TSUKUBA J. MATH.
Vol. 14 No. 2 (1990), 459—487

A CASE OF EXTENSIONS OF GROUP SCHEMES OVER A DISCRETE VALUATION RING

By

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Introduction.

Let $X \to Y$ be a cyclic covering of degree m of normal varieties over a field k. If m is prime to the characteristic of k and k contains all the m-th roots of unity, the Kummer theory asserts that the covering $X \to Y$ is given by a cartesian square:

$$\begin{array}{ccc} X & \dashrightarrow & \boldsymbol{G}_{m,k} \\ \downarrow & & \downarrow \theta \\ Y & \overset{f}{\dashrightarrow} & \boldsymbol{G}_{m,k}\ , \end{array}$$

where θ is the m-th power map and f is a rational map of Y to the multiplicative group $\boldsymbol{G}_{m,k}$. On the other hand, if $m=p^n$ and $p=\text{char}.k>0$, the Witt-Artin-Schreier theory asserts that the covering $X \to Y$ is given by a cartesian square:

$$\begin{array}{ccc} X & \dashrightarrow & \boldsymbol{W}_{n,k} \\ \downarrow & & \downarrow \mathcal{P} \\ Y & \overset{g}{\dashrightarrow} & \boldsymbol{W}_{n,k}\ , \end{array}$$

where $\mathcal{P}(x)=x^p-x$ and g is a rational map of Y to the Witt group $W_{n,k}$. Therefore, if one wishes to deform a cyclic covering $X \to Y$ of degree p^n over a field k of characteristic $p>0$ to a cyclic covering of degree p^n over a field of characteristic 0, it seems natural to consider the deformations of the Witt-Artin-Schereier exact sequence

$$0 \longrightarrow (\boldsymbol{Z}/p^n)_k \longrightarrow W_{n,k} \overset{\mathcal{P}}{\longrightarrow} W_{n,k} \longrightarrow 0$$

over a field k of characteristic $p>0$ to an exact sequence of Kummer type

$$1 \longrightarrow \boldsymbol{\mu}_{p^n,K} \longrightarrow (\boldsymbol{G}_{m,K})^n \longrightarrow (\boldsymbol{G}_{m,K})^n \longrightarrow 1$$

over a field K of characteristic 0. From this point of view, it seems most

Received September 21, 1989. Revised January 17, 1990.
* Partially supported by Grant-in-Aid for Scientific Research #62540066.

appropriate to consider the deformations of Witt groups to tori as the first step. In the one-dimensional case, the deformations of $\boldsymbol{G}_a$ to $\boldsymbol{G}_m$ are completely determined by [9], and later independently by [3]. In fact, every such deformation is given by a group scheme $\mathcal{G}^{(\lambda)}=\mathrm{Spec}\,A[x, 1/(\lambda x+1)]$ over a discrete valuation ring A with a group law $(x, y)\mapsto\lambda xy+x+y$, where λ is a non-zero element of the maximal ideal of A. If we take $A=\boldsymbol{Z}_p[\zeta]$ with a primitive p-th root ζ of unity, and $\lambda=\zeta-1$, then the exact sequence

$$0 \longrightarrow (\boldsymbol{Z}/p)_A \longrightarrow \mathcal{G}^{(\lambda)} \xrightarrow{\psi} \mathcal{G}^{(\lambda^p)} \longrightarrow 0$$

where ψ is the A-homomorphism defined by $x\mapsto\{(\lambda x+1)^p-1\}/\lambda^p$, gives the unique deformation of the Artin-Schreier sequence to the Kummer sequence. This exact sequence is first noticed by [3] and [4], and later independently by [8]. In [3], the above sequence is used to lift an automorphism of order p of a smooth projective curve over an algebraically closed field of characteristic p to one over a field of characteristic 0.

In the higher dimensional cases, some examples of deformations of Witt groups to tori have been illustrated by [4]. Later [5] has generalized the argument of [4] and has developed a method for computing $\mathrm{Ext}_A^1(\mathcal{G}^{(\lambda)}, \mathcal{G}^{(\mu)})$, the group of extensions of $\mathcal{G}^{(\lambda)}$ by $\mathcal{G}^{(\mu)}$. Furthermore, [5] has explicitly computed $\mathrm{Ext}_A^1(\mathcal{G}^{(\lambda)}, \mathcal{G}^{(\mu)})$ under the condition that $\mu | p$ (cf. Ex. 4.1).

In this paper, we shall compute the group $\mathrm{Ext}_A^1(\mathcal{G}^{(\lambda)}, \mathcal{G}^{(\mu)})$ for arbitrary λ, $\mu\neq 0$ of the maximal ideal $\mathfrak{m}$ of A, developing the argument of [5] and analyzing such an extension by means of successive Néron blow-ups from a torus. Our main result is as follows:

THEOREM (cf. 2.3, Cor. 3.5 and Th. 3.10). *Let A be a discrete valuation ring dominating $\boldsymbol{Z}_{(p)}$. Let λ, μ be non-zero elements of the maximal ideal $\mathfrak{m}$ of A with the order of $\mu=m$. Then every extension $\mathcal{E}$ of $\mathcal{G}^{(\lambda)}$ by $\mathcal{G}^{(\mu)}$ is given by a group S-scheme*

$$\mathcal{E}=\mathrm{Spec}\,A[X_0, X_1, 1/(\lambda X_1+1), 1/(\mu X_1+F(X_0))]$$

with the law of multiplication

$$X_0 \longmapsto \lambda X_0\otimes X_0+X_0\otimes 1+1\otimes X_0,$$

$$X_1 \longmapsto \mu X_1\otimes X_1+X_1\otimes F(X_0)+F(X_0)\otimes X_1$$
$$+\frac{1}{\mu}[F(X_0)\otimes F(X_0)-F(\lambda X_0\otimes X_0+X_0\otimes 1+1\otimes X_0)],$$

where $F(X)=1+\sum_{i\geq 1} c_i X^i$ is a polynomial with $c_i\in\mathfrak{m}$ satisfying the equalities

$$c_j=\frac{1}{\binom{j}{p^r}}\left\{c_{p^r}c_{j-p^r}-\sum_{i=0}^{p^r-1}\binom{j-p^r+i}{j-2p^r+2i}\binom{j-2p^r+2i}{i}c_{j-p^r+i}\lambda^{p^r-i}\right\}$$

for j *with* $\mathrm{ord}_p j=r$ *and* $j\neq p^r$, *and*

$$c_{p^r}c_{p^{r+1}-p^r}\equiv\sum_{i=0}^{p^r}\binom{p^{r+1}-p^r+i}{p^{r+1}-2p^r+2i}\binom{p^{r+1}-2p^r+2i}{i}c_{p^{r+1}-p^r+i}\lambda^{p^r-i}\bmod \mathfrak{m}^m$$

for each $r\geqq 0$.

It will be noted that our method is applicable also to the case $\lambda=0, 1$ or $\mu=0, 1$. In particular, we recover the work of Weisfeiler [10] when $\lambda=0$ and μ is a non-zero element of $\mathfrak{m}$.

We now explain briefly the plan of this paper. In § 1, some general facts are discussed, concerning the Néron blow-ups. In § 2, we analyze an extension of $\mathcal{G}^{(\lambda)}$ by $\mathcal{G}^{(\mu)}$ by means of successive Néron blow-ups starting from a torus. Our main theorem is proven in § 3; after establishing an analogue of Lazard's comparison lemma [2], we determine step by step the polynomials $F(X)$, satisfying the condition $F(X)F(Y)=F(\lambda XY+X+Y)$ mod. μ. Some examples concerning the extensions are given in § 4. We conclude this article by noting that a smooth affine 2-dimensional S-group scheme is not necessarily obtained by an extension of smooth 1-dimensional S-group schemes, even though its generic fibre and its special fibre are extensions of smooth 1-dimensional group schemes each.

Notation.

Throughout the article, A denotes a discrete valuation ring and $\mathfrak{m}$ (resp. K, k) denotes the maximal ideal (resp. the fraction field, resp. the residue field) of A, if there are no restrictions. We denote by v the valuation on A and by π a uniformizing parameter of A. We put $S=\mathrm{Spec}\,A$.

An S-group (resp. an S-homomorphism) means a group S-scheme of finite type (resp. an S-morphism between group S-schemes, compatible with the group structures).

For an S-group G, we denote by G_K(resp. G_k) the generic (resp. closed) fibre of G over S. Moreover, when G is affine, we denote by $A[G]$ (resp. $K[G]$, resp. $k[G]$) the coordinate ring of G (resp. G_K, resp. G_k) and by $A[G]^+$ (resp. $K[G]^+$, resp. $k[G]^+$) the augmentation ideal of $A[G]$ (resp. $K[G]$, resp. $k[G]$).

For non-negative integers n, l with $n\geqq l$, we denote by $\binom{n}{l}$ the number $\frac{n!}{(n-l)!\,l!}$. In particular $\binom{0}{0}=1$.

Contents.

1. Nérons blow-ups

We recall first Néron blow-ups. For details, see [1], [9].

1.1. Let G be a flat affine S-group and H a closed k-subgroup of G_k. Let $J(H)$ be the inverse image in $A[G]$ of the defining ideal of H in $k[G]$. Then the structure of Hopf algebra on $K[G]$ induces a structure of Hopf A-algebra on the A-subalgebra $A[\pi^{-1}J(H)]$ of $K[G]$. Then $G^H=\mathrm{Spec}\,A[\pi^{-1}J(H)]$ is a flat affine S-group. The injection $A[G]\subset A[G^H]=A[\pi^{-1}J(H)]$ induces an S-homomorphism $G^H\to G$. By the definition, the generic fiber $(G^H)_K\to G_K$ is an isomorphism. We call the S-group G^H or the canonical S-homomorphism $G^H\to G$ the Néron blow-up of H in G.

REMARK 1.2. It is readily seen that $A[G^H]^+=K[G]^+\cap A[G^H]$.

PROPOSITION 1.3. *Let $\varphi: G'\to G$ be an S-homomorphism of flat affine S-groups and let H' (resp. H) be a closed k-subgroup of G'_k (resp. G_k) such that $\varphi_k(H')\subset H$. Then there exists canonically an S-homomorphism $\tilde{\varphi}=\varphi^{(H',H)}: G'^{H'}\to G^H$ such that $\tilde{\varphi}_K=\varphi_K: G'_K\to G_K$.*

PROOF. Let $\alpha: G'^{H'}\to G'$ denote the canonical S-homomorphism. By the assumption, the image of $(\varphi\circ\alpha)_k: (G'^{H'})_k\to G_k$ is contained in H. Therefore, by the universal property of Néron blow-ups ([9], Prop. 1.2), we get a unique homomorphism $\tilde{\varphi}$ which makes the diagram

$$\begin{array}{ccc} G'^{H'} & \xrightarrow{\tilde{\varphi}} & G^H \\ \downarrow & & \downarrow \\ G' & \xrightarrow{\varphi} & G \end{array}$$

commutative.

PROPOSITION 1.4. *Let G be a flat affine S-group, G' a closed flat S-subgroup of G, H a closed k-subgroup of G_k and $H'=H\cap G_k$. Then the canonical homo-*

morphism $\tilde{\varphi}=\varphi^H=\varphi^{(H',H)}: G'^{H'}\to G^H$ *induced by the inclusion* $G'\to G$ *is a closed immersion.*

PROOF. Since $H'=H\cap G_k$, $J(H')$ is generated by $J(H)$ in $A[G']$. Let π, $f_1, \cdots, f_r$ be generators of $J(H)$ and $g_i(1\leqq i\leqq r)$ be the image of f_i in $A[G']$. Then

$$A[G'^{H'}]=A[G'][\pi^{-1}g_1, \cdots, \pi^{-1}g_r],$$

$$A[G^H]=A[G][\pi^{-1}f_1, \cdots, \pi^{-1}f_r].$$

Hence the canonical surjection $K[G]\to K[G']$ induces a surjection $A[G^H]\to A[G'^{H'}]$.

REMARK 1.5. (1) The defining ideal of $G'^{H'}$ in G^H is given by $J(G')_K\cap A[G^H]$.

(2) In general, the square

$$\begin{array}{ccc} G'^{H'} & \xrightarrow{\tilde{\varphi}} & G^H \\ \downarrow & & \downarrow \\ G' & \xrightarrow{\varphi} & G \end{array}$$

is not cartesian.

PROPOSITION 1.6. *Let* G *be a flat affine* S*-group,* H *a closed* k*-subgroup of* G_k *and* $\tilde{G}=G^H$ *the Néron blow-up. Then, by taking the flat closure, we get bijections among the closed* K*-subgroups of* $G_K=\tilde{G}_K$*, the closed flat* S*-subgroups of* G *and the closed flat* S*-subgroups of* $\tilde{G}$.

