y}$ is an isomorphism, because X/Y is Q_S-cogenerated.

DEFINITION. Let $Q\in R-$TMod.

(i) We will say that Q has no small submodules in case there exists a neighbourhood U of 0 in Q that does not contain any non-zero submodule of ${}_RQ$.

(ii) A $Q\in R-$TMod is called self-slender if the canonical S-homomorphism $\mathrm{CEnd}_R(Q)^{(I)}\to\mathrm{CHom}_R(Q^I, Q)$ is an isomorphism, for every set I.

(iii) For N and M in $R-$TMod, we say that N is C_Q-M-injective if every continuous homomorphism from a Q-closed submodule of M to N extends to one in $\mathrm{CHom}_R(M, N)$. We will say that Q is $\mathrm{C}_Q-\Pi$-quasi-injective if Q is C_Q-Q^I-injective, for every set I.

Note that, by lemma 1.2, if $M\in\mathrm{Ref}({}_RQ)$, then Q is C_Q-M-injective if, and only if, every Q-cogenerated quotient of M^* is Q-reflexive.

1.3. EXAMPLES. (1) *A discrete or compact without small submodules quasi-injective module is always* $\mathrm{C}_Q-\Pi$*-quasi-injective* [4; 3.9].

(2) *A discrete* C_Q-Π*-quasi-injective need not be quasi-injective* [2]. Indeed, if R is a complete discrete valuation domain, then $\mathrm{CHom}_R(-, R)$ and $\mathrm{Hom}_R(-, R)$ define inverse dualities between $\mathrm{CEm}({}_RR)$ and $\mathrm{Cogen}(R_R)$ and Theorem 1.4 below

shows that it is $C_Q - \Pi$-quasi-injective. However, it is well-known that R is self-injective if, and only if, it is a field.

(3) *A compact (self-slender)* $C_Q - \Pi$*-quasi-injective need not be quasi-injective.* To see this, we only have to take the Pontryagin dual of the group $\boldsymbol{Q}_{\boldsymbol{Z}}$ of rational numbers and realize that $\mathrm{Hom}_{\boldsymbol{Z}}(\boldsymbol{Q}, \boldsymbol{Q}^{(I)}) \cong \mathrm{Hom}_{\boldsymbol{Z}}(\boldsymbol{Q}, \boldsymbol{Q})^{(I)}$ and every $\boldsymbol{Q}$-generated $\boldsymbol{Z}$-submodule of $\boldsymbol{Q}^{(I)}$ is injective.

Now, we characterize when $\mathrm{Ref}(Q_S) = \mathrm{Cogen}(Q_S)$.

1.4. THEOREM. *Let ${}_RQ_S$ be a bimodule, such that ${}_RQ$ is a topological module for which any $s \in S$ defines a continuous R-endomorphism and Q_S is faithful. The following statements are equivalent:*

(1) $S \cong \mathrm{CEnd}_R(Q)$ *canonically and ${}_RQ$ is* C_Q-Π*-quasi-injective and self-slender.*

(2) *For every* $X \in \mathrm{Mod}-S$, σ_X *is an epimorphism.*

(3) $\mathrm{Cogen}(Q_S) = \mathrm{Ref}(Q_S)$.

(4) *The functor* $\mathrm{CHom}_R(-, Q)$ *is a duality between* $\mathrm{Copres}({}_RQ)$ *and* $\mathrm{Cogen}(Q_S)$.

If those conditions hold, then $\mathrm{Ref}({}_RQ) = \mathrm{Copres}({}_RQ)$.

PROOF. (1)$\Rightarrow$(2) Let $X \in \mathrm{Mod}-S$. If $S^{(\mathrm{Ker}\, p)} \to S^{(X)} \xrightarrow{p} X \to 0$ is the canonical presentation of X_S, then its dual $0 \to X^* \to Q^X \to Q^{\mathrm{Ker}\, p}$ is a Q-copresentation of ${}_RX^*$, showing that X^* is a Q-closed submodule of Q^X. If $\alpha \in X^{**}$, then it extends to a continuous homomorphism $\beta : Q^X \to Q$. Since Q is self-slender, the compositions $s_x = u_x \cdot \beta \in S$ ($u_x : Q \to Q^X$ the canonical injection in the x^{th} component) are almost all zero. Then $(f)\alpha = (f)\beta = \sum_{x \in X} f(x)s_x = f(\sum_{x \in X} x s_x) = (f)[\sigma_X(\sum_{x \in X} x s_x)]$, for every $f \in X^*$. Thus $\alpha = \sigma_X(\sum_{x \in X} x s_x) \in \mathrm{Im}\sigma_X$.

(2)$\Rightarrow$(3) It is obvious.

(3)$\Rightarrow$(1) If $\mathrm{Cogen}(Q_S) = \mathrm{Ref}(Q_S)$, since $S \in \mathrm{Cogen}(Q_S)$, then every free right S-module is Q-reflexive and hence every product of copies of ${}_RQ$ is Q-reflexive. Consequently, the dual of Q^I is necessarily isomorphic to $S^{(I)}$. This means that Q is self-slender. Let $\alpha \in M^*$, with $M \in C_Q({}_RQ^I)$, and $Y = M'$. Since $S^{(I)}/Y$ is Q-reflexive and $Y' = M$, lemma 1.2 applies.

(1)$\Rightarrow$(4) By (1)$\Rightarrow$(3), we only need to prove that $\mathrm{Copres}({}_RQ) \subseteqq \mathrm{Ref}({}_RQ)$. If $M \in \mathrm{Copres}({}_RQ)$ and $0 \to M \xrightarrow{j} Q^I \to Q^J$ is an exact sequence with j a topological monomorphism, then, by the $C_Q - \Pi$-quasi-injectivity of ${}_RQ$, j^{**} is a topological monomorphism and so is $\sigma_{M'}$ because $j \cdot \sigma_{Q^I} = \sigma_M \cdot j^{**}$ and σ_{Q^I} is a topological isomorphism. If $f : Q^I \to Q$ is a continuous homomorphism that vanishes on M, then $f|_{M^{**}} \in r_{M^{***}}(M) = r_{M^*}(M) = 0$. Consequently, $M^{**} \subseteqq M'' = M$.

