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ON THE THEORY OF MULTIVALENT FUNCTIONS

By

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I would like to dedicate this paper to the late Professor Shigeo Ozaki.

1. Introduction.

Let $A(p)$ be the class of functions of the form

(1) $$f(z)=\sum_{n=p}^{\infty}a_{n}z^{n} \qquad (a_{p}\neq 0\,;\ p\in N=\{1,2,3,\cdots\})$$

which are regular in $|z|<1$.

A function $f(z)$ in $A(p)$ is said to be p-valently starlike iff

$$\operatorname{Re}\frac{zf'(z)}{f(z)}>0 \qquad (|z|<1).$$

We denote by $S(p)$ the subclass of $A(p)$ consisting of functions which are p-valently starlike in $|z|<1$.

Further, a function $f(z)$ in $A(p)$ is said to be p-valently convex iff

$$1+\operatorname{Re}\frac{zf''(z)}{f'(z)}>0 \qquad (|z|<1).$$

Also we denote by $C(p)$ the subclass of $A(p)$ consisting of all p-valently convex functions in $|z|<1$.

2. Preliminaries.

At first, we prove the following lemma by using the method of Ozaki [10].

LEMMA 1. *Let* $f(z)\in A(p)$ *and*

(2) $$\operatorname{Re}\frac{zf'(z)}{f(z)}>K \qquad in \quad |z|<1$$

where K is a real bounded constant, then we have

$$f(z)\neq 0 \qquad \text{in} \quad 0<|z|<1.$$

PROOF. Suppose that $f(z)$ has a zero of order n $(n\geqq 1)$ at a point α that satisfies $0<|\alpha|<1$. Then $f(z)$ can be written as $f(z)=(z-\alpha)^{n}g(z)$, $g(\alpha)\neq 0$ and

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it follows that

$$\frac{zf'(z)}{f(z)}=\frac{nz}{z-\alpha}+\frac{zg'(z)}{g(z)}$$

By a brief calculation, we have

$$\lim_{z\to\alpha}(z-\alpha)\frac{zf'(z)}{f(z)}=\lim_{z\to\alpha}\left(nz+(z-\alpha)\frac{zg'(z)}{g(z)}\right)$$
$$=n\alpha\neq 0$$

which result contradicts (2), because (2) shows that $zf'(z)/f(z)$ has no pole in $0<|z|<1$. Therefore $f(z)$ can not have any zero in $0<|z|<1$.

Applying the same method as the proof of Lemma 1, we have the following lemma.

LEMMA 2. *Let* $f(z)\in A(p)$ *and*

$$1+\mathrm{Re}\frac{zf''(z)}{f'(z)}>K \qquad in \quad |z|<1,$$

where K *is a real bounded constant, then*

$$f'(z)\neq 0 \qquad in \quad 0<|z|<1.$$

We owe this lemma to Ozaki [10] and we owe the following lemma to Ozaki [10, 11].

LEMMA 3. *Let the function* $f(z)$ *defined by* (1) *be in the class* $A(p)$ *and* $f^{(k)}(z)\neq 0$ *for* $k=0,1,2,\cdots,p$ *on* $|z|=1$.

Then we have

$$\int_{|z|=1}|d\arg d^{j}f(z)|\leqq\int_{|z|=1}|d\arg d^{j+1}f(z)|$$

for $j=0,1,2,\cdots,p-1$, *or, by a modification of the above inequalities,*

$$\int_0^{2\pi}\left|j+\mathrm{Re}\frac{zf^{(j+1)}z)}{f^{(j)}(z)}\right|d\theta\leqq\int_0^{2\pi}\left|j+1+\mathrm{Re}\frac{zf^{(j+2)}(z)}{f^{(j+1)}(z)}\right|d\theta$$

for $j=0,1,2,\cdots,p-1$, *where* $z=e^{i\theta}$ *and* $0\leqq\theta\leqq 2\pi$.

LEMMA 4. *Let* $f(z)$ *be regular in* $|z|\leqq 1$ *and* $f'(z)\neq 0$ *on* $|z|=1$.

