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ASYMPTOTIC BEHAVIOR OF SOLUTIONS TO THE KLEIN-GORDON EQUATION WITH A NONLINEAR DISSIPATIVE TERM

Dedicated to Professor Mutsuhide Matsumura on his sixtieth birthday

By

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Abstract. We study the asymptotic behavior of the Klein-Gordon equation with a nonlinear dissipative term $|\partial_t w(t)|^{p-1}\partial_t w(t)$ $(p>1)$ in $x\in \boldsymbol{R}^n$ $(n\geqq 1)$ and $t\geqq 0$. We prove that the energy of solutions does not converge to 0 as $t\to\infty$ for $p>1+2/n$ if Cauchy data are suffciently small. We also prove that solutions of the above equation converge to suitable solutions of the linear Klein-Gordon equation in the energy space as $t\to\infty$ for $p>1+4/n$ if $1\leqq n\leqq 6$ and $1+4/n<p<n/(n-6)$ if $n\geqq 7$.

Key Words. Klein-Gordon equation, nonlinear dissipative term, asymptotic behavior.

1. Introduction and Results

We consider the Cauchy problem for the nonlinear Klein-Gordon equation;

$$\left\{\begin{array}{l}\partial_t^2 w(t)-\Delta w(t)+w(t)+f(\partial_t w(t))=0\\ w(0)=\phi\,,\qquad \partial_t w(0)=\psi\,,\end{array}\right. \tag{1.1}$$

where $x\in \boldsymbol{R}^n$, $t\in \boldsymbol{R}^+=[0,\infty)$, $f(u)=|u|^{p-1}u$ and Δ is the n-dimensional Laplacian. The asymptotic behavior of solutions of (1.1) was considered in Nakao [9]. He showed that

$$\|W(t)\|_e\leqq C\|\Phi\|_e(1+t)^{-(2-n(p-1))/(p-1)} \tag{1.2}$$

for $1<p<1+2/n$ and

$$\|W(t)\|_e\leqq C\|\Phi\|_e(\log(1+t))^{-2/(p-1)} \tag{1.3}$$

for $p=1+2/n$. Here $W(t)=\begin{pmatrix} w(t)\\ \partial_t w(t)\end{pmatrix}$, $\Phi=\begin{pmatrix}\phi\\ \psi\end{pmatrix}$ and $\|\cdot\|_e$ is the energy norm

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defined by

$$\|W(t)\|_e^2=\frac{1}{2}\{\|Hw(t)\|_2^2+\|\partial_t w(t)\|_2^2\},$$

where $\|\cdot\|_2$ is $L_2(\boldsymbol{R}^n)$-norm and H is the positive selfadjoint operator $\sqrt{-\Delta+1}$ in L_2. Our aim of this paper is to investigate how the energy of solutions of (1.1) behaves as $t\to\infty$ in the case $p>1+2/n$.

In order to state our results, we give the main notations used in this paper. We denote by $\|\cdot\|_q$ the norm in $L_q=L_q(\boldsymbol{R}^n)$. Let $H_q^s=H_q^s(\boldsymbol{R}^n)$ with $s\in\boldsymbol{R}$ and $1\leqq q<\infty$ be the Sobolev spaces which are the completion of $C_0^\infty(\boldsymbol{R}^n)$ with norms

$$\|u\|_{s,q}=\|\mathcal{F}^{-1}(1+|\xi|^2)^{s/2}\hat{u}(\xi))\|_q.$$

Here ^ denotes the Fourier transformation and $\mathcal{F}^{-1}$ is its inverse. Especially we denote by H^s the usual Sobolev spaces. We note that $H^s=H_2^s$. For any interval $I\subset\boldsymbol{R}$ and any Banach space B, we denote by $C^k(I;B)$ the space of B-valued C^k-functions over I, and by $C_w(I;B)$ the space of weakly continuous functions from I to B, and by $C_L(I;B)$ the space of functions from I to B that are strongly Lipschitz continuous. For any q, $1\leqq q\leqq\infty$, we denote by $L_q(I;B)$ the space of B-valued L_q-functions on I.