(4)$\Rightarrow$(3) Since $\mathrm{Hom}_S(X, Q) \in \mathrm{Copres}({}_RQ)$ for every $X \in \mathrm{Mod}-S$, the same argument of the proof of [15; Theorem 1.3] is valid to show that $\mathrm{Hom}_S(-, Q)$ is an adjoint of $\mathrm{CHom}_R(-, Q)$, so that they are mutually inverse dualities between $\mathrm{Copres}({}_RQ)$ and $\mathrm{Cogen}(Q_S)$. Thus, by 1.1, $\mathrm{Cogen}(Q_S) \subseteq \mathrm{Ref}(Q_S)$. The converse inclusion is well-known.

Whenever we have a general result involving a certain full subcategory $\mathcal{C}$ of R-TMod, we can consider the particular case where $\mathcal{C}$ consists of Hausdorf compact R-modules. In this situation, the Pontryagin Duality (see, e.g., [4; Ch. 5] for the details) allows us to dualize the result in order to find a corresponding statement for a full subcategory of $\mathrm{Mod}-R$. As an example, we do this with Theorem 1.4, while the dualization of the other propositions and theorems are left to the reader.

In the following Theorem we denote by $\mathrm{CD}(P_R)$ the class of right R-modules with P-codominant dimension $\geqq 2$ in the terminology of [8].

1.4.* TEEOREM. *Let ${}_SP_R$ be a bimodule such that ${}_SP$ is faithful and ${}_RQ_S = \mathrm{Hom}_Z(P, \boldsymbol{R}/\boldsymbol{Z})$ its Pontryagin dual. Then, the following assertions are equivalent:*

(1) $\mathrm{Hom}_R(P, P^{(I)})$ *is canonically isomorphic to* $S^{(I)}$, *for every set* I, *and if* $P^{(I)} \to P^{(J)} \xrightarrow{f} M \to 0$ *is an exact sequence in* $\mathrm{Mod}-R$ *and* $\alpha \in \mathrm{Hom}_R(P, M)$, *then there exists* $\beta \in \mathrm{Hom}_R(P, P^{(J)})$ *such that* $\alpha = f \cdot \beta$.

(2) *For every* $X \in \mathrm{Mod}-S$, *the canonical* $X \to \mathrm{Hom}_R(P, X \otimes_S P)$ *is epic.*

(3) $\mathrm{Cogen}(Q_S) = \{X_S \mid$ *the canonical* $X \to \mathrm{Hom}_R(P, X \otimes_S P)$ *is iso*$\}$.

(4) $\mathrm{Hom}_R(P, -)$ *is an equivalence between* $\mathrm{CD}(P_R)$ *and* $\mathrm{Cogen}(Q_S)$.

DEFINITION. We will refer to the following condition as the (*) property.

(*) *For any continuous homomorphisms* $f: Q^I \to Q^J$ *and* $\alpha: Q^I \to Q$, *with* $\mathrm{Ker} f \subseteq \mathrm{Ker}\, \alpha$, *there exists a continuous homomorphism* $\beta: Q^J \to Q$ *so that* $f \cdot \beta = \alpha$.

1.5. EXAMPLE. If Q is Π-quasi-injective and every morphism $f: Q^I \to Q^J$ is open on its image, then Q satisfies (*) property. In particular, that is the case when Q is quasi-injective, compact with no small submodules or when Q is quasi-injective, artinian and discrete. This last assertion is due to the fact that a discrete artinian module Q is strictly linearly compact and so every topological product of copies of Q is strictly compact. If $f: Q^I \to Q^J$ is continuous, then $Q^I/\mathrm{Ker} f$ has a minimal topology and thus it is topologically isomorphic to $\mathrm{Im} f$.

Nevertheless, the property of every continuous homomorphism $f: Q^I \to Q^J$ being open on its image does not imply (*) property even if ${}_RQ$ is self-slender and $C_Q-\Pi$-quasi-injective. The example of this is due to M. Sato. There exists a module P_R such that $\mathrm{Hom}_R(P, -)$ defines an equivalence of categories between $\mathrm{Gen}(P_R)$ and $\mathrm{Cogen}(K_S)\neq\mathrm{Mod}-S$, where $S=\mathrm{End}_R(P)$ and $K=\mathrm{Hom}_R(P, E)$ for an injective cogenerator E of $\mathrm{Mod}-R$(see [5]). Thus, if ${}_RQ$ is the Pontryagin dual of P_R, then $\mathrm{CHom}_R(-, Q)$ defines a duality between $\mathrm{CEm}({}_RQ)$ and $\mathrm{Cogen}(K_S)$. By Proposition 1.1, $\mathrm{Cogen}(K_S)\subseteqq\mathrm{Ref}(Q_S)$. But, if $X\in\mathrm{Ref}(Q_S)$, then $X^*\in\mathrm{CEm}({}_RQ)$, thus $X\cong X^{**}\in\mathrm{Cogen}(K_S)$, showing that $\mathrm{Ref}(Q_S)=\mathrm{Cogen}(K_S)$. Thus $Q_S\in\mathrm{Cogen}(K_S)$ and so $\mathrm{Cogen}(Q_S)\subseteqq\mathrm{Cogen}(K_S)$, which is in fact an equality. By Theorem 1.4, ${}_RQ$ is $C_Q-\Pi$-quasi-injective and self-slender. Moreover, every continuous homomorphism $f: Q^I\to Q^J$ is open on its image (since Q is compact). However, Theorem 1.7 below shows that ${}_RQ$ does not have (*) property.

Definition. Let ${}_RQ\in R-\mathrm{TMod}$. An $M\in R-\mathrm{TMod}$ is called Q-cogenerated if there exists a continuous (non necessarily topological) monomorphism $M\to Q^I$, for some set I. We will say that ${}_RQ$ is a self-cogenerator if every Hausdorf topological quotient of ${}_RQ$ is Q-cogenerated.

1.6. Lemma. *Let $f: M\to N$ be a continuous homomorphism and assume that N is Q-cogenerated. The following conditions are equivalent:*

(a) $\mathrm{Im} f^*$ *is a Q-closed submodule of* $M^*{}_S$.