If the next relation

$$\int_0^{2\pi}\left|1+\mathrm{Re}\frac{zf''(z)}{f'(z)}\right|d\theta<2\pi(p+1)$$

holds, then $f(z)$ *is at most* p*-valent in* $|z|\leqq 1$.

We owe this lemma to Umezawa [15, 17].

LEMMA 5. *If $F(z)$ and $G(z)$ are regular in $|z|<1$, $F(0)=G(0)=0$, $G(z)$ maps $|z|<1$ onto a many-sheeted region which is starlike with respect to the origin, and* $\mathrm{Re}(F'(z)/G'(z))>0$ *in $|z|<1$, then*

$$\mathrm{Re}(F(z)/G(z))>0 \qquad \text{in} \quad |z|<1.$$

We owe the above lemma to Sakaguchi [12] and Libera [4, Lemma 1].

Applying the same method as the proof of [4, Lemma 2], we can prove the following lemma.

LEMMA 6. *Let $f(z)\in S(p)$. Then*

$$F(z)=\int_0^z f(t)dt\in S(p+1)$$

or

$$\mathrm{Re}\frac{zF'(z)}{F(z)}>0 \qquad \text{in } |z|<1.$$

PROOF. Put $D(z)=zF'(z)=zf(z)$ and $N(z)=F(z)$, then $D(z)$ is (p+1)-valently starlike with respect to the origin, since

$$\mathrm{Re}\frac{zD'(z)}{D(z)}=1+\mathrm{Re}\frac{zf'(z)}{f(z)}>1>0 \qquad \text{in} \quad |z|<1.$$

By an easy calculation, we can have

$$\mathrm{Re}\frac{D'(z)}{N'(z)}=1+\mathrm{Re}\frac{zf'(z)}{f(z)}>0 \qquad \text{in} \quad |z|<1.$$

Therefore we have

$$\mathrm{Re}\frac{N'(z)}{D'(z)}>0 \qquad \text{in} \quad |z|<1.$$

Applying Lemma 5, we have

$$\mathrm{Re}\frac{N(z)}{D(z)}>0 \qquad \text{in} \quad |z|<1$$

or

$$\mathrm{Re}\frac{D(z)}{N(z)}>0 \qquad \text{in} \quad |z|<1.$$

This shows that

$$\mathrm{Re}\frac{zF'(z)}{F(z)}>0 \qquad \text{in} \quad |z|<1.$$

This completes our proof.

LEMMA 7. *If $f(z)\in S(p)$, then $f(z)$ is p-valent in $|z|<1$.*

PROOF. From the definition of $S(p)$ and Lemma 1, we have

$$f(z)\neq 0 \qquad \text{in} \quad 0<|z|<1.$$

Therefore we have

$$\int_0^{2\pi}\mathrm{Re}\frac{re^{i\theta}f'(re^{i\theta})}{f(re^{i\theta})}d\theta=2p\pi$$

for an arbitrary r, $0<r<1$.

This shows that $f(z)$ is p-valent in $|z|<1$ [1, p. 212].

From the definition of $C(p)$, Lemma 2 and [1, p. 211], we have the following lemma.

LEMMA 8. *If $f(z)\in C(p)$, then $f(z)$ is p-valent in $|z|<1$.*

REMARK 1. *Let $f(z)\in A(p)$. Then we can easily confirm that $f(z)$ is p-valently convex if and only if $zf'(z)$ is p-valently starlike.*

LEMMA 9. *Let $f(z)\in A(p)$ and suppose there exists a positive integer j for which*

$$j+\mathrm{Re}\frac{zf^{(j+1)}(z)}{f^{(j)}(z)}>0 \qquad \textit{in} \quad |z|<1$$

where $1\leqq j\leqq p$.

Then we have

$$j-1+\mathrm{Re}\frac{zf^{(j)}(z)}{f^{(j-1)}(z)}>0 \qquad \textit{in} \quad |z|<1.$$

PROOF. For the case $p=1$, from [5, 14] it is clear.