We define an inner product in the energy space $H^1\times L_2$ by

$$\left\langle\begin{pmatrix}u_1\\u_2\end{pmatrix},\begin{pmatrix}v_1\\v_2\end{pmatrix}\right\rangle_e=\frac{1}{2}\{\langle Hu_1, Hv_1\rangle+\langle u_2, v_2\rangle\},$$

where $\langle\,,\rangle$ is $L_2(\boldsymbol{R}^n)$-inner product. We note that $\|W(t)\|_e^2=\langle W(t), W(t)\rangle_e$.

We shall use the operator $\zeta(H)$ for suitable functions $\zeta(\cdot)$ as follows:

$$\zeta(H)u=\mathcal{F}^{-1}\langle\zeta(\langle\xi\rangle)\hat{u}(\xi))\qquad\text{in } \mathcal{S}',$$

where $\langle\xi\rangle=(1+|\xi|^2)^{1/2}$ and $\mathcal{S}'$ means the tempered distribution. We denote by $\{U(t)\}$ $(t\in\boldsymbol{R})$ an unitary group in $H^1\times L_2$ defined by

$$U(t)=\begin{pmatrix}\cos\{Ht\} & H^{-1}\sin\{Ht\}\\ -H\sin\{Ht\} & \cos\{Ht\}\end{pmatrix}.$$

First we state a result of existence and uniqueness.

THEOREM 1. *Let* $n\geqq1$ *and* $p>1$. *Assume that* $\Phi=\begin{pmatrix}\phi\\\psi\end{pmatrix}\in H^2\times H^1\cap L_{2p}$. *Then there exists a unique solution* $w(t)$ *of* (1.1) *which satisfies the following:*

$$w(t)\in L_\infty(\boldsymbol{R}^+;H^2)\cap C(\boldsymbol{R}^+;H^2)\cap C^1(\boldsymbol{R}^+;H^1)\cap C^2(\boldsymbol{R}^+;L_2),\tag{1.4}$$

$$\partial_t w(t)\in L_\infty(\boldsymbol{R}^+;H^1)\cap L_{p+1}(\boldsymbol{R}^+\times\boldsymbol{R}^n)\cap L_\infty(\boldsymbol{R}^+;L_{2p}),\tag{1.5}$$

$$\partial_t^2 w(t)\in L_\infty(\boldsymbol{R}^+;L_2).\tag{1.6}$$

And the following energy equality and inequalities hold:

(1.7) $$\|W(t)\|_e^2+\int_0^t\|\partial_t w(\tau)\|_{p+1}^{p+1}d\tau=\|\Phi\|_e^2,$$

(1.8) $$\|HW(t)\|_e^2+\int_0^t\langle|\partial_t w(\tau)|^{p-1},\ p|\nabla\partial_t w(\tau)|^2+|\partial_t w(\tau)|^2\rangle d\tau\leqq\|H\Phi\|_e^2,$$

(1.9) $$\|\partial_t W(t)\|_e^2+p\int_0^t\langle|\partial_t w(\tau)|^{p-1},\ |\partial_t^2 w(\tau)|^2\rangle d\tau\leqq\left\|\begin{pmatrix}\psi\\ -H^2\phi-f(\psi)\end{pmatrix}\right\|_e^2$$

for $t\in\boldsymbol{R}^+$, *where* $HW(t)=\begin{pmatrix}Hw(t)\\ H\partial_t w(t)\end{pmatrix}$, $\partial_t W(t)=\begin{pmatrix}\partial_t w(t)\\ \partial_t^2 w(t)\end{pmatrix}$ *and* $H\Phi=\begin{pmatrix}H\phi\\ H\psi\end{pmatrix}$.

The main results can be stated as follows:

Theorem 2. *Let* $w(t)$ *be a solution of* (1.1) *with Cauchy data* $\Phi=\begin{pmatrix}\phi\\ \psi\end{pmatrix}\in H^2\cap H_1^{s+1}\times H^1\cap H_1^s\cap L_{2p}$, *where* $s>n/2$. *Suppose that* $p>1+2/n$ $(n\geqq 1)$ *and* $\|\Phi\|_e\neq 0$. *Then there exists a* $\delta>0$ *such that if* $\|\phi\|_{s+1,1}+\|\psi\|_{s,1}\leqq\delta$, *then* $\|W(t)\|_e$ *does not converge to* 0 *as* $t\to\infty$.