(b) *If* $\alpha\in M^*$ *and* $\mathrm{Ker} f\subseteqq\mathrm{Ker}\,\alpha$, *then there exists* $\beta\in N^*$ *so that* $f\cdot\beta=\alpha$.

Proof. $(\mathrm{Im} f^*)''=\{\alpha\in M^*\,|\,(x)\alpha=0\ \forall x\in M$ such that $((x)f)\beta=0\ \forall\beta\in N^*\}$. Since N is Q-cogenerated, the last submodule equals $\{\alpha\in M^*\,|\,\mathrm{Ker} f\subseteqq\mathrm{Ker}\alpha\}$. Now the equivalence is evident.

1.7. Theorem. *Let ${}_RQ_S$ be a bimodule such that ${}_RQ$ is a topological left R-module for which any $s\in S$ defines a continuous R-endomorphism. The following are equivalent:*

(1) $S\cong\mathrm{CEnd}_R(Q)$ *and ${}_RQ$ is self-slender,* C_Q-Π-*quasi-injective and satisfies* (*) *property.*

(2) $\mathrm{Ref}(Q_S)=\mathrm{Mod}-S$.

(3) *The functor* $\mathrm{CHom}_R(-, Q)$ *is a duality between* $\mathrm{Copres}({}_RQ)$ *and* $\mathrm{Mod}-S$.

Proof. $(2)\Rightarrow(3)$ is obvious by taking the duality between $\mathrm{Ref}({}_RQ)$ and $\mathrm{Ref}(Q_S)$ and applying the last assertion in 1.4.

$(1)\Rightarrow(2)$ By 1.4 we know that $\mathrm{Ref}(Q_S)=\mathrm{Cogen}(Q_S)$. Let X be a right S-module and $S^{(J)}\xrightarrow{v}S^{(I)}\xrightarrow{p}X\to 0$ a free presentation of X. Then, $0\to X^*\xrightarrow{p^*}Q^I\xrightarrow{v^*}Q^J$ is an exact sequence with p^* a topological monomorphism. There exists a commutative diagram

$$\begin{tikzcd} S^{(J)} \arrow[r, "v"] \arrow[d, "\sigma", "\cong"'] & S^{(I)} \arrow[r, "p"] \arrow[d, "\sigma", "\cong"'] & X \arrow[r] \arrow[d, "\sigma"] & 0 \\ S^{(J)**} \arrow[r, "v^{**}"] & S^{(I)**} \arrow[r, "p^{**}"] & X^{**} \arrow[r] & 0 \end{tikzcd}$$

where the bottom sequence is exact since Q is C_Q-Q^I-injective and Q has (*) property. We conclude that X_S is Q-reflexive.

(3)$\Rightarrow$(1) We can use 1.4 to see that $S\cong \mathrm{CEnd}_R(Q)$ canonically, ${}_RQ$ is self-slender and $C_Q-\Pi$-quasi-injective and $\mathrm{Hom}_S(-, Q)$ is the inverse functor of $\mathrm{CHom}_R(-, Q)$. On the other hand, let $f: Q^I \to Q^J$ and $\alpha: Q^I \to Q$ be continuous with $\mathrm{Ker} f \subseteq \mathrm{Ker}\,\alpha$. Since $S^{(I)}/\mathrm{Im} f^*$ is Q_S-cogenerated, then $\mathrm{Im} f^*$ is a Q-closed submodule of $S^{(I)}$ and lemma 1.6 applies.

REMARKS. (1) The equivalent conditions of the foregoing theorem are satisfied by the quasi-injective compact modules with no small submodules. But example 1.3.3 shows that those conditions are general because, for instance, the Pontryagin dual of $\boldsymbol{Q}_{\boldsymbol{Z}}$ has small submodules (see [4; Corollary 4.12]) and, however, satisfies the conditions of the above theorem.

(2) The dualization of the foregoing theorem for the case of ${}_RQ$ being Haussdorf compact yields [12; Theorem 2.1].

1.8. EXAMPLE. Theorem 1.7 allows us to give another example of a quasi-injective discrete module that has (*) property. Let R be the upper triangular matrix ring with indices in $\boldsymbol{Z}$ and entries in a field k and Q the set of finite columns matrices with indices in $\boldsymbol{Z}$ and entries in k. Q is canonically a left R-module with the matrix product. It is easy to see that ${}_RQ$ is quasi-injective, $\mathrm{End}_R(Q)\cong k$ and the submodules of ${}_RQ$ are of the form $Q_m=\{x=(x_n)_{n\in \boldsymbol{Z}} \mid x_n=0$ for every $n\geqq m\}$, thus $-^*$ induces a duality between $\mathrm{Copres}({}_RQ)$ and $\mathrm{Cogen}(Q_k)$ $=\mathrm{Mod}-k$. By theorem 1.7, ${}_RQ$ satisfies (*) property and is clearly not artinian, in fact it is neither finitely cogenerated nor linearly compact.

1.9. COROLLARY. *Let ${}_RQ$ be a Haussdorf linearly topological module which is a self-cogenerator. The following statements are equivalent:*

(1) $\mathrm{Ref}(Q_S)=\mathrm{Mod}-S$.

(2) *${}_RQ$ is discrete, quasi-injective and satisfies* (*) *property.*

If the above conditions hold, then ${}_RQ$ must be finitely cogenerated.

PROOF. (1)$\Rightarrow$(2) The quasi-injectivity is a consequence of theorem 1.4 and the fact that $C_Q-\Pi$-quasi-injective+self-cogenerator implies quasi-injectivity. The discreteness can be proven as in [15; 2.3]. By Theorem 1.7, (*) property also holds.

(2)$\Rightarrow$(1) By [4; Theorems 2.9], Q is Π-quasi-injective and, in particular, $C_Q-\Pi$-quasi-injective. Now, Theorem 1.7 applies.

The last assertion is a consequence of S_S being finitely generated, so that $C_Q({}_RQ)$(which is the whole lattice of submodules of ${}_RQ$) has the finite intersection property.