Therefor we assume $p\geqq 2$. Put

$$g(z)=\frac{f^{(j-1)}(z)}{p(p-1)\cdots(p-j+2)a_p}=z^{p-j+1}+\cdots.$$

Then we have

$$1+\mathrm{Re}\frac{zg''(z)}{g'(z)}=1+\mathrm{Re}\frac{zf^{(j+1)}(z)}{f^{(j)}(z)}>1-j \qquad \text{in} \quad |z|<1.$$

From Lemma 2, we have

$$g'(z)=\frac{f^{(j)}(z)}{p(p-1)\cdots(p-j+2)a_p}\neq 0 \qquad \text{in} \quad 0<|z|<1. \tag{3}$$

On the other hand, if $f^{(j-1)}(z)$ has such a zero as $z=\alpha$ of multiplicity $l(l\geqq 1)$ in $0<|z|<1$, then we can choose ρ such that $0<|\alpha|<\rho<1$ and so

$$f^{(j-1)}(z)\neq 0 \qquad \text{on} \quad |z|=\rho,$$

because if this reasoning is impossible, then from elementary analytic function theory (for emample [2, Theorem 8.1.3, p. 198], we have

$$f^{(j-1)}(z)\equiv 0 \qquad \text{in} \quad |\alpha|<|z|<1,$$

which contradicts

$$f^{(j-1)}(z)\not\equiv \text{constant}.$$

Applying the principle of the argument, Lemma 3, (3) and the assumption of Lemma 9, we have the following inequalities:

$$\begin{aligned}2\pi(p+l)&\leqq\int_0^{2\pi}\left(j-1+\mathrm{Re}\frac{zf^{(j)}(z)}{f^{(j-1)}(z)}\right)d\theta\\&\leqq\int_0^{2\pi}\left|j-1+\mathrm{Re}\frac{zf^{(j)}(z)}{f^{(j-1)}(z)}\right|d\theta\\&=\int_{|z|=r}|d\arg d^{j-1}f(z)|\\&\leqq\int_{|z|=r}|d\arg d^{j}f(z)|\\&=\int_0^{2\pi}\left|j+\mathrm{Re}\frac{zf^{(j+1)}(z)}{f^{(j)}(z)}\right|d\theta\\&=\int_0^{2\pi}\left(j+\mathrm{Re}\frac{zf^{(j+1)}(z)}{f^{(j)}(z)}\right)d\theta\\&=2p\pi\end{aligned}$$

where $z=\rho e^{i\theta}$ and $0\leqq\theta\leqq 2\pi$.

But this result contradicts $2p\pi<2\pi(p+l)$.

This shows that $f^{(j-1)}(z)\neq 0$ in $0<|z|<1$ ($f^{(j-1)}(z)$ has a zero $z=0$ of order $p-j+1$).

Therefore we have

$$\begin{aligned}2p\pi&=\int_0^{2\pi}\left(j-1+\mathrm{Re}\frac{zf^{(j)}(z)}{f^{(j-1)}(z)}\right)d\theta\\&=\int_0^{2\pi}\left|j-1+\mathrm{Re}\frac{zf^{(j)}(z)}{f^{(j-1)}(z)}\right|d\theta\\&=2p\pi\end{aligned}$$

for an arbitrary r, $0<r<1$, $z=re^{i\theta}$ and $0\leqq\theta\leqq 2\pi$.

This shows

(4) $$j-1+\mathrm{Re}\frac{zf^{(j)}(z)}{f^{(j-1)}(z)}\geqq 0 \qquad \text{in} \quad |z|<1.$$

But if there is a point z_0 satisfying $|z_0|<1$ and

$$j-1+\mathrm{Re}\frac{z_0f^{(j)}(z_0)}{f^{(j-1)}(z_0)}=0,$$

then we can choose a point z in some neighborhood of z_0 in $|z|<1$ such that

$$j-1+\mathrm{Re}\frac{zf^{(j)}(z)}{f^{(j-1)}(z)}<0.$$

This contradicts (4). Therefore we have

$$j-1+\mathrm{Re}\frac{zf^{(j)}(z)}{f^{(j-1)}(z)}>0 \qquad \text{in} \quad |z|<1.$$

3. Statement of results.

THEOREM 1. *Let $f(z)\in A(p)$ and suppose*

$$p+\mathrm{Re}\frac{zf^{(p+1)}(z)}{f^{(p)}(z)}>0 \qquad \text{in} \quad |z|<1. \tag{5}$$

Then $f(z)$ is p-valent in $|z|<1$ and

$$k+\mathrm{Re}\frac{zf^{(k+1)}(z)}{f^{(k)}(z)}>0 \qquad \text{in} \quad |z|<1$$

for $k=0,1,2,\cdots,p-1$.
This shows that $f(z)\in C(p)$ and $f(z)\in S(p)$.