Theorem 3. *Let* $w(t)$ *be a solution of* (1.1) *with Cauchy data* $\Phi=\begin{pmatrix}\phi\\ \psi\end{pmatrix}\in H^2\times H^1\cap L_{2p}$.

(i) *Suppose that* $p>1+2/n$ $(n\geqq 1)$. *Then there exists* $\Phi^+=\begin{pmatrix}\phi^+\\ \psi^+\end{pmatrix}\in H^2\times H^1$ *such that*

(1.10) $$U(-t)W(t)-\Phi^+\longrightarrow 0 \quad \text{weakly in } H^2\times H^1 \text{ as } t\to\infty.$$

(ii) *Suppose that* $1+4/n<p<\infty$ *if* $1\leqq n\leqq 6$ *and* $1+4/n<p<n/(n-6)$ *if* $n\geqq 7$. *Then the above* Φ^+ *satisfies*

(1.11) $$\|W(t)-U(t)\Phi^+\|_e\longrightarrow 0 \quad \text{as } t\to\infty.$$

The theory of monotone operators provides the existence of a global solution. Uniqueness, energy equality and inequalites are obtained by standard methods. So we may give a sketch of proof of Theorem 1. The energy decay properties of the linear wave equations with a dissipative term are investigated by Mochizuki [7, 8] and Matsumura [6]. For the proof of Theorem 2 we use the same energy method used in Mochizuki [7, 8]. In order to prove Theorem 3, the Strichartz estimate (See Proposition 4.1.) and the energy inequality (1.8) play an important role.

2. Proof of Theorem 1

Since $f'(u)>0$ and $f(u)u\geqq 0$, the theory of monotone operators provides a unique solution of (1.1). Noting that $\Phi=\begin{pmatrix}\phi\\ \psi\end{pmatrix}\in H^2\times H^1\cap L_{2p}$ implies $\|\Phi\|_e^2$, $\|H\phi\|_e^2$ and $\left\|\begin{pmatrix}\psi\\ -H^2\phi-f(\psi)\end{pmatrix}\right\|_e$ are finite, (1.7), (1.8) and (1.9) are obtained by standard methods. So there exist a solution $w(t)$ of (1.1) as follows:

$$w(t)\in L_\infty(\boldsymbol{R}^+;H^2)\cap C_w(\boldsymbol{R}^+;H^2)\cap C_L(\boldsymbol{R}^+;H^1), \tag{2.1}$$

$$\partial_t w(t)\in L_\infty(\boldsymbol{R}^+;H^1)\cap C_w(\boldsymbol{R}^+;H^1)\cap C_L(\boldsymbol{R}^+;L_2)\cap L_{p+1}(\boldsymbol{R}^+\times\boldsymbol{R}^n), \tag{2.2}$$

$$\partial_t^2 w(t)\in L_\infty(\boldsymbol{R}^+;L_2)\cap C_w(\boldsymbol{R}^+;L_2). \tag{2.3}$$

Since $\|f(\partial_t w(t))\|_2=\|\partial_t^2 w(t)-\Delta w(t)+w(t)\|_2$, we have $\partial_t w(t)\in L_\infty(\boldsymbol{R}^+;L_{2p})$ by (2.1) and (2.3).

Employing the same arguments as in Kato [4], Shibata [10] and Shibata and Kikuchi [11], we can obtain

$$w(t)\in C(\boldsymbol{R}^+;H^2)\cap C^1(\boldsymbol{R}^+;H^1)\cap C^2(\boldsymbol{R}^+;L_2). \tag{2.4}$$

Thus Theorem 1 is proved.