REMARKS. (1) Note that if the condition of ${}_RQ$ being a self-cogenerator is dropped, then, as example 1.8 shows, both conditions in the foregoing corollary may occur without ${}_RQ$ being finitely cogenerated.

(2) In [15; Theorems 2.3 and 3.1] the authors show that the two conditions below are equivalent for a linearly topological left R-module Q with $S=\mathrm{CEnd}_R(Q)$.

(i) $\mathrm{Ref}(Q_S)=\mathrm{Mod}-S$ and every Q-quotient (see op. cit for the definition) of ${}_RQ$ is Q-reflexive.

(ii) ${}_RQ$ is a discrete linearly compact finitely cogenerated quasi-injective self-cogenerator.

The condition (2) of our Corollary 1.9 together with the previous hypothesis of ${}_RQ$ being a self-cogenerator are strictly weaker than condition (ii) of Zelmanowitz and Jansen. Indeed if, in our situation, ${}_RQ$ were linearly compact, then Q_S would be an injective cogenerator (see, e.g., [4, Theorem 5.10]). However, there are examples of discrete quasi-injective self-cogenerators satisfying (*) property which are not injective over their endomorphism rings (see, e.g., [13,3.1]).

2. Applications

In what follows, we assume that ${}_RQ$ is a topological module and $S=\mathrm{CEnd}_R(Q)$. The main goal of this part is to study the global and weak dimensions of S, by using the results obtained in part 1. For all concepts and terminology about those dimensions, we refer the reader to [10, Ch. 9]. Let us start with some previous results.

2.1. LEMMA. *Let M and Q be topological left R-modules. Assume that Q is C_Q-M-injective and let $L\subseteq N\subseteq {}_RM$ be topological submodules. If L is Q-closed in ${}_RN$ and N is Q-closed in ${}_RM$, then L is Q-closed in ${}_RM$. Consequently, Q is C_Q-N-injective for every Q-closed submodule N of ${}_RM$.*

PROOF. Let L be a Q-closed topological submodule of ${}_RN$, where N is a Q-closed topological submodule of ${}_RM$. Let $m \in l_M r_{M^*}(L)$ and $f \in r_{N^*}(L)$. Then the C_Q-M-injectivity of Q entails that f extends to a $g \in r_{M^*}(L)$, so that $(m)g = 0$. Since $l_M r_{M^*}(L) \subseteq N$, because N is Q-closed in ${}_RM$ and contains L, $(m)f = (m)g = 0$ and hence $m \in l_N r_{N^*}(L) = L$.

2.2. PROPOSITION. *Let M, Q be topological left R-modules, being the second one self-slender, $S = \mathrm{CEnd}({}_RQ)$ and let us consider the following assertions:*

(1) *${}_RM$ is a topological direct summand of some topological product of copies of ${}_RQ$.*

(2) *$M \in \mathrm{Ref}({}_RQ)$ and M^*_S is projective.*

(3) *${}_RM$ is Q-reflexive and C_Q-Q^J-injective for every set J.*

(4) *$M \in \mathrm{Ref}({}_RQ)$ and is a topological direct summand of any Q-reflexive left R-module containing M as a Q-closed submodule.*

(5) *${}_RM$ is an injective object of $\mathrm{Ref}({}_RQ)$.*

Then $(5) \Rightarrow (4) \Rightarrow (1) \Leftrightarrow (2)$ *and, when Q is $C_Q-\Pi$-quasi-injective,* (1), (2), (3) *and* (4) *are equivalent. If, moreover, Q satisfies* (*) *property, then the five assertions are equivalent.*

PROOF. The equivalence of (1) and (2) is a consequence of the additiveness of the dual functors $(-)^*$ and the fact that topological products of copies of ${}_RQ$ correspond to direct sums of copies of S_S, and viceversa, by means of those functors.

$(5) \Rightarrow (4)$ is obvious.

$(4) \Rightarrow (1)$ Since any Q-reflexive left R-module is topologically isomorphic to a Q-closed submodule of a topological product of copies of ${}_RQ$, the implication is clear.

Let us consider now that ${}_RQ$ is also $C_Q-\Pi$-quasi-injective. Then:

$(1) \Rightarrow (3)$ Since Q is C_Q-Q^J-injective, for any set J, we only have to show that a topological product of C_Q-Q^J-injective R-modules and a topological direct summand of a C_Q-Q^J-injective R-modules are again C_Q-Q^J-injective. This can be done by using standard arguments (see, e.g., [1, Proposition 16.10]).

$(3) \Rightarrow (4)$ Assume that M is a Q-closed topological submodule of $N \in \mathrm{Ref}({}_RQ)$. Since N is topologically isomorphic to a Q-closed submodule of Q^{N^*}, the previous lemma tells us that M is C_Q-N-injective and hence a topological direct summand of N.

When, moreover, ${}_RQ$ satisfies (*) property, the duality between $\mathrm{Ref}({}_RQ)$ and $\mathrm{Ref}(Q_S) = \mathrm{Mod}-S$ (1.7) gives the equivalence between (2) and (5).

REMARKS. (a) The above condition (3) is not equivalent to (1) and (2) when ${}_RQ$ is not $C_Q-\Pi$-quasi-injective. Indeed in that case $M=Q$ satisfies (1) and (2) and does not satisfy (3).

(b) The (*) property is needed in order to ensure that (5) is equivalent to the other assertions. Indeed if $Q=R$ is a complete discrete valuation domain and $x\neq 0$ is a non-invertible element of R, then the multiplication by x provides a topological monomorphism $f: R\rightarrow R$ whose cokernel R/Rx satisfies that $\mathrm{CHom}_R(R/Rx, R)=0$ (I). If R were an injective object of $\mathrm{Ref}({}_RR)$, then f would be a split monomorphism and this contradicts (I). However R is $C_Q-\Pi$-quasi-injective (see Example 1.3 (2)) and hence C_Q-R^J-injective, for every set J.