PROOF. From Lemma 9 and (5), we easily have

$$k+\mathrm{Re}\frac{zf^{(k+1)}(z)}{f^{(k)}(z)}>0 \qquad \text{in} \quad |z|<1$$

for $k=0,1,2,\cdots,p-1$.
This shows that $f(z)$ is p-valent in $|z|<1$, $f(z)\in C(p)$ and $f(z)\in S(p)$.

THEOREM 2. *Let $f(z)\in A(p)$ and*

$$p+\mathrm{Re}\frac{zf^{(p+1)}(z)}{f^{(p)}(z)}>-\frac{1}{2} \qquad \text{in} \quad |z|<1.$$

Then $f(z)$ is p-valent in $|z|<1$.

PROOF. For the case $p=1$, this is due to Umezawa [15, 17].
If we put

$$g(z)=\frac{f^{(p-1)}}{p(p-1)\cdots 3\cdot 2\cdot a_p}=z+\cdots, \qquad p\geqq 2,$$

then we have

$$1+\mathrm{Re}\frac{zg''(z)}{g'(z)}>\frac{1}{2}-p \qquad \text{in} \quad |z|<1.$$

From Lemma 2, we have

$$f^{(p)}(z)=p(p-1)\cdots 3\cdot 2\cdot a_p g'(z)\neq 0 \qquad \text{in} \quad |z|<1.$$

On the other hand, if $f^{(p-1)}(z)$ has a zero $z=\alpha$ of multiplicity $l(l\geqq 1)$ in $0<|z|<1$, then we can choose r satisfying $0<|\alpha|<r<1$ such that

$$f^{(p-1)}(z)\neq 0 \qquad \text{on} \quad |z|=r,$$

because if this supposition is impossible, then from elementary analytic function theory (for example [2, Theorem 8.1.3, p. 198]), we have

$$f^{(p-1)}(z)\equiv 0 \qquad \text{in} \quad |\alpha|<|z|<1.$$

This contradicts

$$f^{(p-1)}(z)\not\equiv \text{constant} \qquad \text{in} \quad |\alpha|<|z|<1.$$

Applying the principle of the argument and Lemma 3, we have the following inequalities:

$$
\begin{aligned}
(6) \qquad 2\pi(p+l) &\leqq \int_0^{2\pi}\left(p-1+\mathrm{Re}\frac{zf^{(p)}(z)}{f^{(p-1)}(z)}\right)d\theta \\
&\leqq \int_0^{2\pi}\left|p-1+\mathrm{Re}\frac{zf^{(p-1)}(z)}{f^{(p-1)}(z)}\right|d\theta \\
&= \int_{|z|=r}|d \arg d^{p-1}f(z)| \\
&\leqq \int_{|z|=r}|d \arg d^{p}f(z)| \\
&= \int_0^{2\pi}\left|p+\mathrm{Re}\frac{zf^{(p+1)}(z)}{f^{(p)}(z)}\right|d\theta \\
&= \int_0^{2\pi}\left|p+\frac{1}{2}+\frac{zf^{(p+1)}(z)}{f^{(p)}(z)}-\frac{1}{2}\right|d\theta \\
&< \int_0^{2\pi}\left|p+\frac{1}{2}+\frac{zf^{(p+1)}(z)}{f^{(p)}(z)}\right|d\theta+\pi \\
&= \int_0^{2\pi}\left(p+\frac{1}{2}+\mathrm{Re}\frac{zf^{(p+1)}(z)}{f^{(p)}(z)}\right)d\theta+\pi \\
&= 2\pi(p+1),
\end{aligned}
$$

where $z=re^{i\theta}$ and $0\leqq\theta\leqq 2\pi$.