PROOF. This is a direct consequence of EGA IV, Prop. 2.8.5.

Combining Proposition 1.5 and Proposition 1.6, we obtain the following assertion.

COROLLARY 1.7. *Let* G *be a flat affine* S*-group and* H *a closed* k*-subgroup of* G_k. *Let* G' *be a closed flat* S*-subgroup of* G *and* $\tilde{G}'$ *the flat closure of* G'_K *in* $\tilde{G}=G^H$. *Then* $\tilde{G}'$ *is the Néron blow-up of* $H\cap G'_k$ *in* G'.

PROPOSITION 1.8. *Let* G *be a flat affine* S*-group,* $H_1\supset H$ *closed* k*-subgroups of* G_k *and* $\tilde{H}$ *the inverse image of* H *in* $(G^{H_1})_k$. *Then there exists a canonical isomorphism* $(G^{H_1})^{\tilde{H}}\simeq G^H$.

PROOF. Take generators $\pi, f_1, \cdots f_r, g_1, \cdots, g_s$ of $J(H)$ such that $J(H_1)$ is generated by $\pi, f_1, \cdots, f_r$. Since the square

$$\begin{array}{ccc} \tilde{H} & \longrightarrow & (G^{H_1})_k \\ \downarrow & & \downarrow \\ H & \xrightarrow{\psi} & G_k \end{array}$$

is cartesian, $J(\tilde{H})$ is generated by $J(H)$ in $A[G^{H_1}]$. Since $f_1, \cdots, f_r$ are divisible by π in $A[G^{H_1}]$, $J(\tilde{H})$ is generated by $\pi, g_1, \cdots, g_s$ in $A[G^{H_1}]$. Hence we have

$$\begin{aligned} A[(G^{H_1})^{\tilde{H}}] &= A[G^{H_1}][\pi^{-1}g_1, \cdots, \pi^{-1}g_s] \\ &= A[G][\pi^{-1}f_1, \cdots, \pi^{-1}f_r][\pi^{-1}g_1, \cdots, \pi^{-1}g_s] = A[G^H]. \end{aligned}$$

THEOREM 1.9. *Let*

$$(\#) \qquad 0 \longrightarrow G' \xrightarrow{\varphi} G \xrightarrow{\psi} G'' \longrightarrow 0$$

be an exact sequence of flat affine S-groups, and let H *a closed k-subgroup of* G_k, H' *the inverse image of* H *in* G'_k *and* H'' *the image of* H *in* G''_k. *Then the sequence of S-groups*

$$(\widetilde{\#}) \qquad 0 \longrightarrow G'^{H'} \xrightarrow{\tilde{\varphi}=\varphi^H} G^H \xrightarrow{\tilde{\psi}=\psi^H} G''^{H'} \longrightarrow 0$$

induced from (#) *is exact if one of the following conditions is satisfied*:

(1) $H \supset \varphi(G'_k)$; *that is to say*, $H = (\psi_k)^{-1}(H'')$.

(2) G' *is smooth over* S.

PROOF. Since $(\tilde{\psi}\circ\tilde{\varphi})_K = (\psi\circ\varphi)_K = 0$ and $\tilde{G}'$ is flat over S, $\tilde{\psi}\circ\tilde{\varphi} = 0$. Hence we obtain a canonical S-homomorphism $G^H/G'^{H'} \to G''^{H'}$. Obviously the generic fiber $(G^H/G'^{H'})_K \to (G''^{H'})_K$ is an isomorphism.

We prove that $\tilde{\psi}_k : (G^H)_k \to (G''^{H'})_k$ is faithfully flat under the condition (1) or (2), which implies that $G^H/G'^{H'} \to G''^{H'}$ is an isomorphism ([9], Lemma 1.3).

Case (1). We identify $A[G''] \subset A[G]$ by ψ: $G \to G''$. We prove that the sqaure

$$\begin{array}{ccc} G^H & \xrightarrow{\tilde{\psi}} & G''^{H'} \\ \downarrow & & \downarrow \\ G & \xrightarrow{\psi} & G \end{array}$$

is cartesian, which implies that $G^H \to G''^{H'}$ is faithfully flat.

By the assumption (1), the square

$$\begin{array}{ccc} H & \longrightarrow & H'' \\ \downarrow & & \downarrow \\ G & \xrightarrow{\psi} & G'' \end{array}$$

is cartesian. Hence the defining ideal of H'' in $k[G'']$ generates in $k[G]$ the defining ideal of H. Therefore $J(H)$ is generated by $J(H'')$ in $A[G]$. Let π, $f_1, \cdots, f_r$ be generators of $J(H'')$. Then

$$A[G^H]=A[G][\pi^{-1}f_1, \cdots, \pi^{-1}f_r],$$
$$A[G''^{H''}]=A[G''][\pi^{-1}f_1, \cdots, \pi^{-1}f_r].$$

Since $A[G]$ is flat over $A[G'']$,

$$A[G][\pi^{-1}f_1, \cdots, \pi^{-1}f_r]=A[G''][\pi^{-1}f_1, \cdots, \pi^{-1}f_r]\otimes_{A[G'']}A[G].$$

Case (2). Let B be a complete discrete valuation ring, unramified over A with residue field $\bar{k}$. Then obviously we have

$$G^H\otimes_A B\cong(G\otimes_A B)^{H\otimes_k \bar{k}}.$$

Since B is faithfully flat over A, the sequence $(\widetilde{\#})$ is exact if and only if so is the sequence induced from $(\widetilde{\#})$ by the base change B/A. Hence for our purpose we may assume that A is a complete discrete valuation ring with algebraically closed residue field k.

Moreover, we may assume $H''=G''_k$. In fact, let H_1 be the inverse image of H'' in G_k. By (1), we get an exact sequence of S-groups

$$0 \longrightarrow G' \longrightarrow G^{H_1} \longrightarrow G''^{H''} \longrightarrow 0.$$

Let $\tilde{H}$ be the inverse image of H in $(G^{H_1})_k$. By Proposition 1.8, $(G^{H_1})^{\tilde{H}}$ is isomorphic to G^H. Moreover, $\tilde{H}\cap G'_k=H'$ and $\tilde{H}$ is mapped onto $(G''^{H''})_k$.

Under these assumption, we prove first that the canonical map $\tilde{\phi}(k)$: $G^H(k)\to G''(k)$ is surjective.

Let $a\in G''(k)$. Since G'' is faithfully flat over $S=\operatorname{Spec} A$, there exist a complete discrete valuation ring B, dominating A and finite over A, and $\tilde{a}\in G''(B)$ such that the diagram

$$\begin{array}{ccc} \operatorname{Spec} B & \xrightarrow{\tilde{a}} & G'' \\ \uparrow & & \uparrow \\ \operatorname{Spec} k & \xrightarrow{a} & G''_k \end{array}$$

is commutative (EGA. IV, Prop. 14.5.8). Since B is strictly Henselian and G' is smooth over $S=\operatorname{Spec} A$, the canonical map $\phi(B)$: $G(B)\to G''(B)$ is surjective (cf. [11], Th. 11.7). Take $\tilde{b}\in G(B)$ such that $\phi(B)(\tilde{b})=\tilde{a}$.

Furthermore, since $H\to G''_k$ is faithfully flat, there exist a complete discrete valuation ring B', dominating A and finite over A, and $b\in H(B\otimes_A k)$ such that the diagram

$$\begin{array}{ccccc} \operatorname{Spec} B'\otimes_A k & & \xrightarrow{\quad b \quad} & & H \\ \downarrow & & & & \downarrow \phi_k \\ \operatorname{Spec} B\otimes_A k & \longrightarrow & \operatorname{Spec} k & \xrightarrow{a} & G''_k \end{array}$$

is commutative (EGA. IV, Cor. 17.16.2). Replacing B' by B, we may assume that $B=B'$.

Then $\tilde{b}_k-b$ is contained in $G'(B\otimes_A k)=\operatorname{Ker}(G(B\otimes_A k)\to G''(B\otimes_A k))$. Since A is strictly Henselian, B is finite flat over A and G' is smooth over A, the canonical map $G'(B)\to G'(B\otimes_A k)$ is surjective (cf. EGA. IV, Th. 18.5.17). Take $\tilde{c}\in G'(B)$ such that $\tilde{c}_k=\tilde{b}_k-b$ in $G'(B\otimes_A k)$. Then $(\tilde{b}-\tilde{c})_k=\tilde{b}_k-\tilde{c}_k=b\in H(B\otimes_A k)$; that is to say, $\tilde{b}-\tilde{c}$ is contained in $G^H(B)$. Let x be the image of $\tilde{b}-\tilde{c}$ by the canonical map $G^H(B)\to G^H(k)$. Then we have $\tilde{\phi}(k)(x)=a$. Therefore we see that $G^H(k)\to G''(k)$ is surjective.

We prove now that for any $t\in\operatorname{Lie}(G''_k)$, there exist an integer $n>0$ and $y\in\operatorname{Ker}(G^H(k[\varepsilon^{1/n}])\to G^H(k))$ such that the diagram

$$\begin{array}{ccc} \operatorname{Spec} k[\varepsilon^{1/n}] & \xrightarrow{\quad y \quad} & (G^H)_k \\ \downarrow & & \downarrow \\ \operatorname{Spec} k[\varepsilon] & \xrightarrow{\quad t \quad} & G''_k \end{array}$$

is commutative, where ε is a dual number.

Let $t\in\operatorname{Lie} G''_k=\operatorname{Ker}(G''(k[\varepsilon])\to G''(k))$. Let $\hat{O}$ denote the completion of $A[G'']$ along the zero section, and let $t^*\colon \hat{O}\to k[\varepsilon]$ be the local homomorphism defined by $t\colon \operatorname{Spec} k[\varepsilon]\to G''$. Moreover, let $\widetilde{s^*}\colon \hat{O}\to A$ be the local homomorphism defined by the zero section $s\colon \operatorname{Spec} A\to G''$. Assume that $t\neq 0$. Then $t^*\colon \hat{O}\to k[\varepsilon]$ is surjective, and therefore, there exists a surjective homomorphism $\tilde{t}^*\colon \hat{O}\to A[\varepsilon]$ such that $t^*\colon \hat{O}\to k[\varepsilon]$ and $s^*\colon \hat{O}\to A$ are factorized by $\hat{O}\xrightarrow{\tilde{t}^*}A[\varepsilon]\to k[\varepsilon]$ and $\hat{O}\xrightarrow{\tilde{t}^*}A[\varepsilon]\to A$, respectively. Let $\tilde{t}\colon \operatorname{Spec} A[\varepsilon]\to G''$ be the S-morphism defined by $\tilde{t}^*\colon \hat{O}\to A[\varepsilon]$. Since A is strictly Henselian and G' is smooth over $S=\operatorname{Spec} A$, the canonical map $\phi(A[\varepsilon])\colon G(A[\varepsilon])\to G''(A[\varepsilon])$ is surjective (cf. [11], Th. 11.7). Take $\tilde{u}\in G(A[\varepsilon])$ such that $\phi(A[\varepsilon])(\tilde{u})=\tilde{t}$.

Furthermore, since $H\to G''_k$ is faithfully flat, there exist an integer $n>0$ and $u\in\operatorname{Ker}(H(k[\varepsilon^{1/n}])\to H(k))$ such that the diagram

$$\begin{array}{ccc} \operatorname{Spec} k[\varepsilon^{1/n}] & \xrightarrow{\ u\ } & H \\ \downarrow & & \downarrow \phi_k \\ \operatorname{Spec} k[\varepsilon] & \xrightarrow{\ t\ } & G''_k \end{array}$$

is commutative. In fact, let $\mathfrak{U}=\operatorname{Ker}(k[G'']\xrightarrow{t^*}k[\varepsilon])$, and let $\mathfrak{B}_0$ be a maximal element in the set Σ of ideals $\mathfrak{B}$ in $k[H]$ such that $(\phi_k^*)^{-1}(\mathfrak{B})=\mathfrak{B}\cap k[G'']=\mathfrak{U}$. Note that Σ is not empty because of the faithful flatness of $\phi_k: H\to G''_k$. Then we can see that $k[H]/\mathfrak{B}_0$ is an Artinian local ring. Because look at the inclusions:

$$k[H]/\mathfrak{B}_0\supset k[\varepsilon]\cong k[G'']/\mathfrak{U}\supset k\,.$$

By the normalization theorem, there exist parameters $x_1, \cdots, x_l\in k[H]/\mathfrak{B}_0$ such that $k[H]/\mathfrak{B}_0$ is integral over the polynomial ring $k[x_1, \cdots, x_l]$. Let $\mathfrak{N}$ be a maximal ideal in $k[x_1, \cdots, x_l]$ containing $(0:\varepsilon)\cap k[x_1, \cdots, x_l]\subset k[x_1, \cdots, x_l]$. Then there exists an ideal $\bar{\mathfrak{C}}$ in $k[H]/\mathfrak{B}_0$ lying over $\mathfrak{N}$ and $\bar{\mathfrak{C}}\cap k[\varepsilon]=(0)$. The inverse image $\mathfrak{C}$ of $\bar{\mathfrak{C}}$ by the canonical map $k[H]\to k[H]/\mathfrak{B}_0$ is obviously an element of Σ, and we get that $\mathfrak{B}_0=\mathfrak{C}$. Therefore $k[H]/\mathfrak{B}_0$ should be the type of $k[\varepsilon^{1/n}]$ for some positive integer n. Let $u'\in H(k[\varepsilon^{1/n}])$ be the point defined by the canonical map $k[H]\to k[H]/\mathfrak{B}_0\cong k[\varepsilon^{1/n}]$. Then $u:=u'-u'_k\in H(k[\varepsilon^{1/n}])$ is a required point. Here we note that $u'_k\in\operatorname{Ker}(H(k)\to G''(k))$, and $H(k)\subset H(k[\varepsilon^{1/n}])$ in the canonical way.