DEFINITION. Let ${}_RQ$ be self-slender. A sequence in $R-\mathrm{TMod}$

$$\cdots \longrightarrow M_{n-1} \xrightarrow{f_n} M_n \xrightarrow{f_{n+1}} M_{n+1} \longrightarrow \cdots$$

is said to be Q-exact when $(\mathrm{Im} f_n)''=\mathrm{Ker} f_{n+1}$ (as submodules of M_n), for every $n\in \boldsymbol{Z}$. A Q-resolution of $M\in \mathrm{Copres}({}_RQ)$ is a sequence of continuous homomorphisms $0\rightarrow M \xrightarrow{f_0} E_0 \xrightarrow{f_1} E_1 \rightarrow \cdots \xrightarrow{f_n} E_n \rightarrow \cdots$ such that f_0 is a topological monomorphism with $\mathrm{Im} f_0=\mathrm{Ker} f_1$, E_n is a topological direct summand of some topological product of copies of ${}_RQ$, for any $n\geqq 0$, and $E_0\rightarrow E_1\rightarrow\cdots\rightarrow E_n\rightarrow\cdots$ is a Q-exact sequence.

We will say that Q-codimension of $M\in\mathrm{Copres}({}_RQ)$ is $\leqq n$ $(n\in \boldsymbol{N})$ if there exists a Q-resolution E of M, as above, such that $E_k=0$ for every $k>n$. We denote by $Q-\mathrm{cd}({}_RM)$ the minimum $n\in\boldsymbol{N}$ (if it exists) such that the Q-codimension of M is $\leqq n$. We call self-codimension of ${}_RQ$ to the supremum of the set $\{Q-\mathrm{cd}({}_RM)\mid M\in\mathrm{Ref}({}_RQ)\}$ (note that in general $\mathrm{Ref}({}_RQ)$ need not be equal to $\mathrm{Copres}({}_RQ)$!) and denote it by $\mathrm{scd}({}_RQ)$, assuming that $\mathrm{scd}({}_RQ)=\infty$ if that supremum does not exist.

NOTATION. If $\boldsymbol{C}: \cdots\rightarrow M_{n-1}\rightarrow M_n\rightarrow M_{n+1}\rightarrow\cdots$ is a co-chain complex in $R-\mathrm{TMod}$, then we will denote by $\boldsymbol{C}^*: \cdots\rightarrow M^*_{n+1}\rightarrow M^*_n\rightarrow M^*_{n-1}\rightarrow\cdots$ the dual chain complex in $\mathrm{Mod}-S$ and viceversa.

2.3. PROPOSITION. *Let ${}_RQ$ be a self-slender topological module. The following assertions are equivalent:*

(a) *${}_RQ$ is $C_Q-\Pi$-quasi-injective and satisfies (*) property.*

(b) *The dual of any Q-exact sequence of Q-copresented left R-modules is exact in* $\mathrm{Mod}-S$.

(c) *The dual of each Q-resolution of an $M\in\mathrm{Copres}({}_RQ)$ is a projective re-*

solution of M^*_S.

(d) *For every exact sequence* $0 \to M \xrightarrow{f} Q^I \to Q^J$, *with* f *a topological monomorphism, the dual sequence* $S^{(J)} \to S^{(I)} \to M^* \to 0$ *is exact.*

PROOF. (a)$\Rightarrow$(b) Let $\boldsymbol{M}$: $\cdots \to M_{n-1} \xrightarrow{f_n} M_n \xrightarrow{f_{n+1}} M_{n+1} \to \cdots$ be a Q-exact sequence, with each M_n Q-copresented, and consider its dual sequence in Mod$-S$ $\boldsymbol{M}^*$: $\cdots \to M^*_{n+1} \xrightarrow{f^*_{n+1}} M^*_n \xrightarrow{f^*_n} M^*_{n-1} \to \cdots$. Since Ref$(Q_S)$=Mod$-S$, Im$f^*_{n+1}$ is a Q-closed submodule of M^*_n so that, in order to prove the exactness of $\boldsymbol{M}^*$, we only have to see that Ker$f^*_n \subseteq (\mathrm{Im} f^*_{n+1})''$, for any $n \in \boldsymbol{Z}$. Let us fix n and take $\alpha \in \mathrm{Ker} f^*_n$ and $x \in (\mathrm{Im} f^*_n)'$. We shall prove that $(x)\alpha = 0$ and this will end the proof of this implication. Let us notice that, by the choice of α and x, $f_n \cdot \alpha = 0$ and $(x)(f_{n+1} \cdot \beta) = 0$ for every $\beta \in M^*_{n+1}$. From the fact that M_{n+1} is Q-copresented (hence, in particular, Q-cogenerated) follows that $(x)f_{n+1} = 0$, i.e., $x \in \mathrm{Ker} f_{n+1}$. But $(\mathrm{Im} f_n)\alpha = 0$, thus $\alpha \in (\mathrm{Im} f_n)'$, and the Q-exactness of $\boldsymbol{M}$ entails that $(\mathrm{Im} f_n)' = (\mathrm{Ker} f_{n+1})'$. Consequently, $(x)\alpha = 0$ as desired.

(b)$\Rightarrow$(c) It is a direct consequence of the definition of Q-resolution and the use of 2.2.

(c)$\Rightarrow$(d) If $0 \to M \xrightarrow{f} Q^I \xrightarrow{g} Q^J$ is an exact sequence in $R-$TMod, then it can be extended to a Q-resolution $\boldsymbol{E}$ of $\boldsymbol{M}$ by taking $E_0 = Q^I$, $E_1 = Q^J$, $f_0 = f$ and $f_1 = g$ and defining E_n and f_n, for $n > 1$, by induction: if f_{n-1} and E_{n-1} have been defined, then there exists a continuous monomorphism λ_n from $E_{n-1}/(\mathrm{Im} f_{n-1})''$ to a certain topological product E_n of copies of ${}_RQ$. Then we put $f_n = p_{n-1} \cdot \lambda_n$, where p_{n-1} is the canonical projection from E_{n-1} onto $E_{n-1}/(\mathrm{Im} f_{n-1})''$. Now (c) can be applied to this Q-resolution to get (d).