3. Proof of Theorem 2

We note that $W(t)=\begin{pmatrix}w(t)\\ \partial_t w(t)\end{pmatrix}$ satisfies

$$W(t)=U(t)\Phi-\int_0^t U(t-\tau)F(\partial_t w(\tau))d\tau, \tag{3.1}$$

where $F(u)=\begin{pmatrix}0\\ f(u)\end{pmatrix}$. Since $U(t)$ is an unitary operator on $H^1\times L_2$, we have

$$\begin{aligned}\langle W(t),U(t)\Phi\rangle_e&=\langle U(t)\Phi,U(t)\Phi\rangle_e-\int_0^t\langle U(t-\tau)F(\partial_t w(\tau)),U(t)\Phi\rangle_e d\tau\\ &=\|\Phi\|_e^2-\int_0^t\langle F(\partial_t w(\tau)),U(\tau)\Phi\rangle_e d\tau\\ &=\|\Phi\|_e^2-\frac{1}{2}\int_0^t\langle f(\partial_t w(\tau)),\partial_t w^0(\tau)\rangle d\tau,\end{aligned} \tag{3.2}$$

Here $w^0(t)$ is a solution of the linear Klein-Gordon equation;

$$\begin{cases}\partial_t^2 w^0(t)-\Delta w^0(t)+w^0(t)=0\\ w^0(0)=\phi, \qquad \partial_t w^0(0)=\psi.\end{cases} \tag{3.3}$$

By the Schwarz inequality we obtain

$$\|\Phi\|_e^2 \leqq \|W(t)\|_e \|U(t)\Phi\|_e + \frac{1}{2}\int_0^t |\langle f(\partial_t w(\tau)),\ \partial_t w^0(\tau)\rangle|\, d\tau \tag{3.4}$$

$$= \|W(t)\|_e \|\Phi\|_e + \frac{1}{2}\int_0^t |\langle f(\partial_t w(\tau)),\ \partial_t w^0(\tau)\rangle|\, d\tau$$

$$= I_1 + I_2 .$$

It follows from Hölder's inequality that

$$I_2 \leqq \frac{1}{2}\left\{\int_0^t \int_{R^n} |\partial_t w(\tau)|^{p+1} dx d\tau\right\}^{p/(p+1)} \left\{\int_0^t \int_{R^n} |\partial_t w^0(\tau)|^{p+1} dx d\tau\right\}^{1/(p+1)} . \tag{3.5}$$

We recall the well-known estimate

$$\|w^0(t)\|_\infty \leqq C(1+t)^{-n/2}(\|\phi\|_{s,1} + \|\psi\|_{s-1,1}), \tag{3.6}$$

where $s>n/2$ and $w^0(t)$ is a solution of (3.3). (See Brenner [2] Appendix 2, Bergh and Löfström [1] Theorem 6.2.4 and Brenner, Thomée and Wablbin [3] Theorem 2.1.) By (1.7) and $\|\partial_t w^0(t)\|_2^2 \leqq 2\|\Phi\|_e^2$ we have

$$I_2 \leqq C(\|\phi\|_{s+1,1} + \|\psi\|_{s,1})^{(p-1)/(p+1)} \|\Phi\|_e^{2p/(p+1)} \tag{3.7}$$

$$\times \left\{\int_0^t (1+\tau)^{-n(p-1)/2} \|\partial_t w^0(\tau)\|_2^2 d\tau\right\}^{1/(p+1)}$$

$$\leqq C(\|\phi\|_{s+1,1} + \|\psi\|_{s,1})^{(p-1)/(p+1)} \|\Phi\|_e^2 \left\{\int_0^t (1+\tau)^{-n(p-1)/2} d\tau\right\}^{1/(p+1)} .$$

Since $p>1+2/n$ implies $-n(p-1)/2<-1$, there exists a $\delta>0$ such that

$$C\delta^{(p-1)/(p+1)} \left\{\int_0^\infty (1+\tau)^{-n(p-1)/2} d\tau\right\}^{1/(p+1)} < \frac{1}{2} . \tag{3.8}$$

Then it follows from (3.4), (3.7) and (3.8) that

$$\|\phi\|_e^2 \leqq \|W(t)\|_e \|\Phi\|_e + \frac{1}{2}\|\Phi\|_e^2 \tag{3.9}$$

if $\|\phi\|_{s+1,1} + \|\psi\|_{s,1} \leqq \delta$. Noting that $\|\Phi\|_e \neq 0$, we have $\frac{1}{2}\|\Phi\|_e^2 \leqq \|W(t)\|_e$ for any $t \in \boldsymbol{R}^+$. Therefore Theorem 2 is proved.