We denote again by $\tilde{u}$ the image of $\tilde{u}$ by the canonical map $G(A[\varepsilon]\to G(A[\varepsilon^{1/n}])$. Then $\tilde{u}_k-u$ is contained in $G'(A[\varepsilon^{1/n}])=\operatorname{Ker}(G(A[\varepsilon^{1/n}])\to G''(A[\varepsilon^{1/n}]))$. Since A is strictly Henselian and G' is smooth over A, the canonical map $G'(A[\varepsilon^{1/n}])\to G'(k[\varepsilon^{1/n}])$ is surjective (cf. EGA. IV, Th. 18.5.17). Take $\tilde{v}\in G'(A[\varepsilon^{1/n}])$ such that $\tilde{v}_k=\tilde{u}_k-u$ in $G'(k[\varepsilon^{1/n}])$. Then $(\tilde{u}-\tilde{v})_k=\tilde{u}_k-\tilde{v}_k=u\in H(k[\varepsilon^{1/n}])$; that is to say, $\tilde{u}-\tilde{v}$ is contained in $G^H(A[\varepsilon^{1/n}])$. Let y be the image of $\tilde{u}-\tilde{v}$ by the canonical map $G^H(A[\varepsilon^{1/n}])\to G^H(k[\varepsilon^{1/n}])$. Then we get the required commutative diagram

$$\begin{array}{ccc} \operatorname{Spec} k[\varepsilon^{1/n}] & \xrightarrow{\ y\ } & (G^H)_k \\ \downarrow & & \downarrow \phi_k \\ \operatorname{Spec} k[\varepsilon] & \xrightarrow{\ t\ } & G''_k\,. \end{array}$$

From the above two facts, we can conclude that $\tilde{\phi}_k: (G^H)_k\to(G''^{H''})_k=G''_k$ is faithfully flat (cf. [7], pp. 109-111), and we accomplish the proof of Theorem 1.9.

Now we give two examples supporting the necessity of the conditions of

Theorem 1.9.

EXAMPLE 1.10. Assume that A has equal-characteristic $p>0$. We consider the exact sequences:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \boldsymbol{\alpha}_{p,S} & \longrightarrow & \boldsymbol{G}_{a,S} & \xrightarrow{F} & \boldsymbol{G}_{a,S} & \longrightarrow & 0 \\ & & \cup & & \cup & & \cup & & \\ 0 & \longrightarrow & \boldsymbol{\alpha}_{p,k} & \longrightarrow & \boldsymbol{G}_{a,k} & \xrightarrow{F} & \boldsymbol{G}_{a,k} & \longrightarrow & 0 \\ & & \cup & & \cup & & \cup & & \\ 0 & \longrightarrow & \boldsymbol{\alpha}_{p,k} & \longrightarrow & \boldsymbol{\alpha}_{p,k} & \longrightarrow & \{0\} & \longrightarrow & 0, \end{array}$$

where F denotes the Frobenius homomorphism. Let G (resp. G'') be the Néron blow-up of $\boldsymbol{\alpha}_{p,k}$ in $\boldsymbol{G}_{a,S}$ (resp. of $\{0\}$ in $\boldsymbol{G}_{a,S}$). Then G'' is isomorphic to $\boldsymbol{G}_{a,S}$. By Theorem 1.9. (1), we get an exact sequence

$$0 \longrightarrow \boldsymbol{\alpha}_{p,k} \longrightarrow G \xrightarrow{\tilde{F}} G''=\boldsymbol{G}_{a,S} \longrightarrow 0,$$

where $\tilde{F}$ is the canonical S-homomorphism induced by F. More precisely, $A[G]=A[X, Y]/(\pi Y-X^p)$ and $\tilde{F}$ is defined by

$$X \longmapsto X^p:\ A[G'']=A[X] \longrightarrow A[G]=A[X, Y]/(\pi Y-X^p).$$

Now let H be the closed k-subgroup of G_k defined by the ideal (X) in $k[G]=k[X, Y]/(X^p)=A[X, Y]/(\pi Y-X^p)\otimes_A k$. Then H is isomorphic to $\boldsymbol{G}_{a,k}$. Moreover, we have exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \boldsymbol{\alpha}_{p,S} & \longrightarrow & G & \xrightarrow{\tilde{F}} & \boldsymbol{G}_{a,S} & \longrightarrow & 0 \\ & & \cup & & \cup & & \cup & & \\ 0 & \longrightarrow & \boldsymbol{\alpha}_{p,k} & \longrightarrow & G_k & \longrightarrow & \boldsymbol{G}_{a,k} & \longrightarrow & 0 \\ & & \cup & & \cup & & \cup & & \\ 0 & \longrightarrow & \{0\} & \longrightarrow & H & \longrightarrow & \boldsymbol{G}_{a,k} & \longrightarrow & 0. \end{array}$$

Let $\tilde{G}$ (resp. $\tilde{G}'$) be the Néron blow-up of H in G (resp. of $\{0\}$ in $\boldsymbol{\alpha}_{p,S}$). Then $\tilde{G}'$ is isomorphic to $\boldsymbol{\alpha}_{p,S}$. In this case, the sequence

$$0 \longrightarrow \boldsymbol{\alpha}_{p,S} \longrightarrow \tilde{G} \xrightarrow{\tilde{F}^H} G''=\boldsymbol{G}_{a,S} \longrightarrow 0$$

is not exact. In fact, we can can easily see that $\tilde{F}^H:\ \tilde{G}\to\boldsymbol{G}_{a,S}$ is defined by

$$X \longmapsto X^p:\ A[X] \longrightarrow A[\tilde{G}]=A[X, Y, Z]/(\pi Z-X,\ \pi^{p-1}Z^p-Y).$$

Therefore $(\tilde{F}^H)_k:\ \tilde{G}_k\to\boldsymbol{G}_{a,k}$ is defined by

$$X \longmapsto 0:\ k[X] \longrightarrow k[\tilde{G}]=k[Z]=A[X, Y, Z]/(\pi Z-X,\ \pi^{p-1}Z^p-Y)\otimes_A k\,;$$

that is to say, $(\tilde{F}^H)_k=0$.

REMARK 1.10.1. $\tilde{G}$ is isomorphic to $\boldsymbol{G}_{a,S}$. In fact,

$$X \longmapsto \pi Z,\ Y \longmapsto \pi^{p-1}Z^p,\ Z \longmapsto Z:$$
$$A[X, Y, Z]/(\pi Z - X, \pi^{p-1}Z^p - Y) \longrightarrow A[Z]$$

defines an isomorphism of $\tilde{G}$ to $\boldsymbol{G}_{a,S}$. Then the S-homomorphism $\tilde{F}^H : \tilde{G} \to \boldsymbol{G}_{a,S}$ is simply written $\pi^{p-1}F$: $\boldsymbol{G}_{a,S} \to \boldsymbol{G}_{a,S}$.

EXAMPLE 1.11. Assume that k is of characteristic $p>0$. We consider the exact sequences:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \boldsymbol{\mu}_{p,S} & \longrightarrow & \boldsymbol{G}_{m,S} & \xrightarrow{p} & \boldsymbol{G}_{m,S} & \longrightarrow & 0 \\ & & \cup & & \cup & & \cup & & \\ 0 & \longrightarrow & \boldsymbol{\mu}_{p,k} & \longrightarrow & \boldsymbol{G}_{m,k} & \xrightarrow{p} & \boldsymbol{G}_{m,k} & \longrightarrow & 0 \\ & & \cup & & \cup & & \cup & & \\ 0 & \longrightarrow & \boldsymbol{\mu}_{p,k} & \longrightarrow & \boldsymbol{\mu}_{p,k} & \longrightarrow & \{1\} & \longrightarrow & 0, \end{array}$$

where p denotes the p-th power map. Let G (resp. G'') be the Néron blow-up of $\boldsymbol{\mu}_{p,k}$ in $\boldsymbol{G}_{m,S}$ (resp. of $\{1\}$ in $\boldsymbol{G}_{m,S}$). Then G'' is isomorphic to $\mathcal{G}^{(\pi)} = \operatorname{Spec} A[Y, 1/(\pi Y+1)]$ (cf. 2.1 or [3], Ch. I). By Theorem 1.9. (1), we get an exact sequence

$$0 \longrightarrow \boldsymbol{\mu}_{p,k} \longrightarrow G \xrightarrow{\tilde{p}} \mathcal{G}^{(\pi)} \longrightarrow 0,$$

where $\tilde{p}$ is the canonical S-homomorphism inducc d by p. More precisely, $A[G] = A[X, 1/X, Y]/(\pi Y - X^p + 1)$ and $\tilde{p}$ is defined by

$$Y \longmapsto Y:\ A[Y, 1/(\pi Y+1)] \longrightarrow A[G] = A[X, 1/X, Y]/(\pi Y - X^p + 1).$$

Now let H be the closed k-subgroup of G_k defined by the ideal $(X-1)$ in $k[G] = k[X, Y]/(X^p - 1) = A[X, 1/X, Y]/(\pi Y - X^p + 1) \otimes_A k$. Then H is isomorphic to $\boldsymbol{G}_{a,k}$. Moreover, we have exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \boldsymbol{\mu}_{p,S} & \longrightarrow & G & \xrightarrow{\tilde{p}} & \mathcal{G}^{(\pi)} & \longrightarrow & 0 \\ & & \cup & & \cup & & \cup & & \\ 0 & \longrightarrow & \boldsymbol{\mu}_{p,k} & \longrightarrow & G_k & \longrightarrow & \boldsymbol{G}_{a,k} & \longrightarrow & 0 \\ & & \cup & & \cup & & \cup & & \\ 0 & \longrightarrow & \{1\} & \longrightarrow & H & \longrightarrow & \boldsymbol{G}_{a,k} & \longrightarrow & 0. \end{array}$$

Let $\tilde{G}$ (resp. $\tilde{G}'$) be the Néron blow-up of H in G (resp. of $\{1\}$ in $\boldsymbol{\mu}_{p,S}$). In this case, the sequence

$$0 \longrightarrow G' \longrightarrow \tilde{G} \xrightarrow{\tilde{p}^H} \mathcal{G}^{(\pi)} \longrightarrow 0$$

is not exact. In fact, we can easily see that $\tilde{p}^H : \tilde{G} \to \mathcal{G}^{(\pi)}$ is defined by

$$Y \longmapsto ((\pi Z+1)^p-1)/\pi:$$
$$A[Y, 1/(\pi Y+1)] \longrightarrow A[\tilde{G}]$$
$$=A[X, 1/X, Y, Z]/(\pi Z-X+1, ((\pi Z+1)^p-1)/\pi-Y).$$

Therefore $(\tilde{p}^H)_k: \tilde{G}_k \to \boldsymbol{G}_{a,k}$ is defined by

$$Y \longmapsto 0:$$
$$k[Y] \longrightarrow k[\tilde{G}]$$
$$=k[Z]=A[X, 1/X, Y, Z]/(\pi Z-X+1, ((\pi Z+1)^p-1)/\pi-Y)\otimes_A k,$$

that is to say, $(\tilde{p}^H)_k=0$.