(d)$\Rightarrow$(a) If M is a Q-closed submodule of Q^I, then M is the kernel of a continuous homomorphism $g: Q^I \to Q^J$ and hence $0 \to M \overset{j}{\hookrightarrow} Q^I \xrightarrow{g} Q^J$ is an exact sequence. Thus its dual sequence is exact and this implies that Q is $\mathrm{C}_Q-\Pi$-quasi-injective. If $f: Q^I \to Q^J$ and $\alpha: Q^I \to Q$ are continuous homomorphism with Ker$f \subseteq$ Ker$\,\alpha$, then $0 \to \mathrm{Ker} f \overset{j}{\hookrightarrow} Q^I \xrightarrow{f} Q^J$ is an exact sequence and $\alpha \in$ Ker$j^* = $ Imf^*.

2.4. THEOREM. *Let* ${}_RQ$ *be a self-slender topological module and* $S = \mathrm{CEnd}_R(Q)$. *Then the right global dimension* rD(S) *of* S *is greater or equal than the self-codimension of* ${}_RQ$.

If, in addition, ${}_RQ$ *is* $\mathrm{C}_Q-\Pi$*-quasi-injective and has* (*) *property, then* rD(S)= scd$({}_RQ)$ *and it can be characterized as follows.*

The following statements are equivalent for a natural number n:

(a) $\mathrm{rD}(S)\leqq n$.

(b) *For every Q-resolution* $0\to M\xrightarrow{f_0}E_0\xrightarrow{f_1}\cdots$ *of an* $M\in\mathrm{Copres}(_RQ)$, $\mathrm{Ker}f_{n+1}$ *is topological direct summand of* E_n.

(c) *For every* $M\in\mathrm{Copres}(_RQ)$ *(resp.* $M\in\mathrm{C}_Q(_RQ)$*) there exists a Q-resolution of the form*

$$0\longrightarrow M\xrightarrow{f_0}Q^{I_0}\xrightarrow{f_1}Q^{I_1}\longrightarrow\cdots\xrightarrow{f_{n-1}}Q^{nI_{n-1}}\longrightarrow E_n\longrightarrow 0.$$

PROOF. Let $M\in\mathrm{Ref}(_RQ)$. Let $\boldsymbol{P}$: $0\to P_n\xrightarrow{g_n}\cdots\xrightarrow{g_1}P_0\xrightarrow{g_0}M^*{}_S\to 0$ be a projective resolution of $M^*{}_S$. We are going to prove that $\boldsymbol{E}: 0\to M\xrightarrow{f_0}E_0\xrightarrow{f_1}\cdots\xrightarrow{f_n}E_n\to 0$, where, $E_k=P_k^*(k=0, 1, 2, \cdots n)$, $f_0=\sigma_M\cdot g_0^*$ and $f_k=g_k^*$ for any $k=1, 2, \cdots, n$, is a Q-resolution of M. First, E_n is a topological direct summand of some topological product of copies of $_RQ$, for every $k=0, 1, \cdots, n$ (2.2). Clearly, $\mathrm{Im}f_k\subseteqq\mathrm{Ker}f_{k+1}(k=1, 2, \cdots, n)$ and so $(\mathrm{Im}f_k)''\subseteq\mathrm{Ker}f_{k+1}$, because $\mathrm{Ker}f_{k+1}$ is Q-closed in E_k. On the other hand, if $\alpha\in\mathrm{Ker}f_{k+1}$ and $x\in(\mathrm{Im}f_k)'$, then $\alpha\cdot g_{k+1}=0$ so that $\mathrm{Ker}g_k=\mathrm{Im}g_{k+1}\subseteq\mathrm{Ker}\alpha$ and, for every $\beta\in P_{k-1}^*$, $\beta(g_k(x))=((\beta)f_k)(x)=0$. But, since P_{k-1} is Q-cogenerated, $g_k(x)=0$ and thus $\alpha(x)=0$. We conclude that $\mathrm{Ker}f_{k+1}\subseteq(\mathrm{Im}f_k)''$. Finally, f_0 is a topological monomorphism and $\mathrm{Ker}f_1=\mathrm{Im}f_0$. We have then proved that $Q-\mathrm{cd}(_RM)$ is smaller or equal than the projective dimension of $M^*{}_S$ and thus $\mathrm{scd}(_RQ)\leqq\mathrm{rD}(S)$.

If $_RQ$ is $\mathrm{C}_Q-\Pi$-quasi-injective and has (*) property as well, then any Q-resolution of every $M\in\mathrm{Copres}(_RQ)$ tranforms in a projective resolution of M_S^* by $-^*$ (2.3). Thus the projective dimension of $X_S\cong X^{**}$ is smaller or equal than $Q-\mathrm{cd}(_RX^*)$ so that $\mathrm{scd}(_RQ)\geqq\mathrm{rD}(S)$.

We are going to prove the equivalence between the last assertions for $n\geqq 1$. The case $n=0$ is very easy to see.

(a)$\Rightarrow$(b) Let $\boldsymbol{E}: 0\to M\xrightarrow{f_0}E_0\xrightarrow{f_1}\cdots$ be a Q-resolution. Then its dual $\boldsymbol{P}=\boldsymbol{E}^*: \cdots\xrightarrow{g_1}P_0\xrightarrow{g_0}M^*\to 0$ is a projective resolution of $M^*{}_S$ (2.3). Consequently, $\mathrm{Ker}g_{n-1}=\mathrm{Im}g_n$ is projective, and hence $\mathrm{Ker}g_n$ is a direct summand of P_n. But $\mathrm{Ker}g_n=(\mathrm{Im}f_n)'$ and, therefore, $(\mathrm{Im}f_n)''$ is also a direct summand of E_n.

(b)$\Rightarrow$(c) It is easy to construct a Q-resolution of any $M\in\mathrm{Copres}(_RQ)$ of the form $0\to M\xrightarrow{f_0}Q^{I_0}\xrightarrow{f_1}Q^{I_1}\to\cdots$ (see the proof of (c)$\Rightarrow$(d) in Proposition 2.3), and the hypothesis implies that $(\mathrm{Im}f_{n-1})''=\mathrm{Ker}f_n$ is a topological direct summand of some topological product of copies of $_RQ$.

(c)$\Rightarrow$(a) It is obvious that, when condition (c) holds, $\mathrm{scd}(_RQ)\leqq n$ and (a) is a consequence of the second paragraph of this proof.