But this result contradicts $2\pi(p+1)\leqq 2\pi(p+l)$. Thus it is not possible for $f^{(p-1)}(z)$ to vanish in $0<|z|<1$.

From (6) we have

$$
\begin{aligned}
(7) \qquad &\int_0^{2\pi}\left|p-1+\mathrm{Re}\frac{zf^{(p)}(z)}{f^{(p-1)}(z)}\right|d\theta \\
&= \int_{|z|=r}|d \arg d^{p-1}f(z)|<2\pi(p+1)
\end{aligned}
$$

for an arbitrary r, $0<r<1$, and $z=re^{i\theta}$.

Repeating the same method as the above, we have $f^{(p-2)}(z)$, $f^{(p-3)}(z), \cdots$, $f''(z), f'(z)$ that do not vanish in $0<|z|<1$ and for an arbitrary r, $0<r<1$,

(8)
$$\int_{|z|=r} |d \arg df(z)| = \int_0^{2\pi} \left|1+\mathrm{Re}\frac{zf''(z)}{f'(z)}\right| d\theta < 2\pi(p+1).$$

From Lemma 4, (8) shows that $f(z)$ is p-valent in $|z|<1$.

This is a generalization of the theorem in [11, 15].

Applying the same method as the proof of Theorem 2 and Lemma 4, we have the following theorems.

THEOREM 3. *Let $f(z) \in A(p)$ and suppose*

$$\int_0^{2\pi} \left|p+\mathrm{Re}\frac{zf^{(p+1)}(z)}{f^{(p)}(z)}\right| d\theta < 2\pi(p+1)$$

for an arbitary r, $0<r<1$, and $z=re^{i\theta}$.

Then $f(z)$ is p-valent in $|z|<1$.

This is a generalization of [10, 15, 16, 17].

THEOREM 4. *Let $f(z) \in A(p)$ and*

$$\int_0^{2\pi} \left|1+\mathrm{Re}\frac{zf^{(p+1)}(z)}{f^{(p)}(z)}\right| d\theta < 4\pi$$

for an arbitrary r, $0<r<1$, and $z=re^{i\theta}$.

Then $f(z)$ is p-valent in $|z|<1$.

THEOREM 5. *Let $f(z) \in A(p)$ and suppose*

$$\mathrm{Re}\frac{zf^{(p)}(z)}{f^{(p-1)}(z)} > 0 \qquad in \quad |z|<1.$$

Then we have

$$\mathrm{Re}\frac{zf^{(k)}(z)}{f^{(k-1)}(z)} > 0 \qquad in \quad |z|<1$$

or

$$f^{(p-k)}(z) \in S(k)$$

for $k=1, 2, 3, \cdots, \mathrm{p}$.

PROOF. For the case $p=1$, the theorem is trivial, so we assume $p \geqq 2$.
Put

$$g(z) = \frac{f^{(p-1)}(z)}{p(p-1)\cdots 3\cdot 2\cdot a_p} = z+\cdots.$$

Then we have

$$\mathrm{Re}\frac{zg'(z)}{g(z)}=\mathrm{Re}\frac{zf^{(p)}(z)}{f^{(p-1)}(z)}>0 \qquad \text{in } |z|<1.$$

This shows that $g(z)$ is univalently starlike in $|z|<1$.

An application of Lemma 6 shows that

$$\int_0^z g(t)dt=\frac{f^{(p-2)}(z)}{p(p-1)\cdots 3\cdot 2\cdot a_p}\in S(2)$$

or

$$\mathrm{Re}\frac{zf^{(p-1)}(z)}{f^{(p-2)}(z)}>0 \qquad \text{in } |z|<1.$$

Applying the same method as the above over again, we have

$$f^{(p-k)}(z)\in S(k)$$

or

$$\mathrm{Re}\frac{zf^{(k)}(z)}{f^{(k-1)}(z)}>0 \qquad \text{in } |z|<1.$$

for $k=1,2,3,\cdots,\mathrm{p}$. This completes our proof.