4. Proof of Theorem 3

We begin with the Strichartz estimate for solutions of the linear Klein-Gordon equation.

Proposition 4.1. *Let* $q \geqq 2$, $r \geqq 2$ *and*

$$\frac{1}{2} - \frac{1}{n} - \frac{1}{nr} < \frac{1}{q} < \frac{1}{2} - \frac{2}{nr} . \tag{4.1}$$

Then we have

$$\|w^0\|_{L_r(R;\,L_q(R^n))}\leqq C\|\Phi\|_e, \tag{4.2}$$

where $\Phi=\begin{pmatrix}\phi\\ \psi\end{pmatrix}$ *and* $w^0(t)$ *is a solution of* (3.3).

See Marshall [5] for a proof.

Using this proposition, we obtain the following

Lemma 4.2. *Under the same assumptions of Proposition* 4.1 *we have*

$$\left\|\int_t^\infty U(-\tau)F(u(\tau))d\tau\right\|_e\leqq C\||u|^{(p-1)/1}(|\nabla u|+|u|)\|_{L_2([t,\infty)\times R^n)} \times\|u\|^{(p-1)/2}_{L_{r(p-1)/(r-2)}([t,\infty);\,L_{q(p-1)/(q-2)}(R^n))} \tag{4.3}$$

for suitable functions u, *where* $F(u)=\begin{pmatrix}0\\ f(u)\end{pmatrix}$.

Proof. For any $V=\begin{pmatrix}v_1\\ v_2\end{pmatrix}\in C_0^\infty(\boldsymbol{R}^n)\times C_0^\infty(\boldsymbol{R}^n)$ we have

$$\begin{aligned}\left\langle\int_t^\infty U(-\tau)F(u(\tau))d\tau,\ V\right\rangle_e&=\int_t^\infty\langle U(-\tau)F(u(\tau))d\tau,\ V\rangle_e d\tau\\ &=\int_t^\infty\langle F(u(\tau)),\ U(\tau)V\rangle_e d\tau\\ &=\frac{1}{2}\int_t^\infty\langle Hf(u(\tau)),\ H^{-1}\partial_t v(\tau)\rangle d\tau,\end{aligned} \tag{4.4}$$

where $v(t)=\cos\{Ht\}v_1+H^{-1}\sin\{Ht\}v_2$. Recalling that $H=\sqrt{-\Delta+1}$, by Hölder's inequality we have

$$\begin{aligned}&\left|\int_t^\infty\langle Hf(u(\tau)),\ H^{-1}\partial_t v(\tau)\rangle d\tau\right|\\ &\leqq C\int_t^\infty\int_{R^n}|u|^{(p-1)/2}(|\nabla u|+|u|)|u|^{(p-1)/2}|H^{-1}\partial_t v(\tau)|dxd\tau\\ &\leqq C\||u|^{(p-1)/2}(|\nabla u|+|u|)\|_{L_2([t,\infty)\times R^n)}\\ &\quad\times\|u\|^{(p-1)/2}_{L_{r(p-1)/(r-2)}([t,\infty);\,L_{q(p-1)/(q-2)})}\|H^{-1}\partial_t v\|_{L_r([t,\infty);\,L_q(R^n))},\end{aligned} \tag{4.5}$$

where $q\geqq 2$ and $r\geqq 2$. Since $H^{-1}\partial_t v(t)$ is a solution of (3.3) with Cauchy data $\begin{pmatrix}H^{-1}v_2\\ -Hv_1\end{pmatrix}$, it follows from Proposition 4.1, (4.4) and (4.5) that

(4.6)
$$\left|\left\langle\int_t^\infty U(-\tau)F(u(\tau))d\tau,\ V\right\rangle_e\right|$$
$$\leqq C\||u|^{(p-1)/2}(|\nabla u|+|u|)\|_{L_2([t,\infty)\times R^n)}$$
$$\times\|u\|^{(p-1)/2}_{L_{r(p-1)/(r-2)}([t,\infty);\,L_{q(p-1)/(q-2)}(R^n))}\|V\|_e\,.$$

Thus by the duality argument we obtain (4.3). Q.E.D.