REMARK 1.11.1. $\tilde{G}$ is isomorphic to $\mathcal{G}^{(\pi)}$. In fact,

$$X \longmapsto \pi Z+1,\ Y \longmapsto ((\pi Z+1)^p-1)/\pi,\ Z \longmapsto Z:$$
$$A[X, 1/X, Y, Z]/(\pi Z-X+1, ((\pi Z+1)^p-1)/\pi-Y) \longrightarrow A[Z, 1/(\pi Z+1)]$$

defines an isomorphism of $\tilde{G}$ to $\mathcal{G}^{(\pi)}$. Then the S-homomorphism $\tilde{p}^H: \tilde{G} \to \mathcal{G}^{(\pi)}$ is defined by

$$Z \longmapsto ((\pi Z+1)^p-1)/\pi:\ A[Z, 1/(\pi Z+1)] \longrightarrow A[Z, 1/(\pi Z+1)].$$

2. Néron blow-ups and $\mathcal{E}^{(\lambda,\mu;F)}$

We recall first some notations and results of [3], [5].

2.1. Let $\lambda \in \mathfrak{m}-\{0\}$. We define a smooth affine S-group $\mathcal{G}^{(\lambda)}$ as follows:

$$\mathcal{G}^{(\lambda)}=\operatorname{Spec} A[X_0, 1/(\lambda X_0+1)]$$

1) law of multiplication

$$X_0 \longmapsto \lambda X_0 \otimes X_0 + X_0 \otimes 1 + 1 \otimes X_0;$$

2) unit

$$X_0 \longmapsto 0;$$

3) inverse

$$X_0 \longmapsto -X_0/(\lambda X_0+1).$$

Moreover, we define an S-homomorphism $\alpha^{(\lambda)}: \mathcal{G}^{(\lambda)} \to \boldsymbol{G}_{m,S}$ by

$$T \longmapsto \lambda X_0+1:\ A[T, T^{-1}] \longrightarrow A[X_0, 1/(\lambda X_0+1)].$$

Then the generic fiber $\alpha_K^{(\lambda)}: \mathcal{G}_K^{(\lambda)} \to \boldsymbol{G}_{m,K}$ is an isomorphism. On the other hand, the closed fiber $\mathcal{G}_k^{(\lambda)}$ is isomorphic to $\boldsymbol{G}_{a,k}$.

DEFINITION 2.2. Let $F(X)$ be a polynomial in $A[X]$. We shall say that

$F(X)$ satisfies the condition $(\#_m)$ if

$$F(X)\equiv 1 \text{ mod. } \mathfrak{m} \quad \text{and} \quad F(X)F(Y)\equiv F(\lambda XY+X+Y) \text{ mod. } \mathfrak{m}^m.$$

2.3. Let λ, $\mu\in\mathfrak{m}-\{0\}$ and $m=v(\mu)$, and let $F(X)$ be a polynomial in $A[X]$, satisfying the condition $(\#_m)$. We define a smooth affine S-group $\mathcal{E}^{(\lambda,\mu;F)}$ as follows:

$$\mathcal{E}^{(\lambda,\mu;F)}=\mathrm{Spec}\, A[X_0,\ X_1,\ 1/(\lambda X_0+1),\ 1/(\mu X_1+F(X_0))]$$

1) law of multiplication

$$X_0 \longmapsto \lambda X_0\otimes X_0+X_0\otimes 1+1\otimes X_0,$$

$$X_1 \longmapsto \mu X_1\otimes X_1+X_1\otimes F(X_0)+F(X_0)\otimes X_1 + \frac{1}{\mu}[F(X_0)\otimes F(X_0)-F(\lambda X_0\otimes X_0+X_0\otimes 1+1\otimes X_0)];$$

2) unit

$$X_0 \longmapsto 0,\ X_1 \longmapsto \frac{1}{\mu}[1-F(0)];$$

3) inverse

$$X_0 \longmapsto -X_0/(\lambda X_0+1),$$

$$X_1 \longmapsto \frac{1}{\mu}[1/(\mu X_1+F(X_0))-F(-X_0/(\lambda X_0+1))].$$

2.4. We define an S-homomorphism $\mathcal{G}^{(\mu)}\to\mathcal{E}^{(\lambda,\mu;F)}$ by

$$X_0 \longmapsto 0,\ X_1 \longmapsto X+\frac{1}{\mu}[1-F(0)]:$$

$$A[X_0,\ X_1,\ 1/(\lambda X_0+1),\ 1/(\mu X_1+F(X_0))] \longrightarrow A[X,\ 1/(\mu X+1)]$$

and an S-homomorphism $\mathcal{E}^{(\lambda,\mu;F)}\to\mathcal{G}^{(\lambda)}$ by

$$X \longmapsto X_0:$$

$$A[X,\ 1/(\lambda X+1)] \longrightarrow A[X_0,\ X_1,\ 1/(\lambda X_0+1),\ 1/(\mu X_1+F(X_0))].$$

Then the sequence of S-groups

$$0 \longrightarrow \mathcal{G}^{(\mu)} \longrightarrow \mathcal{E}^{(\lambda,\mu;F)} \longrightarrow \mathcal{G}^{(\lambda)} \longrightarrow 0$$

is exact, i.e. $\mathcal{E}^{(\lambda,\mu;F)}$ is an extension of $\mathcal{G}^{(\lambda)}$ by $\mathcal{G}^{(\mu)}$. Conversely, any extension of $\mathcal{G}^{(\lambda)}$ by $\mathcal{G}^{(\mu)}$ takes the form of

$$0 \longrightarrow \mathcal{G}^{(\mu)} \longrightarrow \mathcal{E}^{(\lambda,\mu;F)} \longrightarrow \mathcal{G}^{(\lambda)} \longrightarrow 0,$$

where $F(X)$ is a polynomial in $A[X]$, satisfying the condition $(\#_m)$ ([5], Cor. 3.6).

2.5. Let $F(X)$, $\tilde{F}(X)$ be polynomials in $A[X]$, satisfying the condition $(\#_m)$. If $F(X)\equiv\tilde{F}(X) \bmod \mathfrak{m}^m$, then we can define an isomorphism of extensions:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{G}^{(\mu)} & \longrightarrow & \mathcal{E}^{(\lambda,\mu;F)} & \longrightarrow & \mathcal{G}^{(\lambda)} & \longrightarrow & 0 \\ & & \| & & \downarrow \wr & & \| & & \\ 0 & \longrightarrow & \mathcal{G}^{(\mu)} & \longrightarrow & \mathcal{E}^{(\lambda,\mu;\tilde{F})} & \longrightarrow & \mathcal{G}^{(\lambda)} & \longrightarrow & 0 \end{array}$$

by

$$Y_0 \longmapsto X_0,\ Y_1 \longmapsto X_1+\frac{1}{\mu}[F(X_0)-\tilde{F}(X_0)]:$$
$$A[Y_0, Y_1, 1/(\lambda Y_0+1), 1/(\mu Y_1+\tilde{F}(Y_0))] \longrightarrow$$
$$A[X_0, X_1, 1/(\lambda X_0+1), 1/(\mu X_1+F(X_0))].$$

2.6. We define an S-homomorphism $\alpha^{(\lambda,\mu;F)}: \mathcal{E}^{(\lambda,\mu;F)}\rightarrow(\boldsymbol{G}_{m,S})^2$ by

$$T_0 \longmapsto \lambda X_0+1,\ T_1 \longmapsto \mu X_1+F(X_0):$$
$$A[T_0, T_0^{-1}, T_1, T_1^{-1}] \longrightarrow A[X_0, X_1, 1/(\lambda X_0+1), 1/(\mu X_1+F(X_0))].$$

Then we obtain a morphism of extensions of S-gronps:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{G}^{(\mu)} & \longrightarrow & \mathcal{E}^{(\lambda,\mu;F)} & \longrightarrow & \mathcal{G}^{(\lambda)} & \longrightarrow & 0 \\ & & \downarrow \alpha^{(\mu)} & & \downarrow \alpha^{(\lambda,\mu;F)} & & \downarrow \alpha^{(\lambda)} & & \\ 0 & \longrightarrow & \boldsymbol{G}_{m,S} & \longrightarrow & (\boldsymbol{G}_{m,S})^2 & \longrightarrow & \boldsymbol{G}_{m,S} & \longrightarrow & 0. \end{array}$$

Hence the generic fiber $\alpha_K^{(\lambda,\mu;F)}: \mathcal{E}_K^{(\lambda,\mu;F)}\rightarrow(\boldsymbol{G}_{m,K})^2$ is an isomorphism. On the other hand, the closed fiber $\mathcal{E}_k^{(\lambda,\mu;F)}$ is a unipotent k-group; more precisely, $\mathcal{E}_k^{(\lambda,\mu;F)}$ is an extension of $\boldsymbol{G}_{a,k}$ by $\boldsymbol{G}_{a,k}$ defined by the 2-cocycle $\varphi(X, Y)=\frac{1}{\mu}[F(X)F(Y)-F(\lambda XY+X+Y)] \bmod. \mathfrak{m}$.

Now we describe the S-homomorphism $\alpha^{(\lambda,\mu;F)}: \mathcal{E}^{(\lambda,\mu;F)}\rightarrow(\boldsymbol{G}_{m,S})^2$ using Néron blow-ups.

2.7. Let $F(X)=\Sigma a_i X^i$ be a polynomial in $A[X]$, satisfying the condition $(\#_m)$. Put

$$F_j(X)=\sum_{v(a_i)\leq j} a_i X^i$$

and

$$G_j(X)=\sum_{v(a_i)=j} a_i X^i.$$

Then we see readily that

$$F_j(X)=\sum_{0\leq i\leq j} G_i(X)$$

LEMMA 2.7.1. (1) $F_{j-1}(X)F_{j-1}(Y)\equiv F_{j-1}(\lambda XY+X+Y) \bmod. \mathfrak{m}^j$ *for each* j, $1\leqq j\leqq m$.

(2) $F_{j-1}(X)F_{j-1}(Y)-F_{j-1}(\lambda XY+X+Y)\equiv G_j(X+Y)-G_j(X)-G_j(Y) \bmod \mathfrak{m}^{j+1}$ *for each* j, $1\leqq j\leqq m-1$.

PROOF. The first assertion follows from the second. Assume that (1) holds for $j=i+1$ $(i\geqq 1)$, i.e.

$$F_i(X)F_i(Y)-F_i(\lambda XY+X+Y)\equiv 0 \bmod \mathfrak{m}^{i+1}.$$

Then

$$\{F_{i-1}(X)+G_i(X)\}\{F_{i-1}(Y)+G_i(Y)\}$$
$$-\{F_{i-1}(\lambda XY+X+Y)+G_i(\lambda XY+X+Y)\}\equiv 0 \bmod \mathfrak{m}^{i+1}.$$

Hence

$$\{F_{i-1}(X)F_{i-1}(Y)-F_{i-1}(\lambda XY+X+Y)\}$$
$$-\{G_i(\lambda XY+X+Y)-F_{i-1}(Y)G_i(X)-F_{i-1}(X)G_i(Y)\}\equiv 0 \bmod \mathfrak{m}^{i+1}.$$

Since $G_i(X)\equiv 0 \bmod \mathfrak{m}^i$ and $F_{i-1}(X)\equiv 1 \bmod \mathfrak{m}$ (resp. $\lambda\equiv 0 \bmod \mathfrak{m}$),

$$F_{i-1}(Y)G_i(X)\equiv G_i(X),\ F_{i-1}(X)G_i(Y)\equiv G_i(Y) \bmod \mathfrak{m}^{i+1}$$
$$(\text{resp. } G_i(\lambda XY+X+Y)\equiv G_i(X+Y) \bmod \mathfrak{m}^{i+1}).$$

Hence we obtain

$$F_{i-1}(X)F_{i-1}(Y)-F_{i-1}(\lambda XY+X+Y)\equiv G_i(X+Y)-G_i(X)-G_i(Y) \bmod \mathfrak{m}^{i+1}.$$

Therefore we get our assertion by induction on j counting down from $j=m$.

2.8. Hereafter, we assume that $\lambda=\pi^n$ and $\mu=\pi^m$, where π is a uniformizing parameter of A, for simplicity.

We start off with the first step.

Let $\mathcal{G}_i$ denote the S-group $\mathcal{G}^{(\pi^i)}\times_S \boldsymbol{G}_{m,S}$.