THEOREM 6. *Let $f(z)\in A(p)$ and if there exists a positive integer $q(1\leqq q\leqq p)$ that satisfies*

$$\int_0^{2\pi}\left|\mathrm{Re}\frac{zf^{(q)}(z)}{f^{(q-1)}(z)}\right|d\theta\leqq 2\pi(p+1-q)$$

for an arbitrary r, $0<r<1$, and $z=re^{i\theta}$, then we have

$$\mathrm{Re}\frac{zf^{(k)}(z)}{f^{(k-1)}(z)}>0 \qquad \textit{in } |z|<1$$

or

$$f^{(k-1)}(z)\in S(p+1-k)$$

for $k=1,2,3,\cdots,\mathrm{q}$.

PROOF. From the principle of the argument and the assumption, we have

$$2\pi(p+1-q)\leqq\int_0^{2\pi}\mathrm{Re}\frac{zf^{(q)}(z)}{f^{(p-1)}(z)}d\theta$$

$$\leqq\int_0^{2\pi}\left|\mathrm{Re}\frac{zf^{(q)}(z)}{f^{(q-1)}(z)}\right|d\theta\leqq 2\pi(p+1-q)$$

for an arbitrary r, $0<r<1$, and $z=re^{i\theta}$.

Therefore we must have

$$\mathrm{Re}\frac{zf^{(q)}(z)}{f^{(q-1)}(z)}\geqq 0 \qquad \text{in} \quad |z|<1.$$

Applying the same method as the proof of Theorem 1, we can show

$$\mathrm{Re}\frac{zf^{(q)}(z)}{f^{(q-1)}(z)}>0 \qquad \text{in} \quad |z|<1$$

or

$$f^{(q-1)}(z)\in S(p+1-q).$$

Integrating $f^{(q-1)}(z)$, then from Lemma 6, we have

$$f^{(p-2)}(z)\in S(p+2-q).$$

Repeating the same method as the above, we can complete the proof of Theorem 6.

Applying the same method as the proof of Theorem 1 and 2, we can easily prove

THEOREM 7. *Suppose $f(z)\in C(p)$. Then we have $f(z)\in S(p)$.*

REMARK 2. *For the case $p=1$, $C(p)$ and $S(p)$ are the subclasses of classical univalent functions which are convex and starlike respectively, and $S(1)\supset C(1)$.*

It is worth noting that for $p\geqq 2$, then $S(p)\not\supset C(p)$, if $f(z)$ is not normalized such that $f(z)=\sum_{n=p}^{\infty}a_n z^n$, $(a_p\neq 0)$.

A. W. Goodman noticed Remark 2 [1, p. 212].

THEOREM 8. *Let $f(z)\in A(p)$ and if there exists a $(p-k+1)$-valent starlike function $g(z)=\sum_{n=p-k+1}^{\infty} b_n z^n$, $(b_{p-k+1}\neq 0)$ that satisfies*

$$\text{(9)} \qquad \mathrm{Re}\frac{zf^{(k)}(z)}{g(z)}>0 \qquad \textit{in} \quad |z|<1,$$

then $f(z)$ is p-valent in $|z|<1$.

PROOF. For the case $p=1$, it is well-known in [3]. So we assume $p\geqq 2$.

If we put $g(z)=z\varphi'(z)$, then from Remark 1, $\varphi(z)$ is a $(p-k+1)$-valently convex function. From Theorem 7, $\varphi(z)$ is $(p-k+1)$-valently starlike in $|z|<1$ and from (9) we can have

$$\mathrm{Re}\frac{f^{(k)}(z)}{\varphi'(z)}>0 \qquad \text{in} \quad |z|<1.$$

Applying Lemma 5 repeatedly, we have

$$\mathrm{Re}\frac{f'(z)}{\phi(z)}>0 \qquad \text{in} \quad |z|<1$$

where $\phi^{(k-2)}(z)=\varphi(z)$, $\phi(0)=\phi'(0)=\phi''(0)=\cdots=\phi^{(k-2)}(0)=0$.