The case $n=1$ does not need the hypothesis of Q satisfying (*) property.

We need the following previous definition.

Definition. A contionuous R-homomorphism $f: M \to N$ is said to be a Q-epimorphism if $(\mathrm{Im} f)''=N$.

2.5. Proposition. *Let ${}_RQ$ be a self-slender $C_Q-\Pi$-quasi-injective topological module and $S=\mathrm{CEnd}_R(Q)$. The following conditions are equivalent:*

(a) *S is right hereditary.*

(b) *For every Q-epimorphism $f: Q \to M$, with M Q-reflexive, M is a topological direct summand of a topological product of copies of ${}_RQ$.*

(c) *For every continuous homomorphism $f: Q \to Q^I$, $(\mathrm{Im} f)''$ is a topological direct summand of Q^I.*

Proof. Let us consider two homomorphisms, $f: Q \to Q^I$ in $R-\mathrm{TMod}$ and $g: S^{(I)} \to S$ in $\mathrm{Mod}-S$. If they are dual one of each other, then $(\mathrm{Im} f)'=\mathrm{Ker} g$ so that $(\mathrm{Im} f)''$ is a topological direct summand of Q^I if, and only if, $\mathrm{Ker} g$ is a direct summand of $S^{(I)}$. From this observation the equivalence (a)$\Leftrightarrow$(c) becomes clear.

(b)$\Rightarrow$(c) If $f: Q \to Q^I$ is a continuous homomorphism, then we get (c) by applying (b) to $M=(\mathrm{Im} f)''$ and the canonical homomorphism $\tilde{f}: Q \to M$, bearing in mind 2.2.

(c)$\Rightarrow$(b) Let $Q \xrightarrow{f} M$ be a Q-epimorphism, with M Q-reflexive. Since M is Q-copresented (1.4), it can be viewed as a Q-closed topological submodule of some Q^I. If we apply (c) to the composition $Q \xrightarrow{f} M \xrightarrow{j} Q^I$, where j is the inclusion, then we get (b).

Remark. The assumption of ${}_RQ$ having (*) property cannot be avoided in order to characterize, by means of injective Q-resolutions, when S is semisimple. For instance if ${}_RQ$ is an infinite dimensional vector space, then every submodule of ${}_RQ$ is a direct summand of it. However S is not semisimple.

In order to study the weak dimension of S we must study which topological left R-modules have a flat dual. To do this, we give the following proposition that leans upon the well-known result of D. Lazard [3, Théorème 1.2] stating that a module X is flat if, and only if, any homomorphism from a finitely presented module to X factors through a free module of finite rank.

2.6. Proposition. *Let ${}_RM$ be a topological module. The following assertions are equivalent:*

(a) *M^*_S is flat.*

(b) *Given a continuous homomorphism $f: M \to N$, with N finitely Q-copresented, there exist, for some $k \geqq 1$, continuous homomorphisms $u: M \to Q^k$ and $v: Q^k \to N$ such that $u \cdot v = f$.*

PROOF. (a)$\Rightarrow$(b) Let $f: M \to N$ be a continuous homomorphism and $0 \to N \xrightarrow{j} Q^n \xrightarrow{g} Q^m$ be a finite Q-copresentation. By applying the functor $-^*$, we get a homomorphism $f^*: N^* \to M^*$. Let $p: Q^{n*} = S^n \to X = S^n/\mathrm{Im}\, g^*$ be the canonical projection. There exists a homomorphism $h: X \to N^*$ such that $j^* = h \cdot p$. Since M^*_S is flat, there exist homomorphisms $u: X \to S^k$ and $v: S^k \to M^*$ such that $v \cdot u = f^* \cdot h$. Now, we have the following commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & N \arrow[r, "j"] \arrow[d, "\sigma_N \cdot h^*"] & Q^n \arrow[r, "g"] \arrow[d, "\cong"] & Q^m \arrow[d, "\cong"] \\
0 \arrow[r] & X^* \arrow[r, "p^*"] & Q^{n**} \arrow[r, "g^{**}"] & Q^{m**}
\end{tikzcd}$$

Thus $\sigma_N \cdot h^*$ is a topological isomorphism. Finally, $f \cdot \sigma_N \cdot h^* = \sigma_M \cdot f^{**} \cdot h^* = \sigma_M \cdot v^* \cdot u^*$ and hence f factors through Q^k.

(b)$\Rightarrow$(a) Let $f: X \to M^*$ be a homomorphism and $S^n \xrightarrow{g} S^m \xrightarrow{p} X \to 0$ a finite presentation of X_S. Then, by applying the functor $-^*$, we get that ${}_RX^*$ is finitely Q-copresented and there exist continuous homomorphisms $u: M \to Q^k$ and $v: Q^k \to X^*$ such that $\sigma_M \cdot f^* = u \cdot v$. Thus $u^* \cdot v^* \cdot \sigma_X = \sigma_M^* \cdot f^{**} \cdot \sigma_X = \sigma_M^* \cdot \sigma_{M^*} \cdot f = f$.

DEFINITION. We will say that ${}_RM \in R$—TMod is Q-coflat when it satisfies condition (b) in the above proposition. A Q-coflat resolution of ${}_RM$ is a sequence of continuous homomorphisms $\boldsymbol{C}: 0 \to M \xrightarrow{f_0} C_0 \xrightarrow{f_1} \cdots$ where C_n is Q-coflat for every $n \geqq 0$, f_0 is a topological monomorphism with $\mathrm{Im}\, f_0 = \mathrm{Ker}\, f_1$ and the sequence $C_0 \to C_1 \to \cdots \to C_n \to \cdots$ is Q-exact.