Proof of (i). We note that $W(t)$ satisfies

(4.7)
$$W(t)=U(t)\Phi-\int_0^t U(t-\tau)F(\partial_t w(\tau))\,d\tau\,,$$

and then

(4.8)
$$U(-t)W(t)=\Phi-\int_0^t U(-\tau)F(\partial_t w(\tau))d\tau$$
$$=\Phi^+ +\int_t^\infty U(-\tau)F(\partial_t w(\tau))d\tau\,,$$

where

(4.9)
$$\Phi^+=\Phi-\int_0^\infty U(-\tau)F(\partial_t w(\tau))d\tau\,.$$

For $V=\begin{pmatrix}v_1\\ v_2\end{pmatrix}\in C_0^\infty(R^n)\times C_0^\infty(R^n)$ we have

(4.10)
$$\langle U(-t)W(t)-\Phi^+,\ V\rangle_e=\int_0^\infty\langle U(-\tau)F(\partial_t w(\tau)),\ V\rangle_e d\tau$$
$$=\int_t^\infty\langle F(\partial_t w(\tau)),\ U(\tau)V\rangle_e d\tau$$
$$=\frac{1}{2}\int_t^\infty\langle f(\partial_t w(\tau)),\ \partial_t v(\tau)\rangle d\tau\,,$$

where $v(t)=\cos\{Ht\}v_1+H^{-1}\sin\{Ht\}v_2$. In the same way as in obtaining (3.5) and (3.7), it holds that for $p>1+2/n$

(4.11)
$$|\langle U(-t)W(t)-\phi^+,\ V\rangle_e|\longrightarrow 0\qquad \text{as } t\to\infty\,.$$

Since $U(t)$ is an unitary operator on $H^1\times L_2$, it follows from (1.8) that $\{U(-t)W(t)\}$ is uniformly bounded on t in $H^2\times H^1$. Therefore we have

(4.12)
$$U(-t)W(t)\longrightarrow\Phi^+\qquad \text{weakly in } H^2\times H^1 \text{ as } t\to\infty$$

and $\Phi^+\in H^2\times H^1$.

Proof of (ii). If $1/r$ and $1/q$ satisfy (4.1), by (4.8) and Lemma 4.2 we have

$$\|W(t)-U(t)\Phi^+\|_e=\|U(-t)W(t)-\Phi^+\|_e \tag{4.13}$$

$$\leqq\left\|\int_t^\infty U(-\tau)F(\partial_t w(\tau))d\tau\right\|_e$$

$$\leqq C\|\,|\partial_t w|^{(p-1)/2}(|\nabla\partial_t w|+|\partial_t w|)\|_{L_2([t,\infty)\times R^n)}$$

$$\times\|\partial_t w\|^{(p-1)/2}_{L_{r(p-1)/(r-2)}([t,\infty);\,L_{q(p-1)/(q-2)}(R^n))}\,.$$

On the other hand by (1.8) and (1.5) we have

$$|\partial_t w|^{(p-1)/2}(|\nabla\partial_t w|+|\partial_t w|)\in L_2(\boldsymbol{R}^+\times\boldsymbol{R}^n)$$

and $\partial_t w\in L_{r'}(\boldsymbol{R}^+;\,L_{q'}(\boldsymbol{R}^n))$, where $1/r'=\theta/p+1$ and $1/q'=(\theta/p+1)+(1-\theta)/2p$ $(0\leqq\theta\leqq1)$. Thus if

$$\frac{r-2}{r(p-1)}=\frac{1}{r'},\qquad \frac{q-2}{q(p-1)}=\frac{1}{q'}, \tag{4.14}$$

it follows from (4.13) that

$$\|W(t)-U(t)\phi^+\|_e\longrightarrow 0\qquad\text{as } t\to\infty\,. \tag{4.15}$$