(1) $\mathcal{G}_1$ is the Néron blow-up of $\{1\}\times \boldsymbol{G}_{m,k}$ in $\boldsymbol{G}_{m,S}\times_S \boldsymbol{G}_{m,S}$. The canonical homomorphism $\mathcal{G}_1\to \boldsymbol{G}_{m,S}\times_S \boldsymbol{G}_{m,S}$ is defined by

$$X_0\longmapsto \pi Y_0+1,\ X_1\longmapsto Y_1:$$
$$A[X_0, X_1, 1/X_0, 1/X_1]\longrightarrow A[Y_0, Y_1, 1/(\pi Y_0+1), 1/Y_1].$$

(2) For each i, $1\leqq i\leqq n-1$, $\mathcal{G}_{i+1}$ is the Néron blow-up of $\{0\}\times \boldsymbol{G}_{m,k}$ in $\mathcal{G}_i$. The canonical homomorphism $\mathcal{G}_{i+1}\to\mathcal{G}_i$ is defined by

$$X_0\longmapsto \pi Y_0,\ X_1\longmapsto Y_1:$$
$$A[X_0, X_1, 1/(\pi^i X_0+1), 1/X_1]\longrightarrow A[Y_0, Y_1, 1/(\pi^{i+1}Y_0+1), 1/Y_1].$$

(3) $\mathcal{G}^{(\lambda)}\times_S\mathcal{G}^{(\mu)}$ is the Néron blow-up of $\boldsymbol{G}_{a,k}\times\{1\}$ in $\mathcal{G}_n=\mathcal{G}^{(\lambda)}\times_S \boldsymbol{G}_{m,S}$. The canonical homomorphism $\mathcal{E}_1\to\mathcal{G}_n$ is defined by

$$X_0 \longmapsto Y_0,\ X_1 \longmapsto \pi Y_1+1:$$

$$A[X_0, X_1, 1/(\lambda X_0+1), 1/X_1] \longrightarrow A[Y_0, Y_1, 1/(\lambda Y_0+1), 1/(\pi Y_1+1)].$$

Now we pass to the second step.

Let $\mathcal{E}_j$ denote the S-group $\mathcal{E}^{(\lambda, \pi^j; F_{j-1})}$. Note that $\mathcal{E}_1=\mathcal{G}^{(\lambda)}\times_S\mathcal{G}^{(\mu)}$. By Lemma 2.7.1, for j with $1 \leqq j \leqq m-1$,

$$\varphi_j(X, Y):=[F_{j-1}(X)F_{j-1}(Y)-F_{j-1}(\lambda XY+X+Y)]/\pi^j \bmod \mathfrak{m}$$
$$=G_j(X+Y)/\pi^j-G_j(X)/\pi^j-G_j(Y)/\pi^j \bmod \mathfrak{m},$$

and therefore, the closed fiber $(\mathcal{E}_j)_k$ is isomorphic to $(G_{a,k})^2$. Let Γ_j be the closed k-subgroup of $(\mathcal{E}_j)_k$ defined by the ideal $(X_1-G_j(X_0)/\pi^j)$ in

$$k[X_0, X_1]=A[X_0, X_1, 1/(\lambda X_0+1), 1/(\pi^j X_1+F_{j-1}(X_0))]\otimes_A k.$$

Then Γ_j is isomorphic to $G_{a,k}$, and $\mathcal{E}_{j+1}$ is the Néron blow-up of Γ_j in $\mathcal{E}_j$. The canonical homomorphism $\mathcal{E}_{j+1}\to\mathcal{E}_j$ is defined by

$$X_0 \longmapsto Y_0,\ X_1 \longmapsto \pi Y_1+G_j(Y_0)/\pi^j:$$

$$A[X_0, X_1, 1/(\lambda X_0+1), 1/(\pi^j X_1+F_{j-1}(X_0)] \longrightarrow$$
$$A[Y_0, Y_1, 1/(\lambda Y_0+1), 1/(\pi^{j+1}Y_1+F_j(Y_0)].$$

Summing up the above argument, we conclude that *the S-homomorphism $\alpha^{(\lambda,\mu;F)}: \mathcal{E}^{(\lambda,\mu;F)}\to(G_{m,S})^2$ is obtained by the sequence of Néron blow-ups*

$$\mathcal{E}^{(\lambda,\mu;F)}=\mathcal{E}_m \longrightarrow \mathcal{E}_{m-1} \longrightarrow \cdots \longrightarrow \mathcal{E}_1 \longrightarrow \mathcal{G}_n$$
$$=\mathcal{G}^{(\lambda)}\times_S G_{m,S} \longrightarrow \mathcal{G}_{n-1} \longrightarrow \cdots \longrightarrow \mathcal{G}_1 \longrightarrow (G_{m,S})^2.$$

REMARK 2.9. We can see that

$$\mathcal{E}^{(\lambda,\mu;F)}\cong\mathcal{E}_m \longrightarrow \mathcal{E}_{m-1} \longrightarrow \cdots \longrightarrow \mathcal{E}_1 \longrightarrow \mathcal{G}^{(\lambda)}\times_S G_{m,S}$$

and

$$\mathcal{G}^{(\lambda)}\times_S G_{m,S}=\mathcal{G}_n \longrightarrow \mathcal{G}_{n-1}\times_S G_{m,S} \longrightarrow \cdots \longrightarrow \mathcal{G}_1\times_S G_{m,S} \longrightarrow (G_{m,S})^2$$

are standard blow-up sequences in the sense of [9], p. 552, Remark 1. However,

$$\mathcal{E}^{(\lambda,\mu;F)}=\mathcal{E}_m \longrightarrow \mathcal{E}_{m-1} \longrightarrow \cdots \longrightarrow \mathcal{E}_1 \longrightarrow \mathcal{G}_n$$
$$=\mathcal{G}^{(\lambda)}\times_S G_{m,S} \longrightarrow \mathcal{G}_{n-1} \longrightarrow \cdots \longrightarrow \mathcal{G}_1 \longrightarrow (G_{m,S})^2$$

is not so.

REMARK 2.10. One may note that

$$0 \longrightarrow \mathcal{G}^{(\pi^{j+1})} \longrightarrow \mathcal{E}^{(\lambda,\pi^{j+1};F_j)} \longrightarrow \mathcal{G}^{(\lambda)} \longrightarrow 0$$

is the Néron blow-up of the exact sequence of k-groups

$$0 \longrightarrow \{0\} \longrightarrow \Gamma_j \longrightarrow G_a \longrightarrow 0$$

in

$$0 \longrightarrow \mathcal{G}^{(\pi^j)} \longrightarrow \mathcal{E}^{(\lambda, \mu^j; F_{j-1})} \longrightarrow \mathcal{G}^{(\lambda)} \longrightarrow 0$$

(cf. Theorem 1.9).

3. $\mathrm{Ext}^1_S(\mathcal{G}^{(\lambda)}, \mathcal{G}^{(\mu)})$

In this section, we suppose that the residue field k is of characteristic $p>0$. We fix λ, $\mu \in \mathfrak{m}-\{0\}$. Put $m=v(\mu)$.

LEMMA 3.1. (Comparison lemma) *Let $F(X)=1+c_1X+c_2X^2+\cdots+c_nX^n$ be a polynomial in $A[X]$ with $c_i \in \mathfrak{m}$. The following conditions are equivalent.*

(a) *$F(X)$ satisfies the condition $(\#_m)$.*

(b) $$c_{p^{r-1}}c_{j-p^{r-1}} \equiv \sum_{i=0}^{p^{r-1}} \binom{j-p^{r-1}+i}{j-2p^{r-1}+2i}\binom{j-2p^{r-1}+2i}{i} c_{j-p^{r-1}+i}\lambda^{p^{r-1}-i} \bmod \mathfrak{m}^m$$

if $j=p^r>1$, and

$$c_{p^r}c_{j-p^r} \equiv \sum_{i=0}^{p^r} \binom{j-p^r+i}{j-2p^r+2i}\binom{j-2p^r+2i}{i} c_{j-p^r+i}\lambda^{p^r-i} \bmod \mathfrak{m}^m$$

if $\mathrm{ord}_p j=r$ *and* $j \neq p^r$.

PROOF. (a)$\Rightarrow$(b): It is enough to remark that

$$\begin{aligned} &F(X)F(Y)-F(\lambda XY+X+Y) \\ &= \sum_{\substack{k\geq 0\\ l\geq 1}} \left\{c_kc_{k+l} - \sum_{i=0}^{k}\binom{k+l+i}{l+2i}\binom{l+2i}{i} c_{k+l+i}\lambda^{k-i}\right\}(XY)^k(X^l+Y^l) \\ &\quad + \sum_{k\geq 0}\left\{c_k^2 - \sum_{i=0}^{k}\binom{k+i}{2i}\binom{2i}{i}c_{k+i}\lambda^{k-i}\right\}(XY)^k. \end{aligned}$$

Here we understand that $c_f=0$ if $f>n$ and $c_0=1$.

(b)$\Rightarrow$(a): Assume that $F(X)F(Y) \not\equiv F(\lambda XY+X+Y) \bmod \mathfrak{m}^m$. Take the greatest s such that $F(X)F(Y) \equiv F(\lambda XY+X+Y) \bmod \mathfrak{m}^s$. Choose k, l such that

$$c_kc_{k+l} - \sum_{i=0}^{k}\binom{k+l+i}{l+2i}\binom{l+2i}{i}c_{k+l+i}\lambda^{k-i} \not\equiv 0 \bmod \mathfrak{m}^{s+1}.$$

Put

$$j=2k+l$$

and

$$g(X, Y):=[F(X)F(Y)-F(\lambda XY+X+Y)]/\pi^s \bmod \mathfrak{m}.$$

Let $g_j(X, Y)$ denote the homogeneous component of degree j of $g(X, Y)$. Then

1) $g_j(X, Y)=g_j(Y, X)$;

2) $g_j(X+Y, Z)+g_j(X, Y)=g_j(X, Y+Z)+g_j(Y, Z)$ (cf. 2.6).

By Lazard's comparison lemma ([2], lemma 3),

$$g_j(X, Y)=\begin{cases} c\{(X+Y)^j-X^j-Y^j\} & \text{if } j \text{ is not a power of } p. \\ \frac{c}{p}\{(X+Y)^j-X^j-Y^j\} & \text{if } j \text{ is a power of } p. \end{cases}$$

where c is a constant $\neq 0$. Hence the coefficient of $X^{p^{r-1}}Y^{j-p^{r-1}}+X^{j-p^{r-1}}Y^{p^{r-1}}$ (resp. $X^{p^r}Y^{j-p^r}+X^{j-p^r}Y^{p^r}$) does not vanish when $j=p^r>1$ (resp. $\mathrm{ord}_p j=r$ and $j\neq p^r$); that is to say,

$$c_{p^{r-1}}c_{j-p^{r-1}}-\sum_{i=0}^{p^{r-1}}\binom{j-p^{r-1}+i}{j-2p^{r-1}+2i}\binom{j-2p^{r-1}+2i}{i}c_{j-p^{r-1}+i}\lambda^{p^{r-1}-i}\not\equiv 0 \mod \mathfrak{m}^{s+1}$$

when $j=p^r>1$, and

$$c_{p^r}c_{j-p^r}-\sum_{i=0}^{p^r}\binom{j-p^r+i}{j-2p^r+2i}\binom{j-2p^r+2i}{i}c_{j-p^r+i}\lambda^{p^r-i}\not\equiv 0 \mod \mathfrak{m}^{s+1}$$

when $\mathrm{ord}_p j=r$ and $j\neq p^r$. Note that $s+1\leqq m$.

DEFINITION 3.2. Let $a_0, a_1, \cdots, a_l\in\mathfrak{m}$. We define the polynomial $F(\lambda, a_0, a_1, \cdots, a_l; X)=1+\sum_{i\geqq 1}c_iX^i$ in $A[X]$ by

$$c_1=a_0,\ c_p=a_1,\ \cdots,\ c_{p^l}=a_l$$

and

$$c_j=\frac{1}{\binom{j}{p^r}}\left\{c_{p^r}c_{j-p^r}-\sum_{i=0}^{p^r-1}\binom{j-p^r+i}{j-2p^r+2i}\binom{j-2p^r+2i}{i}c_{j-p^r+i}\lambda^{p^r-i}\right\}$$

if $\mathrm{ord}_p j=r$ and $j\neq p^r$.

EXAMPLE 3.3. (1) $p=2$.

$$F(\lambda, a_0, a_1; X)=1+a_0X+a_1X^2+a_1(a_0-2\lambda)X^3/3.$$

(2) $p=3$.

$$\begin{aligned}F(\lambda, a_0, a_1; X)=&1+a_0X+a_0(a_0-\lambda)X^2/2+a_1X^3\\&+a_1(a_0-3\lambda)X^4/4+a_1(a_0-3\lambda)(a_0-4\lambda)X^5/20\\&+a_1\left\{(a_1-\lambda^3)-\frac{3}{2}(a_0-2\lambda)(a_0-3\lambda)\lambda\right\}X^6/20\end{aligned}$$

$$+a_1\left\{(a_1-\lambda^3)-\frac{3}{2}(a_0-2\lambda)(a_0-3\lambda)\lambda\right\}(a_0-6\lambda)X^7/140$$

$$+a_1\left\{(a_1-\lambda^3)-\frac{3}{2}(a_0-2\lambda)(a_0-3\lambda)\lambda\right\}(a_0-6\lambda)(a_0-7\lambda)X^8/1120.$$

REMARK 3.4. The coefficient c_i of X^i in $F(\lambda, a_0, a_1, \cdots, a_l; X)$ can been seen as a polynomial in $\lambda, a_0, a_1, \cdots, a_r$ $(r=[\log_p i])$ with coefficients in $\boldsymbol{Z}_{(p)}$.