2.7. LEMMA. *Let $f: M \to N$ be a continuous homomorphism of topological left R-modules and assume that, for every $\alpha \in M^*$ such that* $\mathrm{Ker}\, f \subseteqq \mathrm{Ker}\, \alpha$, *there exists $\beta \in N^*$ so that $f \cdot \beta = \alpha$. Then f induces a canonical isomorphism $\bar{f}: N^*/(\mathrm{Im}\, f)' \to (M/\mathrm{Ker}\, f)^*$.*

PROOF. Note that $\mathrm{Ker}\, f^* = (\mathrm{Im}\, f)'$. We shall prove that $\mathrm{Im}\, f^* = \mathrm{Im}\, p^*$, where

$p: M \to M/\mathrm{Ker}f$ is the canonical projection. This will end the proof. If $\alpha \in \mathrm{Im}f^*$, then $\alpha = f \cdot \beta$, for some $\beta \in N^*$, and thus $(\mathrm{Ker}f)\alpha = 0$ and there exists a unique homomorphism $\gamma: M/\mathrm{Ker}f \to Q$ such that $p \cdot \gamma = \alpha$ and is necessarily continuous. Conversely, if $\alpha = p^*(\gamma) = p \cdot \gamma$, for some $\gamma \in (M/\mathrm{Ker}f)^*$, then $\mathrm{Ker}f \subseteq \mathrm{Ker}\,\alpha$ and the hypothesis implies that $\alpha \in \mathrm{Im}f$.

2.8. LEMMA. *Let $Y \leqq X_S \in \mathrm{Mod}-S$. There exists a topological isomorphism between $(X/Y)^*$ and $Y' = l_{X^*}(Y)$.*

PROOF. The canonical short exact sequence $0 \to Y \xrightarrow{j} X \xrightarrow{p} X/Y \to 0$ induces an exact sequence of continuous homomorphisms $0 \to (X/Y)^* \xrightarrow{p^*} X^* \xrightarrow{j^*} Y^*$ and p^* is a topological monomorphism. Thus p^* induces a topological isomorphism between $(X/Y)^*$ and $\mathrm{Ker}j^* = Y'$.

2.9. THEOREM. *Let ${}_RQ$ be a self-slender $\mathrm{C}_Q-\Pi$-quasi-injective topological module which satisfies (*) property. Then the following assertions are equivalent for a natural number n:*

(a) $\mathrm{wD}(S) \leqq n$.

(b) *For every Q-coflat resolution $0 \to M \xrightarrow{f_0} C_0 \xrightarrow{f_1} \cdots$, with $C_k \in \mathrm{Ref}({}_RQ)$ for any $k \in \boldsymbol{N}$, $\mathrm{Ker}f_{n+1}$ is Q-coflat.*

(c) *For every $M \in \mathrm{Copres}({}_RQ)$ (resp. $M \in \mathrm{C}_Q({}_RQ)$) there exists a Q-coflat resolution of the form*

$$0 \longrightarrow M \xrightarrow{f_0} Q^{I_0} \xrightarrow{f_1} Q^{I_1} \longrightarrow \cdots \xrightarrow{f_{n-1}} Q^{I_{n-1}} \xrightarrow{f_n} C_n \longrightarrow 0$$

with $C_n \in \mathrm{Ref}({}_RQ)$.

(d) *For every $M \in \mathrm{Copres}({}_RQ)$ (resp. $M \in \mathrm{C}_Q({}_RQ)$), there exists a Q-coflat resolution $\boldsymbol{C}: 0 \to M \to C_0 \to \cdots$ with $C_k \in \mathrm{Ref}({}_RQ)$ for any $k \in \boldsymbol{N}$ and $f_k = 0$ for any $k > n$.*

PROOF. (a) $\Rightarrow$ (b) Let $\boldsymbol{C}: 0 \to M \xrightarrow{f_0} C_0 \xrightarrow{f_1} \cdots$ be a Q-coflat resolution with $C_k \in \mathrm{Ref}({}_RQ)$ for any $k \in \boldsymbol{N}$. Then its dual sequence $\boldsymbol{F} = \boldsymbol{C}^*: \cdots \xrightarrow{g_1} F_0 \xrightarrow{g_0} M^* \to 0$ is a flat resolution of M^*_S (2.3 and 2.6) and thus $\mathrm{Ker}g_{n-1}$ is flat. But $\mathrm{Ker}g_{n-1} = (\mathrm{Im}f_{n-1})' = (\mathrm{Im}f_{n-1})'' \cong (C_{n-1}/(\mathrm{Im}f_{n-1})'')^* = (C_{n-1}/\mathrm{Ker}f_n)^*$ and, therefore, $(C_{n-1}/\mathrm{Ker}f_n)^{**}$ is Q-coflat (2.6). But $(\mathrm{Im}f_n)'' \cong (C_n^*/(\mathrm{Im}f_n)')^* \cong (C_{n-1}/\mathrm{Ker}f_n)^{**}$ (2.7).

(b) $\Rightarrow$ (c) can be proved as (b) $\Rightarrow$ (c) in 2.4, when we realize that, for every set I, Q^I is Q-coflat (2.6).

(c) $\Rightarrow$ (d) is trivial.

(d) $\Rightarrow$ (a) is a direct consequence of 2.6 and 2.3.

REMARK. (1) For $n=0$, the foregoing theorem gives a characterization of when S is regular von Neumann. However, the hypothesis may be weakened. For instance, self-slenderness of ${}_RQ$ may be avoided and the other two conditions can be replaced by the condition that $-^*$ preserves exact sequences of the type $0 \to M \xrightarrow{j} Q^n \to Q^m$, with j a topological monomorphism. This is due to the fact that S is regular von Neumann if, and only if, every finitely presented right S-module is flat.

(2) For $n=1$, the hypothesis of ${}_RQ$ being self-slender and having (*) property can be deleted and the condition of ${}_RQ$ being $C_Q-\Pi$-quasi-injective can be changed for the weaker condition that Q is C_Q-Q^n-injective for every $n\geqq 1$. Under this condition, every finitely generated right ideal of S is Q-reflexive (It can be proved in a manner similar to that of (1)$\Rightarrow$(2) in Theorem 1.7). Now, $\mathrm{wD}(S)\leqq 1$ if, and only if, for every continuous homomorphism $f: Q \to Q^n$, $(\mathrm{Im} f)''$ is Q-coflat. This is due to the fact that $\mathrm{wD}(S)\leqq 1$ if, and only if, every finitely generated right ideal is flat.

The results obtained in this section may be used to get, by means of the Pontryagin Duality, corresponding results on the endomorphism ring of a module P_R that satisfies suitable conditions as, for instance, those of [12].

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