(4.14) implies that

$$\frac{1}{r}=\frac{1}{2}-\frac{(p-1)\theta}{2(p+1)},\qquad \frac{1}{q}=\frac{1}{2}-\frac{(p-1)\theta}{2(p+1)}-\frac{(p-1)(1-\theta)}{4p}. \tag{4.16}$$

Substituting (4.16) for (4.1), we have

$$\{(n+2)\theta+(n-6)\}p^2-2\{(n+1)\theta+3\}p-n(1-\theta)<0\,, \tag{4.17}$$

$$\{(n+4)\theta+(n-4)\}p^2-2\{(n+2)\theta+2\}p-n(1-\theta)>0\,. \tag{4.18}$$

By (4.17) we have

$$1<p<\infty\qquad\text{if } \theta\leqq\frac{6-n}{n+2}, \tag{4.19}$$

$$1<p<\alpha_n(\theta)\qquad\text{if } \theta>\frac{6-n}{n+2}. \tag{4.20}$$

Here $\alpha_n(\theta)$ is a positive solution of

$$\{(n+2)\theta+(n-6)\}p^2-2\{(n+1)\theta+3\}p-n(1-\theta)=0\,.$$

On the other hand by (4.18) we have

$$p>\beta_n(\theta)\qquad\text{if } \theta>\frac{4-n}{n+4}. \tag{4.21}$$

Here $\beta_n(\theta)$ is a positive solution of

$$\{(n+4)\theta+(n-4)\}p^2-2\{(n+2)\theta+2)p-n(1-\theta)=0\,.$$

Noting that

$$\{(n+2)\theta+(n-6)\}p^2-2\{(n+1)\theta+3\}p-n(1-\theta)$$
$$<\{(n+4)\theta+(n-4)\}p^2-2\{(n+2)\theta+2\}p-n(1-\theta)$$

for $p>1$, we see that $\beta_n(\theta)<\alpha_n(\theta)$ for $(6-n)/(n+2)<\theta$.

First we consider the case $1\leqq n\leqq 6$. Since $\alpha_n(\theta)\uparrow\infty$ as $\theta\downarrow(6-n)/(n+2)$, there exists an $\varepsilon>0$ such that

$$\beta_n\left(\frac{6-n}{n+2}\right)<\alpha_n\left(\frac{6-n}{n+2}+\varepsilon\right). \tag{4.22}$$

Since $0\leqq\theta\leqq 1$, it follows from (4.19), (4.20) and (4.21) that

$$\beta_n(\theta)<p<\infty \qquad \text{for } \mathrm{Max}\left\{0,\frac{4-n}{n+4}\right\}<\theta\leqq\frac{6-n}{n+2}, \tag{4.23}$$

$$\beta_n(\theta)<p<\alpha_n(\theta) \qquad \text{for } \frac{6-n}{n+2}<\theta\leqq 1 \tag{4.24}$$

respectively. Noting that $\alpha_n(\theta)$ and $\beta_n(\theta)$ are monotone decreasing functions, we have

$$\beta_n\left(\frac{6-n}{n+2}\right)<p<\infty, \tag{4.25}$$

$$\beta_n(1)<p<\alpha_n\left(\frac{6-n}{n+2}+\varepsilon\right). \tag{4.26}$$

Thus by (4.22), (4.25) and (4.26) we have $1+4/n=\beta_n(1)<p<\infty$ if $1\leqq n\leqq 6$.

Next we consider the case $n\geqq 7$. By (4.20) and (4.21) we have

$$\beta_n(\theta)<p<\alpha_n(\theta) \qquad \text{for } 0\leqq\theta\leqq 1. \tag{4.27}$$

Since $\alpha_n(\theta)$ and $\beta_n(\theta)$ are monotone decreasing functions, we have

$$1+\frac{4}{n}=\beta_n(1)<p<\alpha_n(0)=\frac{n}{n-6}.$$

Thus Theorem 3 is proved.

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