By the definition, we obtain immediately the following assertion.

COROLLARY 3.5. *Let* $F(X)=F(\lambda, a_0, a_1, \cdots, a_l; X)=1+\sum_{i\geq 1} c_i X^i$. *The following conditions are equivalent.*

(a) $F(X)$ *satisfies the condition* $(\#_m)$.

(b) $$c_{p^r}c_{p^{r+1}-p^r}\equiv \sum_{i=0}^{p^r}\binom{p^{r+1}-p^r+i}{p^{r+1}-2p^r+2i}\binom{p^{r+1}-2p^r+2i}{i}c_{p^{r+1}-p^r+i}\lambda^{p^r-i} \bmod \mathfrak{m}^m$$

for each $r\geq 0$.

EXAMPLE 3.6. (1) $p=2$. $F(\lambda, a_0, a_1; X)$ satisfies the condition $(\#_m)$ if and only if $a_0(a_0-\lambda)\equiv 2a_1$ and $a_1(a_1-\lambda^2)-2a_1(a_0-2\lambda)\lambda\equiv 0 \bmod \mathfrak{m}^m$.

(2) $p=3$. $F(\lambda, a_0, a_1; X)$ satisfies the condition $(\#_m)$ if and only if $a_0(a_0-\lambda)(a_0-2\lambda)\equiv 6a_1$ and $a_1\{(a_1-\lambda^3)-(3/2)(a_0-2\lambda)(a_0-3\lambda)\lambda\}(a_0-20\lambda^3)-3a_1\{(a_1-\lambda^3)-(3/2)(a_0-2\lambda)(a_0-3\lambda)\lambda\}(a_0-6\lambda)(a_0-2\lambda)\lambda\equiv 0 \bmod \mathfrak{m}^m$.

REMARK 3.7. In [5], $\phi(a, \lambda; X)$ denotes the polynomial

$$1+aX+\frac{a(a-\lambda)}{2}X^2+\cdots+\frac{a(a-\lambda)\cdots(a-(p-2)\lambda)}{(p-1)!}X^{p-1}.$$

(Here we employ a slightly different notation.) We see readily that $F(\lambda, a; X)=\phi(a, \lambda; X)$ and that $F(\lambda, a; X)=\phi(a, \lambda; X)$ satisfies the condition $(\#_m)$ if and only if $a(a-\lambda)\cdots(a-(p-1)\lambda)\equiv 0 \bmod \mathfrak{m}^m$. (cf. [5], 3.7 and 3.9)

The following assertions also can be seen without difficulty.

COROLLARY 3.8. *Suppose that* $F(X)=F(\lambda, a_0, a_1, \cdots, a_l; X)$ *satisfies the condition* $(\#_m)$.

(1) *The closed fiber of* $\mathcal{E}^{(\lambda, \mu; F)}$ *is the extension of* $\boldsymbol{G}_{a,k}$ *by* $\boldsymbol{G}_{a,k}$, *defined by the 2-cocycle* $\sum_{j\geq 1}\xi_j\dfrac{X^{p^j}+Y^{p^j}-(X+Y)^{p^j}}{p}$, *where*

$$\xi_j = -\frac{1}{\mu}\left\{c_{pj-1}c_{pj-pj-1} - \sum_{i=0}^{pj-1}\binom{p^j-p^{j-1}+i}{p^j-2p^{j-1}+2i}\binom{p^j-2p^{j-1}+2i}{i} c_{pj-pj-1+i}\lambda^{p^{j-1}-i}\right\} \bmod \mathfrak{m}.$$

(2) *If the closed fiber of $\mathcal{E}^{(\lambda,\mu;F)}$ is isomorphic to $(\boldsymbol{G}_{a,k})^2$, $F(X)$ satisfies the condition $(\#_{m+1})$.*

LEMMA 3.9. *If $a_0, a_1, \cdots, a_l \in \mathfrak{m}$ and $b_0, b_1, \cdots, b_l \in \mathfrak{m}^s$, then*

$$F(\lambda, a_0+b_0, a_1+b_1, \cdots, a_l+b_l; X) \equiv F(\lambda, a_0, a_1, \cdots, a_l; X) + \sum_{0\le i\le l} b_i X^{p^i} \bmod \mathfrak{m}^{s+1}.$$

Proof. Let

$$F(\lambda, a_0+b_0, a_1+b_1, \cdots, a_l+b_l; X) = 1 + \sum_{i\ge1} \tilde{c}_i X^i$$

and

$$F(\lambda, a_0, a_1, \cdots, a_l; X) = 1 + \sum_{i\ge1} c_i X^i.$$

We first note that, by the definition,

$$\tilde{c}_j = a_r + b_r = c_j + b_r$$

if $j = p^r \geqq 1$.

Now let j be an integer >0, which is not a power of p. We show that $\tilde{c}_j \equiv c_j \bmod \mathfrak{m}^{s+1}$, assuming that $\tilde{c}_i \equiv c_i \bmod \mathfrak{m}^{s+1}$ if $i<j$ and i is not a power of p. Put $r = \mathrm{ord}_p j$. Then, by the defintion,

$$\tilde{c}_j = \frac{1}{\binom{j}{p^r}}\left\{\tilde{c}_{p^r}\tilde{c}_{j-p^r} - \sum_{i=0}^{p^r-1}\binom{j-p^r+i}{j-2p^r+2i}\binom{j-2p^r+2i}{i}\tilde{c}_{j-p^r+i}\lambda^{p^r-i}\right\}$$

and

$$c_j = \frac{1}{\binom{j}{p^r}}\left\{c_{p^r}c_{j-p^r} - \sum_{i=0}^{p^r-1}\binom{j-p^r+i}{j-2p^r+2i}\binom{j-2p^r+2i}{i}c_{j-p^r+i}\lambda^{p^r-i}\right\}.$$

Obviously, $j-p^r+i\ (1\leqq i\leqq p^r-1)$ are not powers of p.

Case 1: $j-p^r$ is a power of p. Put $j-p^r=p^\nu$. Then

$$\begin{aligned}\tilde{c}_j &= \frac{1}{\binom{p^\nu+p^r}{p^r}}\left\{\tilde{c}_{p^r}\tilde{c}_{p^\nu} - \sum_{i=0}^{p^r-1}\binom{p^\nu+i}{p^\nu-p^r+2i}\binom{p^\nu-p^r+2i}{i}\tilde{c}_{p^\nu+i}\lambda^{p^r-i}\right\}\\ &= \frac{1}{\binom{p^\nu+p^r}{p^r}}\left\{(c_{p^r}+b_r)(c_{p^\nu}+b_\nu) - \binom{p^\nu}{p^\nu-p^r}(c_{p^\nu}+b_\nu)\lambda^{p^r}\right.\\ &\qquad \left. - \sum_{i=1}^{p^r-1}\binom{p^\nu+i}{p^\nu-p^r+2i}\binom{p^\nu-p^r+2i}{i}\tilde{c}_{p^\nu+i}\lambda^{p^r-i}\right\}\end{aligned}$$

$$=\frac{1}{\binom{p^\nu+p^r}{p^r}}\Big\{b_r b_\nu+\Big(c_{p^r}-\binom{p^\nu}{p^\nu-p^r}\lambda^{p^r}\Big)b_\nu+c_{p^\nu+p^r}b_r+c_{p^r}c_{p^\nu+p^r}$$
$$-\binom{p^\nu}{p^\nu-p^r}c_{p^\nu}\lambda^{p^r}-\sum_{i=1}^{p^r-1}\binom{p^\nu+i}{p^\nu-p^r+2i}\binom{p^\nu-p^r+2i}{i}\tilde{c}_{p^\nu+i}\lambda^{p^r-i}\Big\}.$$

By the hypothesis of induction, $\tilde{c}_{p^\nu+i}\equiv c_{p^\nu+i}\bmod.\mathfrak{m}^{s+1}$ for each $i\,(1\leqq i\leqq p^r-1)$. Moreover, $b_r,\ b_{\nu+r}\in\mathfrak{m}^s$ and $c_{p^\nu},\ c_{p^\nu+p^r},\ \lambda\in\mathfrak{m}$. Hence we obtain

$$\frac{1}{\binom{p^\nu+p^r}{p^r}}\Big\{b_r b_\nu+\Big(c_{p^r}-\binom{p^\nu}{p^\nu-p^r}\lambda^{p^r}\Big)b_\nu+c_{p^\nu+p^r}b_r+c_{p^r}c_{p^\nu+p^r}$$
$$-\binom{p^\nu}{p^\nu-p^r}c_{p^\nu}\lambda^{p^r}-\sum_{i=1}^{p^r-1}\binom{p^\nu+i}{p^\nu-p^r+2i}\binom{p^\nu-p^r+2i}{i}\tilde{c}_{p^\nu+i}\lambda^{p^r-i}\Big\}$$
$$\equiv\frac{1}{\binom{p^\nu+p^r}{p^r}}\Big\{c_{p^r}c_{p^\nu+p^r}-\binom{p^\nu}{p^\nu-p^r}c_{p^\nu}\lambda^{p^r}$$
$$-\sum_{i=1}^{p^r-1}\binom{p^\nu+i}{p^\nu-p^r+2i}\binom{p^\nu-p^r+2i}{i}c_{p^\nu+i}\lambda^{p^r-i}\Big\}\bmod.\mathfrak{m}^{s+1},$$

and therefore $\tilde{c}_j\equiv c_j\bmod.\mathfrak{m}^{s+1}$.

Case 2: $j-p^r$ is not a power of p.

$$\tilde{c}_j=\frac{1}{\binom{j}{p^r}}\Big\{\tilde{c}_{p^r}\tilde{c}_{j-p^r}-\sum_{i=0}^{p^r-1}\binom{j-p^r+i}{j-2p^r+2i}\binom{j-2p^r+2i}{i}\tilde{c}_{j-p^r+i}\lambda^{p^r-i}\Big\}$$
$$=\frac{1}{\binom{j}{p^r}}\Big\{c_{p^r}\tilde{c}_{j-p^r}+b_r\tilde{c}_{j-p^r}-\sum_{i=0}^{p^r-1}\binom{j-p^r+i}{j-2p^r+2i}\binom{j-2p^r+2i}{i}\tilde{c}_{j-p^r+i}\lambda^{p^r-i}\Big\}.$$

By the hypothesis of induction, $\tilde{c}_{j-p^r+i}\equiv c_{j-p^r+i}\bmod.\mathfrak{m}^{s+1}$ for each $i(0\leqq i\leqq p^r-1)$. Moreover, $b_r\in\mathfrak{m}^s$ and $c_{p^r},\ \tilde{c}_{j-p^r},\ \lambda\in\mathfrak{m}$. Hence we obtain

$$\frac{1}{\binom{j}{p^r}}\Big\{c_{p^r}\tilde{c}_{j-p^r}+b_r\tilde{c}_{j-p^r}-\sum_{i=0}^{p^r-1}\binom{j-p^r+i}{j-2p^r+2i}\binom{j-2p^r+2i}{i}\tilde{c}_{j-p^r+i}\lambda^{p^r-i}\Big\}$$
$$\equiv\frac{1}{\binom{j}{p^r}}\Big\{c_{p^r}c_{j-p^r}-\sum_{i=0}^{p^r-1}\binom{j-p^r+i}{j-2p^r+2i}\binom{j-2p^r+2i}{i}c_{j-p^r+i}\lambda^{p^r-i}\Big\}\bmod.\mathfrak{m}^{s+1},$$

and therefore $\tilde{c}_j\equiv c_j\bmod.\mathfrak{m}^{s+1}$.

THEOREM 3.10. *Let $F(X)$ be a polynomial in $A[X]$, satisfying the condition $(\#_m)$. Then there exist $a_0,\ a_1,\ \cdots,\ a_l\in\mathfrak{m}$ such that $F(X)\equiv F(\lambda,\ a_0,\ a_1,\ \cdots,\ a_l\,;\ X)$*

mod. $\mathfrak{m}^m$.

PROOF. We prove the theorem by induction on m.