Then from Lemma 6, $\phi(z)$ is a $(p-1)$-valently starlike function.

On the other hand, if we put $G(z)=z\phi(z)$, then we have

$$\mathrm{Re}\frac{zG'(z)}{G(z)}=\frac{d\arg G(z)}{d\theta}=\frac{d\arg z\phi(z)}{d\theta}$$
$$=1+\frac{d\arg\phi(z)}{d\theta}>1$$

for an arbitrary r, $0<r<1$, $z=re^{i\theta}$ and $0\leqq\theta\leqq 2\pi$, and furthermore we have

$$\int_0^{2\pi}\mathrm{Re}\frac{zG'(z)}{G(z)}d\theta=\int_0^{2\pi}\left(1+\frac{d\arg\phi(z)}{d\theta}\right)d\theta$$
$$=2p\pi.$$

It shows that $G(z)$ is p-valently starlike in $|z|<1$.

Therefore we have

$$\mathrm{Re}\frac{zf'(z)}{\phi(z)}=\mathrm{Re}\frac{zf'(z)}{G(z)}>0 \qquad \text{in} \quad |z|<1$$

where $G(z)$ is a p-valently starlike function.

From [6, 18], $f(z)$ is p-valent in $|z|<1$. This completes our proof.

Let $f(z)\in A(p)$ and let α be a real number. Then $f(z)$ is said to be p-valently α-convex in $|z|<1$ iff

$$\mathrm{Re}\left[(1-\alpha)\frac{zf^{(p)}(z)}{f^{(p-1)}(z)}+\alpha\left(1+\frac{zf^{(p+1)}(z)}{f^{(p)}(z)}\right)\right]>0 \tag{10}$$

holds in $|z|<1$.

This is a generalization of α-convex functions [7, 8, 9].

THEOREM 9. *Let $f(z)$ defined by* (1) *be p-valently α-convex in $|z|<1$ and let $(\alpha-1)$ not be a positive integer.*

Then we have that $f(z)$ is p-valent in $|z|<1$ and

$$\mathrm{Re}\frac{zf^{(k)}(z)}{f^{(k-1)}(z)}>0 \qquad \text{in } |z|<1$$

for $k=1,2,3,\cdots,\mathrm{p}$.

PROOF. For the case $\alpha=1$, from the assumption we have

$$1+\mathrm{Re}\frac{zf^{(p+1)}(z)}{f^{(p)}(z)}>0 \qquad \text{in} \quad |z|<1. \tag{11}$$

If we put

$$g(z)=\frac{f^{(p-1)}(z)}{p(p-1)\cdots 3\cdot 2\cdot a_p}=z+\cdots,$$

then from (11) we have

$$1+\mathrm{Re}\frac{zg''(z)}{g'(z)}>0 \qquad \text{in} \quad |z|<1,$$

and so $g(z)\in C(1)$.

By Marx-Strohhäcker's theorem [5, 14], we have

$$\mathrm{Re}\frac{zg'(z)}{g(z)}=\mathrm{Re}\frac{zf^{(p)}(z)}{f^{(p-1)}(z)}>\frac{1}{2}>0 \qquad \text{in} \quad |z|<1$$

Then, from Theorem 5, we have

$$\mathrm{Re}\frac{zf^{(k)}(z)}{f^{(k-1)}(z)}>0 \qquad \text{in} \quad |z|<1$$

for $k=1,2,3,\cdots,p$.

Next, we assume that α is not a positive integer. Applying the same method as the proof of [13, Theorem 2] (It is the same idea as the proof of Lemma 1), we can prove that $f^{(p-1)}(z)\neq 0$ in $0<|z|<1$ and $f^{(p)}(z)\neq 0$ in $0<|z|<1$. Because, if $f^{(p-1)}(z)$ has a zero of order n ($n\geqq 1$) at a point β such that $0<|\beta|<1$, then $f^{(p-1)}(z)$ may be put

$$f^{(p-1)}(z)=(z-\beta)^n g(z), \qquad g(\beta)\neq 0.$$

Then by an easy calculation, we can have

$$\lim_{z\to\beta}(z-\beta)\left\{(1-\alpha)\frac{zf^{(p)}(z)}{f^{(p-1)}(z)}+\alpha\left(1+\frac{zf^{(p+1)}(z)}{f^{(p)}(z)}\right)\right\}$$
$$=\beta(n-\alpha)\neq 0$$