Note first that $F(X)\equiv 1$ mod. $\mathfrak{m}$. Assume that there exist $a_0, a_1, \cdots a_l \in \mathfrak{m}$ such that $F(X)\equiv F(\lambda, a_0, a_1, \cdots, a_l; X)$ mod. $\mathfrak{m}^s$. (We take l so that $\deg F(\lambda, a_0, a_1, \cdots, a_l; X)\geqq \deg F(X)$.) Put

$$\tilde{F}(X)=F(\lambda, a_0, a_1, \cdots, a_l; X)$$

and

$$F_{s-1}(X)=\sum_{v(c_j)\leq s-1} c_jX^j,\ G_s(X)=\sum_{v(c_j)=s} c_jX^j,$$
$$\tilde{F}_{s-1}(X)=\sum_{v(\tilde{c}_j)\leq s-1} \tilde{c}_jX^j,\ \tilde{G}_s(X)=\sum_{v(\tilde{c}_j)=s} \tilde{c}_jX^j,$$

where

$$F(X)=\sum_{j\geq 0} c_jX^j,\ \tilde{F}(X)=\sum_{j\geq 0} \tilde{c}_jX^j.$$

Then $F_{s-1}(X)\equiv\tilde{F}_{s-1}(X)$ mod. $\mathfrak{m}^s$ and $F_{s-1}(X)$, $\tilde{F}_{s-1}(X)$ satisfy $(\#_s)$.

Let $\mathcal{E}=\mathcal{E}^{(\lambda, \pi^s; F_{s-1})}$ and $\tilde{\mathcal{E}}=\mathcal{E}^{(\lambda, \pi^s; \tilde{F}_{s-1})}$. We define an S-isomorphism β: $\mathcal{E}\xrightarrow{\sim}\tilde{\mathcal{E}}$ by

$$Y_0\longmapsto X_0,\ Y_1\longmapsto X_1+\frac{1}{\pi^s}[F(X_0)-\tilde{F}(X_0)]:$$
$$A[Y_0, Y_1, 1/(\lambda Y_0+1), 1/(\pi^sY_1+\tilde{F}_{s-1}(Y_0))]\longrightarrow$$
$$A[X_0, X_1, 1/(\lambda X_0+1), 1/(\pi^sX_1+F_{s-1}(X_0))].$$

Since the closed fibers $\tilde{\mathcal{E}}_k\cong\mathcal{E}_k$ are isomorphic to $(\boldsymbol{G}_{a,k})^2$, $\tilde{F}(X)$ satisfies $(\#_{s+1})$ (cf. Corollary 3.8), and therefore

$$\tilde{F}_{s-1}(X)\tilde{F}_{s-1}(Y)-\tilde{F}_{s-1}(\lambda XY+X+Y)\equiv\tilde{G}_s(X+Y)-\tilde{G}_s(X)-\tilde{G}_s(Y) \text{ mod. } \mathfrak{m}^{s+1}$$

(cf. Lemma 2.7.1.). We define now k-isomorphisms α: $\mathcal{E}_k\xrightarrow{\sim}(\boldsymbol{G}_{a,k})^2$ and $\tilde{\alpha}$: $\tilde{\mathcal{E}}_k\xrightarrow{\sim}(\boldsymbol{G}_{a,k})^2$ by

$$T_0\longmapsto X_0,\ T_1\longmapsto(X_1-G_s(X_0)/\pi^s):$$
$$k[T_0, T_1]\longrightarrow k[X_0, X_1]=A[X_0, X_1, 1/(\lambda X_0+1), 1/(\pi^sX_1+F_{s-1}(X_0))]\otimes_A k$$

and by

$$T_0\longmapsto Y_0,\ T_1\longmapsto(Y_1-\tilde{G}_s(Y_0)/\pi^s):$$
$$k[T_0, T_1]\longrightarrow k[Y_0, Y_1]=A[Y_0, Y_1, 1/(\lambda Y_0+1), 1/(\pi^sY_1+\tilde{F}_{s-1}(Y_0))]\otimes_A k,$$

respectively. Then $\tilde{\alpha}\circ\beta_k\circ\alpha^{-1}$ is defined by

$$T_0\longmapsto T_0,\ T_1\longmapsto T_1+[G_s(T_0)-\tilde{G}_s(T_0)+F_{s-1}(T_0)-\tilde{F}_{s-1}(T_0)]/\pi^s$$
$$=T_1+[F(T_0)-\tilde{F}(T_0)]/\pi^s.$$

Hence $T_0 \mapsto [F(T_0)-\tilde{F}(T_0)]/\pi^s \bmod \mathfrak{m}$ defines a k-endomorphism of $\boldsymbol{G}_{a,k}$, and therefore, there exist $b_0, b_1, \cdots, b_l \in \mathfrak{m}^s$ such that

$$F(X)-\tilde{F}(X) \equiv \sum_{0\leq i\leq l} b_i X^{p^i} \bmod \mathfrak{m}^{s+1}.$$

By Lemma 3.9, we obtain

$$F(X) \equiv \tilde{F}(X)+\sum_{0\leq i\leq l} b_i X^{p^i} \equiv F(\lambda,\ a_0+b_0,\ a_1+b_1, \cdots, a_l+b_l\,;\ X) \bmod \mathfrak{m}^{s+1},$$

and we are done.

3.11. Let $\mathfrak{M}_{(\lambda,\mu)}$ be the subset of $(\mathfrak{m}/\mathfrak{m}^m)^{(N)}$ formed by the elements $(a_0, a_1, \cdots)$ such that

$$a_r c_{p^{r+1}-p^r}(a_0, a_1, \cdots, a_{r-1}) \equiv$$
$$\sum_{i=0}^{p^r} \binom{p^{r+1}-p^r+i}{p^{r+1}-2p^r+2i} \binom{p^{r+1}-2p^r+2i}{i} c_{p^{r+1}-p^r+i}(a_0, a_1, \cdots, a_{r-1}) \lambda^{p^r-i} \bmod \mathfrak{m}^m$$

for each $r \geqq 0$. Here $c_j(a_0, a_1, \cdots, a_{r-1})$ is the polynomial defined by the coefficient of X^j in the expansion of $F(\lambda, a_0, a_1, \cdots, a_l : X)$ (cf. Def. 3.2).

We define a law of multiplication on $\mathfrak{M}_{(\lambda,\mu)}$ by

$$(a_0, \cdots, a_r, \cdots)(b_0, \cdots, b_r, \cdots)=$$
$$\left(a_0+b_0, \cdots, a_r+b_r+\sum_{i=1}^{p^r-1} c_i(a_0, a_1, \cdots, a_{r-1}) c_{p^r-i}(b_0, b_1, \cdots, b_{r-1}), \cdots\right).$$

Then $\mathfrak{M}_{(\lambda,\mu)}$ is isomorphic to the subgroup of the multiplicative group $(A/\mathfrak{M}^m[X])^\times$, formed by the polynomials $F(X)$ such that $F(X)F(Y)= F(\lambda XY+X+Y)$.

Moreover, let $\tilde{\mathfrak{M}}_{(\lambda,\mu)}$ denote the quotient of $\mathfrak{M}_{(\lambda,\mu)}$ by the subgroup generated by $(\lambda, 0, 0, \cdots)$. By [5], Cor. 3.6, *$\tilde{\mathfrak{M}}_{(\lambda,\mu)}$ is isomorphic to* $\mathrm{Ext}^1_S(\mathcal{G}^{(\lambda)}, \mathcal{G}^{(\mu)})$.

4. Examples

In this section, we suppose that the residue field k is of characteristic $p>0$.

EXAMPLE 4.1. Suppose that $\mu \mid p$ and $v(\mu)=m$. Let j be an integer >1, which is not a power of p. Put $r=\operatorname{ord}_p j$. The relation

$$c_j=\frac{1}{\binom{j}{p^r}}\left\{c_{p^r}c_{j-p^r}-\sum_{i=0}^{p^r-1}\binom{j-p^r+i}{j-2p^r+2i}\binom{j-2p^r+2i}{i}c_{j-p^r+i}\lambda^{p^r-i}\right\}$$

implies

$$c_j \equiv \frac{1}{\binom{j}{p^r}}\left\{c_{p^r}c_{j-p^r} - \left(\frac{j}{p^r} - 1\right)c_{j-p^r}\lambda^{p^r}\right\} \bmod . \mathfrak{m}^m .$$

(Note that $\binom{kp^r}{p^r} \equiv k \bmod . p$ and $\binom{kp^r+i}{p^r-i} \equiv 0 \bmod . p$ for i, $1 \leq i \leq p^r - 1$.) Hence we obtain

$$c_k \equiv \prod_{r=0}^{l} \frac{a_r(a_r - \lambda^{p^r})\cdots(a_r - (n_r - 1)\lambda^{p^r})}{n_r\,!} \bmod . \mathfrak{m}^m ,$$

where $k = \sum_{r=0}^{l} n_r p^r$ is the p-adic expansion of k, and therefore

$$F(\lambda, a_0, a_1, \cdots, a_l\,; X) \equiv \phi(a_0, \lambda\,; X)\phi(a_1, \lambda^p\,; X^p)\cdots\phi(a_l, \lambda^{p^l}\,; X^{p^l}) \bmod . \mathfrak{m}^m .$$

Moreover, the congruence relation

$$c_{p^r}c_{p^{r+1}-p^r} \equiv \sum_{i=0}^{p^r} \binom{p^{r+1}-p^r+i}{p^{r+1}-2p^r+2i}\binom{p^{r+1}-2p^r+2i}{i} c_{p^{r+1}-p^r+i}\lambda^{p^r-i} \bmod . \mathfrak{m}^m$$

reads

$$c_{p^r}c_{p^{r+1}-p^r} \equiv (p-1)c_{p^{r+1}-p^r}\lambda^{p^r} \bmod . \mathfrak{m}^m .$$

Hence we have

$$a_r(a_r - \lambda^{p^r})(a_r - 2\lambda^{p^r})\cdots(a_r - (p-1)\lambda^{p^r})/(p-1)\,! \equiv 0 \bmod . \mathfrak{m}^m$$

and therefore

$$a_r^p - \lambda^{p^r(p-1)}a_r \equiv 0 \bmod . \mathfrak{m}^m .$$

It follows that $F(\lambda, a_0, a_1, \cdots, a_l\,; X)$ satisfies the condition $(\#_m)$ if and only if $a_r^p - \lambda^{p^r(p-1)}a_r \equiv 0 \bmod . \mathfrak{m}^m$ for each $r \geq 0$.

The closed fiber of $\mathcal{E}^{(\lambda,\mu;F)}$ is the extension of $G_{a,k}$ by $G_{a,k}$, defined by the 2-cocycle $\sum_{j\geq 1} \xi_j \frac{X^{p^j}+Y^{p^j}-(X+Y)^{p^j}}{p}$, where $\xi_j = \frac{1}{\mu}\{a_{j-1}^p - \lambda^{p^{j-1}(p-1)}a_{j-1}\} \bmod . \mathfrak{m}$.

Thus we recover [5], Cor. 3.8 and Th. 4.4, under the assumption that $\mu | p$.

EXAMPLE 4.2. Suppose that $\mu | \lambda$ and $v(\mu) = m$. Let j be an integer >1, which is not a power of p. Put $r = \mathrm{ord}_p j$. The relation

$$c_j = \frac{1}{\binom{j}{p^r}}\left\{c_{p^r}c_{j-p^r} - \sum_{i=0}^{p^r-1}\binom{j-p^r+i}{j-2p^r+2i}\binom{j-2p^r+2i}{i}c_{j-p^r+i}\lambda^{p^r-i}\right\}$$

implies

$$c_j \equiv \frac{1}{\binom{j}{p^r}} c_{p^r}c_{j-p^r} \bmod . \mathfrak{m}^m .$$

Hence we obtain

$$c_k \equiv \prod_{r=0}^{l} \frac{(p^r!\cdot a_r)^{n_r}}{k!} \mod \mathfrak{m}^m .$$

where $k=\sum_{r=0}^{l} n_r p^r$ is the p-adic expansion of k.

Moreover, the congruence relation

$$c_{p^r} c_{p^{r+1}-p^r} \equiv \sum_{i=0}^{p^r} \binom{p^{r+1}-p^r+i}{p^{r+1}-2p^r+2i} \binom{p^{r+1}-2p^r+2i}{i} c_{p^{r+1}-p^r+i} \lambda^{p^r-i} \mod \mathfrak{m}^m$$

reads

$$c_{p^r} c_{p^{r+1}-p^r} \equiv \binom{p^{r+1}}{p^r} c_{p^{r+1}} \mod \mathfrak{m}^m ,$$

and therefore

$$a_r^p / \sum_{i=1}^{p-2} \binom{p^{r+1}-ip^r}{p^r} \equiv \binom{p^{r+1}}{p^r} a_{r+1} \mod \mathfrak{m}^m .$$

Hence $F(\lambda, a_0, a_1, \cdots, a_l; X)$ satisfies the condition $(\#_m)$ if and only if

$$a_r^p \equiv \sum_{i=0}^{p-2} \binom{p^{r+1}-ip^r}{p^r} a_{r+1} \mod \mathfrak{m}^m, \text{ i.e. } a_r^p \equiv \frac{p^{r+1}!}{(p^r!)^r} a_{r+1} \mod \mathfrak{m}^m$$

for each $r \geqq 0$.