But this is a contradiction to (10), because

$$(1-\alpha)\frac{zf^{(p)}(z)}{f^{(p-1)}(z)}+\alpha\left(1+\frac{zf^{(p+1)}(z)}{f^{(p)}(z)}\right)$$

has no zero in $|z|<1$. Therefore $f^{(p-1)}(z)$ can not have any zero in $0<|z|<1$. Then from the assumption (10), $f^{(p)}(z)$ has no zero in $0<|z|<1$ either.

Hence we have that $f^{(p-1)}(z)\neq 0$ in $0<|z|<1$ and $f^{(p)}(z)\neq 0$ in $0<|z|<1$. Therefore, if we put $p(z)=zf^{(p)}(z)/f^{(p-1)}(z)$ in (10), then we can obtain

$$\mathrm{Re}\left[p(z)-i\alpha\frac{\partial}{\partial\theta}\log p(z)\right]>0$$

for an arbitrary r, $0<r<1$ and $z=re^{i\theta}$.

Applying the same method as the proof of [7], we can have

$$\mathrm{Re}\frac{zf^{(p)}(z)}{f^{(p-1)}(z)}>0 \qquad \text{in} \quad |z|<1.$$

From Theorem 5, it follows that

$$\mathrm{Re}\frac{zf^{(k)}(z)}{f^{(k-1)}(z)}>0 \qquad \text{in} \quad |z|<1$$

for $k=1, 2, 3, \cdots, \mathrm{p}$.

This completes our proof.

Applying the same method as the proof of [13, Theorem 2] and Theorem 5, we can prove

THEOREM 10. *Let $f(z)\in A(p)$ and suppose*

$$\mathrm{Re}\left(1+\frac{zf^{(p+1)}(z)}{f^{(p)}(z)}-\frac{zf^{(p)}(z)}{f^{(p-1)}(z)}\right)>-\frac{1}{2} \qquad in \quad |z|<1.$$

Then we have

$$\mathrm{Re}\frac{zf^{(k)}(z)}{f^{(k-1)}(z)}>0 \qquad in \quad |z|<1$$

for $k=1, 2, 3, \cdots, \mathrm{p}$.

THEOREM 11. *Let $f(z)\in A(p)$ and if $f(z)$ satisfies the following condition*

$$\int_0^{2\pi}\left|1+\mathrm{Re}\frac{zf^{(p)}(z)}{f^{(p-1)}(z)}\right|d\theta\leqq 4\pi$$

for an arbitrary r, $0<r<1$ and $z=re^{i\theta}$, then $f^{(k-1)}(z)\in S(p+1-k)$ for $k=1, 2, 3, \cdots, \mathrm{p}-1$.

PROOF. From the principle of the argument and assumption, we have

$$4\pi\leqq\int_0^{2\pi}\left(1+\mathrm{Re}\frac{zf^{(p)}(z)}{f^{(p-1)}(z)}\right)d\theta \leqq\int_0^{2\pi}\left|1+\mathrm{Re}\frac{zf^{(p)}(z)}{f^{(p-1)}(z)}\right|d\theta\leqq 4\pi \tag{12}$$

for an arbitrary r, $0<r<1$ and $z=re^{i\theta}$.

Applying the same reason as in the proof of Theorem 1 and from (12), we can have

$$1+\mathrm{Re}\frac{zf^{(p)}(z)}{f^{(p-1)}(z)}>0 \qquad \text{in} \quad |z|<1.$$

From the definition of the class $C(p)$, this shows $f^{(p-2)}(z)\in C(2)$.

Then from Theorem 7, we have $f^{(p-2)}(z)\in S(2)$.

Applying Theorem 5, we have

$$f^{(k-1)}(z)\in S(p+1-k)$$

for $k=1, 2, 3, \cdots, \mathrm{p}-1$. This completes our proof.

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