The closed fiber of $\mathcal{E}^{(\lambda,\mu;F)}$ is the extension of $G_{a,k}$ by $G_{a,k}$, defined by the 2-cocycle $\sum_{j\geqq 1} \xi_j \frac{X^{p^j}+Y^{p^j}-(X+Y)^{p^j}}{p}$, where

$$\xi_j = -\frac{1}{\mu}\left\{ a_{j-1}^p / \sum_{i=1}^{p-2} \binom{p^{r+1}-ip^r}{p^r} - \binom{p^{r+1}}{p^r} a_j \right\} \mod \mathfrak{m} .$$

COROLLARY 4.2.1. *The canonical map* $\mathrm{Ext}^1_S(\mathcal{G}^{(\lambda)}, \mathcal{G}^{(\mu)}) \to \mathrm{Ext}^1_k(G_{a,k}, G_{a,k})$ *is surjective if* $pv(p) < (p-1)m$.

PROOF. It is sufficient to remark that $\mathrm{Ext}^1_k(G_{a,k}, G_{a,k})$ is generated by the 2-cocycles $\sum_{j\geqq 1} \eta_j \frac{X^{p^j}+Y^{p^j}-(X+Y)^{p^j}}{p}$, $\eta_j \in k$ (see [6], Ch. VII, 2.7). (Compare with [5], example 3.4)

EXAMPLE 4.3. Suppose that $\mu | p$ and $\mu | \lambda$. Then

$$\phi(a_r, \lambda^{p^r}; X) \equiv \sum_{i=0}^{p-1} (a_r X^{p^r})^i / i! \mod \mathfrak{m}^m ,$$

and therefore

$$F(\lambda, a_0, a_1, \cdots, a_l; X) \equiv \prod_{r=0}^{l} \sum_{i=0}^{p-1} (a_r X^{p^r})^i / i! \bmod. \mathfrak{m}^m .$$

Moreover, $F(\lambda, a_0, a_1, \cdots, a_l; X)$ satisfies the condition $(\#_m)$ if and only if $a_r^p \equiv 0 \bmod. \mathfrak{m}^m$ for each $r \geqq 0$. Hence $c_{p^r-i}(a_0, \cdots, a_{r-1}) c_i(b_0, \cdots, b_{r-1}) \equiv 0$ mod. $\mathfrak{m}^m$ for each $r \geqq 1$ and each i with $1 \leqq i \leqq p^{r-1}$ if $(a_0, a_1, \cdots), (b_0, b_1, \cdots) \in \mathfrak{M}_{(\lambda, \mu)}$. Therefore $\mathfrak{M}_{(\lambda, \mu)} = \widetilde{\mathfrak{M}}_{(\lambda, \mu)}$ *is isomorphic to the additive group* $(\mathfrak{m}^s/\mathfrak{m}^m)^{(N)}$, *where*

$$s = \begin{cases} [m/p]+1 & \text{if } (p, m)=1 \\ m/p & \text{if } p \mid m . \end{cases}$$

The closed fiber of $\mathcal{E}^{(\lambda, \mu; F)}$ is the extension of $\boldsymbol{G}_{a,k}$ by $\boldsymbol{G}_{a,k}$, defined by the 2-cocycle $\sum_{j \geq 1} \xi_j \dfrac{X^{p^j} + Y^{p^j} - (X+Y)^{p^j}}{p}$, where $\xi_j = \dfrac{1}{\mu} a_{j-1}^p \bmod. \mathfrak{m}$.

COROLLARY 4.3.1. (1) *Assume that p does not divide m. Then the canonical map* $\mathrm{Ext}_S^1(\mathcal{G}^{(\lambda)}, \mathcal{G}^{(\mu)}) \to \mathrm{Ext}_k^1(\boldsymbol{G}_{a,k}, \boldsymbol{G}_{a,k})$ *is zero.*

(2) *Assume that p divides m. Then the canonical map* $\mathrm{Ext}_S^1(\mathcal{G}^{(\lambda)}, \mathcal{G}^{(\mu)}) \to \mathrm{Ext}_k^1(\boldsymbol{G}_{a,k}, \boldsymbol{G}_{a,k})$ *is surjective if the residue field k is perfect.*

PROOF. We have only to note that the equation $X^p \equiv \mu a \bmod. \mathfrak{m}^m$ has a solution in A for any $a \in A$ if k is perfect. (cf. [5], 4.5.)

We have computed the group of extensions $\mathrm{Ext}_S^1(\mathcal{G}^{(\lambda)}, \mathcal{G}^{(\mu)})$ of smooth affine 1-dimensional S-groups $\mathcal{G}^{(\lambda)}$ and $\mathcal{G}^{(\mu)}$. We conclude this article by noting that a smooth affine 2-dimensional S-group is not necessarily obtained by an extension of smooth 1-dimensional S-groups, even though its generic fiber and its special fibre are extensions of smooth 1-dimensional groups each.

4.4. Suppose that $k \neq \boldsymbol{F}_p$. Let π be a uniformizing parameter of A, and let m be an integer >2. Put $\lambda = \pi^{m-1}$ and $\mu = \pi^m$. We choose an element $a \in A$ such that the image of a in k is not contained in $\boldsymbol{F}_p (\subset k)$. Then the polynomial $F(X) = 1 + a\lambda X$ satisfies the condition $(\#_m)$ (cf. Remark 3.7). Let G denote the smooth affine S-group $\mathcal{E}^{(\lambda, \mu; F)}$:

$$G = \mathrm{Spec}\, A[X_0, X_1, 1/(\lambda X_0 + 1), 1/(\mu X_1 + F(X_0))] .$$

By our assumption on m, the closed fiber G_k is isomorphic to $(\boldsymbol{G}_{a,k})^2$. More precisely, the comultiplication of $k[G] = k[X_0, X_1] = A[X_0, X_1, 1/(\lambda X_0 + 1), 1/(\mu X_1 + F(X_0))] \otimes_A k$ is defined by

$$X_0 \longmapsto X_0 \otimes 1 + 1 \otimes X_0, \quad X_1 \longmapsto X_1 \otimes 1 + 1 \otimes X_1 .$$

Let H be the closed k-subgroup of $G_k = (\boldsymbol{G}_{a,k})^2$ defined by the ideal $(X_1^p - X_0)$ in

$k[G]=k[X_0, X_1]$, and let $\beta: \tilde{G}\to G$ be the Néron blow-up of H in G. Since G is smooth over S and $H\cong \boldsymbol{G}_{a,k}$ is smooth over k, $\tilde{G}$ is smooth over S ([9], Th. 1.7).

Under these notations, we get the following assertion.

4.4.1. *Any flat 1-dimensional closed S-subgroup of $\tilde{G}$ is not smooth.*

PROOF. For integers r, s, we define an injective S-homomorphism

$$\varphi_{r,s}:\ \boldsymbol{G}_{m,S} \longrightarrow (\boldsymbol{G}_{m,S})^2$$

by

$$U\longmapsto T^r,\ V\longmapsto T^s:\ A[U, U^{-1}, V, V^{-1}] \longrightarrow A[T, T^{-1}].$$

By the general theory of algebraic tori, we know that any closed K-subgroup of dimension 1 of $(\boldsymbol{G}_{m,K})^2$ is the form of $\varphi_{r,s}(\boldsymbol{G}_{m,K})$, where r, s are integers with $(r, s)=1$ or $(r, s)=(0, 1), (1, 0)$. We identify the generic fiber $\tilde{G}_K$ (resp. G_K) to $(\boldsymbol{G}_{m,K})^2$ via the isomorphism $\beta_K\circ\alpha_K^{(\lambda,\mu;F)}: \tilde{G}_K \xrightarrow{\sim} (\boldsymbol{G}_{m,K})^2$ (resp. $\alpha_K^{(\lambda,\mu;F)}: G_K \xrightarrow{\sim} (\boldsymbol{G}_{m,K})^2$). Let $\tilde{G}_{r,s}$ (resp. $G_{r,s}$) denote the flat closure of $\varphi_{r,s}(\boldsymbol{G}_{m,K})$ in $\tilde{G}$ (resp. G). We show that the closed fiber $(\tilde{G}_{r,s})_k$ is isomorphic to $\boldsymbol{\alpha}_p\times \boldsymbol{G}_{a,k}$, which implies our assertion together with Prop. 1.6.

Note first that the subgroup $\varphi_{r,s}(\boldsymbol{G}_{m,K})$ of $G_K=(\boldsymbol{G}_{m,K})^2$ is defined by the ideal $(U^s-V^r)=((\lambda X_0+1)^s-(\mu X_1+F(X_0))^r)$ in $K[U, U^{-1}, V, V^{-1}]=K[X_0, X_1, 1/(\lambda X_0+1), 1/(\mu X_1+F(X_0))]$. By our assumption that $v(\mu)=m$ and $v(\lambda)=m-1$,

$$(\lambda X_0+1)^s-(\mu X_1+F(X_0))^r \equiv (s\lambda X_0+1)-(ra\lambda X_0+1) \equiv (s-ra)\lambda X_0 \bmod. \mathfrak{m}^m.$$

By the choice of a, $s-ra$ is invertible in A. Hence $G_{r,s}$ is defined by the ideal $(\{(\lambda X_0+1)^s-(\mu X_1+F(X_0))^r\}/\lambda)$ in $A[G]=A[X_0, X_1, 1/(\lambda X_0+1), 1/(\mu X_1+F(X_0))]$.

Now we define an S-homomorphism $\psi_{r,s}: \mathcal{G}^{(\mu)}\to G=\mathcal{E}^{(\lambda,\mu;F)}$ by

$$X_0\longmapsto \{(\mu X+1)^r-1\}/\lambda,\ X_1\longmapsto \{(\mu X+1)^s-a(\lambda X+1)^r+a-1\}/\mu:$$
$$A[X_0, X_1, 1/(\lambda X_0+1), 1/(\mu X_1+F(X_0))] \longrightarrow A[X, 1/(\mu X+1)].$$

Then $\psi_{r,s}: \mathcal{G}^{(\mu)}\to G$ factors through $\mathcal{G}^{(\mu)}\to G_{r,s}\to G$, and we can see that $\mathcal{G}^{(\mu)}\to G_{r,s}$ is an isomorphism. Hence we obtain a commutative diagram of S-groups:

$$\begin{array}{ccc} G_{r,s} & \xrightarrow{\ \psi_{r,s}\ } & G \\ \Big\downarrow{\scriptstyle \alpha^{(\mu)}} & & \Big\downarrow{\scriptstyle \alpha^{(\lambda,\mu;F)}} \\ \boldsymbol{G}_{m,S} & \xrightarrow{\ \varphi_{r,s}\ } & (\boldsymbol{G}_{m,S})^2 \end{array} .$$

Since $\tilde{G}$ is the Néron blow-up of H in G, $\tilde{G}_{r,s}$ is the Néron blow-up of $(G_{r,s})_k\cap H$ in $G_{r,s}$ (cf. Cor. 1.9). As is shown above,

$$\{(\lambda X_0+1)^s-(\mu X_1+F(X_0))^r\}/\lambda \equiv (s-ra)X_0 \equiv 0 \mod \mathfrak{m}.$$

Hence $(G_{r,s})_k$ is defined by the ideal $((s-ra)X_0)$ in $k[G]=k[X_0, X_1]$, and therefore, $(G_{r,s})_k \cap H$ is defined by the ideal $((s-ra)X_0, X_1^p-X_0)$ in $k[G]=k[X_0, X_1]$. Hence $(G_{r,s})_k \cap H$ is defined by the ideal (X^p) in $k[G_{r,s}]=k[X]$. Then $A[\tilde{G}_{r,s}]=A[X, 1/(\mu X+1), Y]/(\pi Y-X^p)$, and therefore $k[\tilde{G}_{r,s}]=k[X, Y]/(X^p)$.

REMARK 4.4.2. The exact sequence of S-groups

$$0 \longrightarrow \tilde{G}_{0,1} \longrightarrow \tilde{G} \longrightarrow \mathcal{G}^{(\lambda)} \longrightarrow 0$$

is the Néron blow-up of the exact sequence of k-groups

$$0 \longrightarrow \boldsymbol{\alpha}_p \longrightarrow \boldsymbol{G}_{a,k} \xrightarrow{F} \boldsymbol{G}_{a,k} \longrightarrow 0$$

in

$$0 \longrightarrow G_{0,1} \longrightarrow G \longrightarrow \mathcal{G}^{(\lambda)} \longrightarrow 0$$

(cf. Theorem 1.9).

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